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ON THE ORDER OF MAGNITUDE OF WALSH-FOURIER TRANSFORM

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Abstract. For a Lebesgue integrable complex-valued function f defined on $\mathbb{R}^+ := [0,\infty)$ let $\hat{f}$ be its Walsh-Fourier transform. The Riemann-Lebesgue lemma says that $\hat{f}(y) \to 0$ as $y \to \infty$. But in general, there is no definite rate at which the Walsh-Fourier transform tends to zero. In fact, the Walsh-Fourier transform of an integrable function can tend to zero as slowly as we wish. Therefore, it is interesting to know for functions of which subclasses of $L^1(\mathbb{R}^+)$ there is a definite rate at which the Walsh-Fourier transform tends to zero. We determine this rate for functions of bounded variation on $\mathbb{R}^+$. We also determine such rate of Walsh-Fourier transform for functions of bounded variation in the sense of Vitali defined on $(\mathbb{R}^+)^N$, $N \in \mathbb{N}$.

Keywords: function of bounded variation over $\mathbb{R}^+$; function of bounded variation over $(\mathbb{R}^+)^2$; function of bounded variation over $(\mathbb{R}^+)^N$; order of magnitude; Riemann-Lebesgue lemma; Walsh-Fourier transform

MSC 2010: 42C20, 26A12, 26A45, 26B30, 26D15

1. Introduction

We consider the Walsh orthonormal system $\{w_m(x)\colon m \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, defined on the unit interval $\mathbb{I} := [0,1)$ in the Paley enumeration (see [19]). To go into some details, let

$$r_0(x) := \begin{cases} 1 & \text{if } x \in [0,\frac{1}{2}), \\ -1 & \text{if } x \in [\frac{1}{2},1), \end{cases}$$

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and extend $r_0(x)$ for the half-real axis $\mathbb{R}^+ := [0,\infty)$ with period 1. The Rademacher orthonormal system $\{r_k(x)\colon\ k \in \mathbb{N}\}$ is defined by

$$r_k(x) := r_0(2^k x), \quad k = 1, 2, \ldots,\ x \in \mathbb{I}.$$

Now, the mth Walsh function $w_m(x)$ in the Paley enumeration is defined as follows: If

$$m = \sum_{k=0}^{\infty} m_k 2^k, \quad \text{where each } m_k = 0 \text{ or } 1,$$

is the binary decomposition of $m \in \mathbb{N}$, then let

$$w_m(x) := \prod_{k=0}^{\infty} r_k^{m_k}(x), \quad x \in \mathbb{I}. \tag{1.1}$$

Clearly, $m_k = 0$ except for a finite number of k's. Thus, the right-hand side of (1.1) is a finite product for each $m \in \mathbb{N}$. In particular, we have

$$w_0(x) \equiv 1 \quad \text{and} \quad w_{2^m} = r_m(x), \quad m \in \mathbb{N}_0.$$

It is well known that $\{w_m(x)\colon\ m \in \mathbb{N}_0\}$ is a complete orthonormal system on $\mathbb{I}$.

Any $x \in \mathbb{I}$ can be written in the form

$$x = \sum_{k=0}^{\infty} x_k 2^{-k-1}, \quad \text{where each } x_k = 0 \text{ or } 1.$$

For each $x \in \mathbb{I} \setminus Q$ there is only one expression of this form, where Q is the collection of dyadic rationals in $\mathbb{I}$. When $x \in Q$, there are two expressions of this form, one which terminates in 0's and other which terminates in 1's. Now the *dyadic sum* of $x, y \in \mathbb{I}$ is defined by

$$x \dot{+} y := \sum_{k=0}^{\infty} |x_k - y_k| 2^{-k-1}.$$

A remarkable property of the Walsh functions is that for each $m \in \mathbb{N}_0$ we have

$$w_m(x \dot{+} y) = w_m(x) w_m(y), \quad x, y \in \mathbb{I},\ x \dot{+} y \notin Q.$$

Next, we consider the generalized Walsh functions ψ_x, $x \in \mathbb{R}^+$ (see [20], Chapter 9), and recall the following properties:

(i) $\psi_k(x) = w_k(x)$ for $k \in \mathbb{N}_0$, $x \in \mathbb{I}$;

(ii) $\psi_y(x \dot{+} t) = \psi_y(x)\psi_y(t)$ for $x, t \in \mathbb{R}^+$ and $x \dot{+} t$ dyadic irrational;

(iii) $\psi_y(x) = \psi_x(y)$, $\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y)$ for $x, y \in \mathbb{R}^+$, where for $u \in \mathbb{R}^+$, $[u]$ represents the greatest integer in u;

(iv) the functions ψ_j, $j \in \mathbb{N}_0$ form a complete orthonormal system in each of the intervals of the form $[k, k+1)$, $k \in \mathbb{N}_0$;

(v) ψ_j is a periodic extension of w_j from $\mathbb{I}$ to $\mathbb{R}^+$.

Now we recall (see e.g. [20], page 421) that the Walsh-Fourier transform of an $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(y) := \int_0^\infty f(x)\psi_y(x)\,\mathrm{d}x, \quad y \in \mathbb{R}^+. \tag{1.2}$$

We also recall that the Riemann-Lebesgue lemma holds for Walsh-Fourier transform (see [20], page 422), that is, $\hat{f}(y) \to 0$ as $y \to \infty$. But in general, there is no definite rate at which the Walsh-Fourier transform tends to zero. In fact, the Walsh-Fourier transform of an integrable function can tend to zero as slowly as we wish (see e.g. [13], 32.47 (b)). Therefore, it is interesting to know for functions of which subclasses of $L^1(\mathbb{R}^+)$ there is a definite rate at which the Walsh-Fourier transform tends to zero.

Looking to the periodic version, for the case of trigonometric Fourier series, that is, for functions on one-dimensional torus $\mathbb{T} := [0, 2\pi)$, the study of order of magnitude of Fourier coefficients is done extensively (see e.g. [15], [21], see also [3], Section 2.3, page 30 and [22], Section 4, page 45). This study in periodic version for trigonometric Fourier series is done even for more general cases, that is, for the case of functions on two-dimensional torus, or more generally, on the N-dimensional torus $\mathbb{T}^N := [0, 2\pi)^N$, $N \in \mathbb{N}$ (see e.g. [16], [5], [6], [8]).

Also looking to the periodic version, for the case of Walsh-Fourier series, that is, for functions defined on $\mathbb{I}$, the study of order of magnitude of Walsh-Fourier coefficients is done (see e.g. [4], [11]). This study in periodic version for Walsh-Fourier series is done even for more general cases, that is, for the case of functions on two-dimensional torus $\mathbb{I}^2$, or more generally, on the N-dimensional torus $\mathbb{I}^N$, $N \in \mathbb{N}$ (see e.g. [7]).

Recently, in 2015 (see [9]), we have studied the order of magnitude of trigonometric Fourier transform for functions of bounded variation on $\mathbb{R}$ and for functions of bounded variation in the sense of Vitali on $\mathbb{R}^N$ and obtained results analogous to the periodic case. But it appears that such a study for the Walsh-Fourier transform has not yet been done. In this paper we carry out this study and determine the rate of decay of Walsh-Fourier transform for functions of bounded variation on $\mathbb{R}^+$. We also determine such rate of Walsh-Fourier transform for functions of bounded variation in the sense of Vitali defined on $(\mathbb{R}^+)^N$, $N \in \mathbb{N}$.

2. One-dimensional case

We recall that a function $f\colon \mathbb{R}^+ \to \mathbb{C}$ is said to be of bounded variation over $\mathbb{R}^+$, in symbol $f \in \mathrm{BV}(\mathbb{R}^+)$, if

$$\sup_{\mathcal{S}} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty, \tag{2.1}$$

where the supremum is extended over all finite sequences

$$\mathcal{S}\colon\ 0 \leqslant x_0 < x_1 < x_2 < \ldots < x_n < \infty \quad \text{and} \quad n = 1, 2, \ldots$$

The supremum in (2.1), denoted by $V(f)$, is called the total variation of f over $\mathbb{R}^+$.

It is clear that the above definition of bounded variation over $\mathbb{R}^+$ can be reformulated equivalently as follows. A function f is of bounded variation over $\mathbb{R}^+$ if and only if f is of bounded variation over any closed and bounded interval $[a, b] \subset \mathbb{R}^+$ in the ordinary sense and the set of the total variations $V(f, [a, b])$ of f over all such closed and bounded intervals $[a, b]$ is bounded. Furthermore, if this is the case, then the supremum of the total variations over all such closed and bounded intervals is equal to $V(f)$ defined above (see e.g. [18], page 238).

In a similar way, one can define the notion of bounded variation over the intervals of the form $[a, \infty)$, where $a \in \mathbb{R}$ is arbitrary.

Given $f \in \mathrm{BV}(\mathbb{R}^+)$, let $V(f, x) := V(f, [0, x])$ denote the total variation of f over the interval $[0, x]$. Then it is evident that

$$\lim_{x\to\infty} V(f, x) = V(f). \tag{2.2}$$

We note that the variation of f over $[x, \infty)$ is given by $V(f, [x, \infty)) = V(f) - V(f, x)$ (see e.g. [9], (9) in Lemma 1) and hence from (2.2) it follows that

$$\lim_{x\to\infty} V(f, [x, \infty)) = 0.$$

In this section we prove a theorem concerning definite rate of decay of Walsh-Fourier transform for functions of bounded variation on $\mathbb{R}^+$. Our main theorem of this section is as follows.

Theorem 2.1. *If $f \in L^1(\mathbb{R}^+) \cap \mathrm{BV}(\mathbb{R}^+)$, then $\hat{f}(y) = O(1/y)$, $y \to \infty$.*

We need the following lemma whose proof is similar to that of Lemma 1 in [9].

Lemma 2.1. *If* $f \in \mathrm{BV}(\mathbb{R}^+)$ *and* $\{a_n\colon n \in \mathbb{N}_0\}$ *is an increasing sequence of non-negative real numbers with* $\lim_{n\to\infty} a_n = \infty$*, then the series* $\sum_{n=1}^{\infty} V(f, [a_{n-1}, a_n])$ *converges and*

$$V(f, [a_0, \infty)) = \sum_{n=1}^{\infty} V(f, [a_{n-1}, a_n]). \tag{2.3}$$

P r o o f of Theorem 2.1. We present a proof using Taibleson-like technique (see [21]), which we have developed for $\mathbb{I}$ in [11] and here for $\mathbb{R}^+$.

Fix $y \in \mathbb{R}^+$, $y \geqslant 1$. Put $n = [y]$. Then $n \in \mathbb{N}$, so there exists a unique $m \in \mathbb{N}_0$ such that $2^m \leqslant n < 2^{m+1}$. Now, put $a_i = i/2^m$ for $i = 0, 1, 2, 3, \ldots, 2^m$. Then by the definition of Walsh functions, w_n takes the value 1 on one half of each of the intervals (a_{i-1}, a_i) and the value -1 on the other half. Therefore we have

$$\int_{a_{i-1}}^{a_i} w_n(x)\,\mathrm{d}x = 0 \quad \text{for } i = 1, 2, 3, \ldots, 2^m.$$

Since ψ_n is a periodic extension of w_n from $\mathbb{I}$ to $\mathbb{R}^+$, for each $k \in \mathbb{N}_0$ and $i = 1, 2, 3, \ldots, 2^m$, we have

$$\int_{k+a_{i-1}}^{k+a_i} \psi_n(x)\,\mathrm{d}x = \int_{a_{i-1}}^{a_i} \psi_n(x)\,\mathrm{d}x = \int_{a_{i-1}}^{a_i} w_n(x)\,\mathrm{d}x = 0. \tag{2.4}$$

Define a step function g on $\mathbb{R}^+$ by $g(x) := f(k + a_{i-1})$ on $[k + a_{i-1}, k + a_i)$, $i = 1, 2, 3, \ldots, 2^m$, $k \in \mathbb{N}_0$. Since $f \in L^1(\mathbb{R}^+)$, it follows that $g \in L^1(\mathbb{R}^+)$. Then in view of (2.4) for each $k \in \mathbb{N}_0$ and $i = 1, 2, 3, \ldots, 2^m$, we have

$$\int_{k+a_{i-1}}^{k+a_i} g(x)\psi_n(x)\,\mathrm{d}x = f(k + a_{i-1}) \int_{k+a_{i-1}}^{k+a_i} \psi_n(x)\,\mathrm{d}x = 0. \tag{2.5}$$

Therefore by Property (iii) of generalized Walsh functions, as stated above and from (2.5) we have

$$\begin{aligned} \int_{k+a_{i-1}}^{k+a_i} g(x)\psi_y(x)\,\mathrm{d}x &= \int_{k+a_{i-1}}^{k+a_i} g(x)\psi_n(x)\psi_{[x]}(y)\,\mathrm{d}x \\ &= \psi_k(y) \int_{k+a_{i-1}}^{k+a_i} g(x)\psi_n(x)\,\mathrm{d}x = 0 \end{aligned} \tag{2.6}$$

for each $k \in \mathbb{N}_0$ and $i = 1, 2, 3, \ldots, 2^m$. Now, by definition (1.2) of $\hat{f}(y)$ and (2.6) we have

$$\begin{aligned}
|\hat{f}(y)| &= \left|\int_0^{\infty} f(x)\psi_y(x)\,\mathrm{d}x\right| = \left|\sum_{k=0}^{\infty}\sum_{i=1}^{2^m}\int_{k+a_{i-1}}^{k+a_i} f(x)\psi_y(x)\,\mathrm{d}x\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{i=1}^{2^m}\int_{k+a_{i-1}}^{k+a_i} (f(x)-g(x))\psi_y(x)\,\mathrm{d}x\right| \leqslant \sum_{k=0}^{\infty}\sum_{i=1}^{2^m}\int_{k+a_{i-1}}^{k+a_i} |f(x)-g(x)|\,\mathrm{d}x \\
&= \sum_{k=0}^{\infty}\sum_{i=1}^{2^m}\int_{k+a_{i-1}}^{k+a_i} |f(x)-f(k+a_{i-1})|\,\mathrm{d}x \\
&\leqslant \sum_{k=0}^{\infty}\sum_{i=1}^{2^m} V(f,[k+a_{i-1},k+a_i])(k+a_i-(k+a_{i-1})) \\
&= \sum_{k=0}^{\infty} V(f,[k,k+1])\frac{1}{2^m} = \frac{1}{2^m}V(f) \leqslant \frac{2}{n}V(f) \leqslant \frac{4}{y}V(f).
\end{aligned}$$

Note that we have used (2.3) of Lemma 2.1 in the last step. Therefore we have

$$|\hat{f}(y)| \leqslant \frac{4\,V(f)}{y}. \tag{2.7}$$

This completes the proof of Theorem 2.1. □

P r o b l e m 2.1. What can be said about the exactness of the constant in (2.7)?

3. Two-dimensional case

There is a number of definitions extending the concept of bounded variation for functions of two variables defined on closed and bounded rectangle (see e.g. [1], [2], [12]). We recall one of them.

Let $R := [a_1, b_1] \times [a_2, b_2]$ be a closed and bounded rectangle on the real plane $\mathbb{R}^2$. We recall that (see e.g. [10], page 21) a collection of points $(x_0, y_0), (x_0, y_1), \ldots, (x_m, y_n)$ in R, where $m, n \in \mathbb{N}$, satisfying

$$a_1 = x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_m = b_1 \quad \text{and} \quad a_2 = y_0 \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n = b_2,$$

is called a collection of grid points of R. If P is any such collection of grid points of R and $f\colon R \to \mathbb{C}$ is any function, we put

$$S(P,f) = \sum_{j=1}^{m}\sum_{k=1}^{n} |f(x_j,y_k) - f(x_{j-1},y_k) - f(x_j,y_{k-1}) + f(x_{j-1},y_{k-1})|. \tag{3.1}$$

Now, such a function $f\colon R \to \mathbb{C}$ is said to be of bounded variation over the rectangle R in the sense of Vitali (-Lebesgue, -Fréchet, -de la Vallée Poussin, as indicated in [2]), in symbol $f \in \mathrm{BV}_V(R)$, if

$$V(f) = V(f, R) := \sup S(P, f) < \infty, \tag{3.2}$$

where the supremum is extended over all collections P of grid points of R, while $V(f)$ defined in (3.2) is called the total variation of f over R.

Next, we recall the concept of bounded variation for functions on $(\mathbb{R}^+)^2$ which is defined as follows (see e.g. [17], Section 2). To do this, analogously to the grid points of a rectangle as defined above, we say that a collection of points $(x_0, y_0), (x_0, y_1), \ldots, (x_m, y_n)$ in $(\mathbb{R}^+)^2$, where $m, n \in \mathbb{N}$, satisfying

$$0 \leqslant x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_m < \infty$$

and

$$0 \leqslant y_0 \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n < \infty,$$

is called a collection of grid points of $(\mathbb{R}^+)^2$. If P is any such collection of grid points of $(\mathbb{R}^+)^2$ and $f\colon (\mathbb{R}^+)^2 \to \mathbb{C}$ is any function, we define $S(P, f)$ as in (3.1).

Now, such a function $f\colon (\mathbb{R}^+)^2 \to \mathbb{C}$ is said to be of bounded variation over the set $(\mathbb{R}^+)^2$ in the sense of Vitali, in symbol $f \in \mathrm{BV}_V((\mathbb{R}^+)^2)$, if

$$V(f) = V(f, (\mathbb{R}^+)^2) := \sup S(P, f) < \infty, \tag{3.3}$$

where the supremum is extended over all collections P of grid points of $(\mathbb{R}^+)^2$, while $V(f)$ defined in (3.3) is called the total variation of f over $(\mathbb{R}^+)^2$.

Similarly to the case of functions $f \in \mathrm{BV}(\mathbb{R}^+)$, the above definition can also be equivalently reformulated as follows. A function $f\colon (\mathbb{R}^+)^2 \to \mathbb{C}$ is of bounded variation over $(\mathbb{R}^+)^2$ if and only if f is of bounded variation over all closed and bounded rectangles

$$[a_1, b_1] \times [a_2, b_2], \quad 0 \leqslant a_1 < b_1 < \infty \text{ and } 0 \leqslant a_2 < b_2 < \infty$$

in the sense of Vitali, and in addition, the set of the total variations of f over all such closed and bounded rectangles $[a_1, b_1] \times [a_2, b_2]$ is bounded. Furthermore, if this is the case, then the supremum of the set of these total variations over all such closed and bounded rectangles $[a_1, b_1] \times [a_2, b_2]$ is equal to $V(f)$ defined in (3.3).

Next, for a complex-valued Lebesgue integrable function f on $(\mathbb{R}^+)^2$, in symbol $f \in L^1((\mathbb{R}^+)^2)$, we consider its Walsh-Fourier transform defined as

$$\hat{f}(\xi, \eta) := \int_0^\infty \int_0^\infty f(x, y) \psi_\xi(x) \psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y, \quad (\xi, \eta) \in (\mathbb{R}^+)^2. \tag{3.4}$$

We observe that a version of Riemann-Lebesgue lemma holds for the Walsh-Fourier transform defined above. In fact, we have the following.

Lemma 3.1 (Riemann-Lebesgue)**.** *If* $f \in L^1((\mathbb{R}^+)^2)$*, then*

$$\lim_{\xi,\eta\to\infty} \hat{f}(\xi,\eta) = 0. \tag{3.5}$$

Proof. We give a proof of this lemma, which is similar to its one-dimensional version (see [20], page 422). Let $\varepsilon > 0$ be given. Since $f \in L^1((\mathbb{R}^+)^2)$, we have

$$\lim_{m,n\to\infty} \int_0^m \int_0^n |f(x,y)| \, \mathrm{d}x \, \mathrm{d}y = \int_0^\infty \int_0^\infty |f(x,y)| \, \mathrm{d}x \, \mathrm{d}y.$$

Therefore, we can choose m, n so large that

$$\int_0^\infty \int_0^\infty |f(x,y)| \, \mathrm{d}x \, \mathrm{d}y - \int_0^m \int_0^n |f(x,y)| \, \mathrm{d}x \, \mathrm{d}y < \varepsilon,$$

that is,

$$\int_0^m \int_n^\infty |f(x,y)| \, \mathrm{d}x \, \mathrm{d}y + \int_m^\infty \int_0^n |f(x,y)| \, \mathrm{d}x \, \mathrm{d}y + \int_m^\infty \int_n^\infty |f(x,y)| \, \mathrm{d}x \, \mathrm{d}y < \varepsilon. \tag{3.6}$$

Now, we notice that

$$\begin{aligned} &\int_0^m \int_0^n f(x,y)\psi_\xi(x)\psi_\eta(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \int_k^{k+1} \int_l^{l+1} f(x,y)\psi_\xi(x)\psi_\eta(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \int_k^{k+1} \int_l^{l+1} f(x,y)\psi_{[\xi]}(x)\psi_{[x]}(\xi)\psi_{[\eta]}(y)\psi_{[y]}(\eta) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \psi_k(\xi)\psi_l(\eta) \int_k^{k+1} \int_l^{l+1} f(x,y)\psi_{[\xi]}(x)\psi_{[\eta]}(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \psi_k(\xi)\psi_l(\eta) \int_0^1 \int_0^1 f(x,y)w_{[\xi]}(x)w_{[\eta]}(y) \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{3.7}$$

In view of (3.6) and (3.7), we see that $\hat{f}(\xi,\eta)$ is dominated by ε plus a fixed sum of double Walsh-Fourier coefficients of order $([\xi],[\eta])$. Since each of these double Walsh-Fourier coefficients tend to zero as $\xi,\eta\to\infty$, we conclude that (3.5) holds. This completes the proof of Lemma 3.1. □

By the Riemann-Lebesgue lemma, as above, it is certain that the Walsh-Fourier transform $\hat{f}(\xi,\eta) \to 0$ as $\xi,\eta \to \infty$. But in general, there is no definite rate at which the Walsh-Fourier transform tends to zero. In fact, the Walsh-Fourier transform of an integrable function on $(\mathbb{R}^+)^2$ can tend to zero as slowly as we wish (see e.g. [13], 32.47 (b)). Therefore, as in the one-dimensional case, it is interesting to know for functions of which subclasses of $L^1((\mathbb{R}^+)^2)$ there is a definite rate at which the Walsh-Fourier transform tends to zero. In this section, we carry out this study for functions of bounded variation on $(\mathbb{R}^+)^2$ in the sense of Vitali. Our main theorem of this section is as follows.

Theorem 3.1. *If $f \in L^1((\mathbb{R}^+)^2) \cap \mathrm{BV}_V((\mathbb{R}^+)^2)$ and $(\xi,\eta) \in (\mathbb{R}^+)^2$ is such that $\xi\eta \neq 0$, then*

$$\hat{f}(\xi,\eta) = O\Big(\frac{1}{\xi\eta}\Big), \quad \xi,\eta \to \infty.$$

We need the following lemma, whose proof is similar to that of Lemma 2 in [9].

Lemma 3.2. *If $f \in \mathrm{BV}_V((\mathbb{R}^+)^2)$ and if $\{a_n\colon n \in \mathbb{N}_0\}$ and $\{b_n\colon n \in \mathbb{N}_0\}$ are two increasing sequences of non-negative real numbers with*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty,$$

then

$$V(f) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V(f, [a_{m-1}, a_m] \times [b_{n-1}, b_n]),$$

the series on the right-hand side being convergent in the Pringsheim's sense.

P r o o f of Theorem 3.1. As in the proof of Theorem 2.1, here we present a proof using Taibleson-like technique (see [21]) developed in [7], and developed here for $(\mathbb{R}^+)^2$.

Fix $(\xi,\eta) \in (\mathbb{R}^+)^2$ with $\xi \geqslant 1$, $\eta \geqslant 1$. Put $m = [\xi]$ and $n = [\eta]$. Then $m, n \in \mathbb{N}$, so there exist unique $s, t \in \mathbb{N}_0$ such that $2^s \leqslant m < 2^{s+1}$ and $2^t \leqslant n < 2^{t+1}$.

Now, put $a_i = i/2^s$ for $i = 0,1,2,3,\ldots,2^s$. Then by the definition of Walsh functions, w_m takes the value 1 on one half of each of the intervals (a_{i-1}, a_i) and the value -1 on the other half. Similarly, if we put $b_j = j/2^t$ for $j = 0,1,2,3,\ldots,2^t$, then again by the definition of Walsh functions, w_n takes the value 1 on one half of each of the intervals (b_{j-1}, b_j) and the value -1 on the other half. Therefore we have

$$\int_{a_{i-1}}^{a_i} w_m(x)\,\mathrm{d}x = 0 \quad \text{for } i = 1,2,3,\ldots,2^s \tag{3.8}$$

and

$$\int_{b_{j-1}}^{b_j} w_n(y)\,\mathrm{d}y = 0 \quad \text{for } j = 1, 2, 3, \ldots, 2^t. \tag{3.9}$$

Since ψ_m and ψ_n are periodic extensions of w_m and w_n, respectively, from $\mathbb{I}$ to $\mathbb{R}^+$, for each $k, l \in \mathbb{N}_0$, $i = 1, 2, 3, \ldots, 2^s$, and $j = 1, 2, 3, \ldots, 2^t$, in view of (3.8)–(3.9), we have

$$\int_{k+a_{i-1}}^{k+a_i} \psi_m(x)\,\mathrm{d}x = \int_{a_{i-1}}^{a_i} \psi_m(x)\,\mathrm{d}x = \int_{a_{i-1}}^{a_i} w_m(x)\,\mathrm{d}x = 0 \tag{3.10}$$

and

$$\int_{l+b_{j-1}}^{l+b_j} \psi_n(y)\,\mathrm{d}y = \int_{b_{j-1}}^{b_j} \psi_n(y)\,\mathrm{d}y = \int_{b_{j-1}}^{b_j} w_n(y)\,\mathrm{d}y = 0. \tag{3.11}$$

Define three functions f_1, f_2, f_3 on $(\mathbb{R}^+)^2$ by setting

$$f_1(x,y) := f(x, l + b_{j-1}), \quad x \in \mathbb{R}^+,\ l + b_{j-1} \leqslant y < l + b_j$$

for $j = 1, 2, 3, \ldots, 2^t$, $l \in \mathbb{N}_0$;

$$f_2(x,y) := f(k + a_{i-1}, y), \quad k + a_{i-1} \leqslant x < k + a_i,\ y \in \mathbb{R}^+$$

for $i = 1, 2, 3, \ldots, 2^s$, $k \in \mathbb{N}_0$; and

$$f_3(x,y) := f(k + a_{i-1}, l + b_{j-1}), \quad k + a_{i-1} \leqslant x < k + a_i,\ l + b_{j-1} \leqslant y < l + b_j$$

for $i = 1, 2, 3, \ldots, 2^s$, $j = 1, 2, 3, \ldots, 2^t$, and $k, l \in \mathbb{N}_0$. Since $f \in L^1((\mathbb{R}^+)^2)$, it follows that $f_1, f_2, f_3 \in L^1((\mathbb{R}^+)^2)$. Now, in view of Fubini's theorem and relations (3.10)–(3.11), for each $i = 1, 2, 3, \ldots, 2^s$, $j = 1, 2, 3, \ldots, 2^t$ and $k, l \in \mathbb{N}_0$, we have

$$\begin{aligned} &\int_{k+a_{i-1}}^{k+a_i} \int_{l+b_{j-1}}^{l+b_j} f_1(x,y)\psi_\xi(x)\psi_n(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad = \int_{k+a_{i-1}}^{k+a_i} \left(f(x, l + b_{j-1}) \int_{l+b_{j-1}}^{l+b_j} \psi_n(y)\,\mathrm{d}y \right) \psi_\xi(x)\,\mathrm{d}x = 0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\int_{k+a_{i-1}}^{k+a_i} \int_{l+b_{j-1}}^{l+b_j} f_2(x,y)\psi_m(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad = \int_{l+b_{j-1}}^{l+b_j} \left(f(k + a_{i-1}, y) \int_{k+a_{i-1}}^{k+a_i} \psi_m(x)\,\mathrm{d}x \right) \psi_\eta(y)\,\mathrm{d}y = 0 \end{aligned} \tag{3.13}$$

and

$$\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_3(x,y)\psi_m(x)\psi_n(y)\,\mathrm{d}x\,\mathrm{d}y = f(k+a_{i-1}, l+b_{j-1})\Big(\int_{k+a_{i-1}}^{k+a_i}\psi_m(x)\,\mathrm{d}x\Big)\Big(\int_{l+b_{j-1}}^{l+b_j}\psi_n(y)\,\mathrm{d}y\Big) = 0. \tag{3.14}$$

Therefore by Property (iii) of generalized Walsh functions and (3.12)–(3.14) for each $i = 1, 2, 3, \dots, 2^s$, $j = 1, 2, 3, \dots, 2^t$, and $k, l \in \mathbb{N}_0$, we have

$$\begin{aligned}\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_1(x,y)\psi_\xi(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_1(x,y)\psi_\xi(x)\psi_n(y)\psi_{[y]}(\eta)\,\mathrm{d}x\,\mathrm{d}y \\ &= \psi_l(\eta)\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_1(x,y)\psi_\xi(x)\psi_n(y)\,\mathrm{d}x\,\mathrm{d}y = 0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_2(x,y)\psi_\xi(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_2(x,y)\psi_m(x)\psi_{[x]}(\xi)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \psi_k(\xi)\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_2(x,y)\psi_m(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y = 0,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_3(x,y)\psi_\xi(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_3(x,y)\psi_m(x)\psi_{[x]}(\xi)\psi_n(y)\psi_{[y]}(\eta)\,\mathrm{d}x\,\mathrm{d}y \\ &= \psi_k(\xi)\psi_l(\eta)\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f_3(x,y)\psi_m(x)\psi_n(y)\,\mathrm{d}x\,\mathrm{d}y = 0.\end{aligned} \tag{3.17}$$

Using (3.15)–(3.17) in definition (3.4) of $\hat{f}(\xi,\eta)$ we get

$$\begin{aligned}|\hat{f}(\xi,\eta)| &= \Big|\int_0^\infty\int_0^\infty f(x,y)\psi_\xi(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &= \Big|\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{i=1}^{2^s}\sum_{j=1}^{2^t}\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j} f(x,y)\psi_\xi(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y\Big|\end{aligned}$$

$$
\begin{aligned}
&= \Bigg|\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=1}^{2^s}\sum_{j=1}^{2^t}\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j}(f(x,y)-f_1(x,y)\\
&\qquad\qquad -f_2(x,y)+f_3(x,y))\psi_\xi(x)\psi_\eta(y)\,\mathrm{d}x\,\mathrm{d}y\Bigg|\\
&\leqslant \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=1}^{2^s}\sum_{j=1}^{2^t}\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j}|f(x,y)-f_1(x,y)\\
&\qquad\qquad -f_2(x,y)+f_3(x,y)|\,\mathrm{d}x\,\mathrm{d}y\\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=1}^{2^s}\sum_{j=1}^{2^t}\int_{k+a_{i-1}}^{k+a_i}\int_{l+b_{j-1}}^{l+b_j}|f(x,y)-f(x,l+b_{j-1})\\
&\qquad\qquad -f(k+a_{i-1},y)+f(k+a_{i-1},l+b_{j-1})|\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=1}^{2^s}\sum_{j=1}^{2^t}V(f,[k+a_{i-1},k+a_i]\times[l+b_{j-1},l+b_j])\\
&\qquad\qquad \times(a_i-a_{i-1})(b_j-b_{j-1})\\
&\leqslant \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}V(f,[k,k+1]\times[l,l+1])\frac{1}{2^s}\frac{1}{2^t}\\
&= \frac{1}{2^s2^t}V(f)\leqslant\frac{4}{mn}V(f)\leqslant\frac{16V(f)}{\xi\eta},
\end{aligned}
$$

in view of Lemma 3.2. Thus, we get

$$
|\hat{f}(\xi,\eta)|\leqslant\frac{16V(f)}{\xi\eta}. \tag{3.18}
$$

The proof of Theorem 3.1 is complete. □

Problem 3.1. How to estimate $\hat{f}(\xi,0)$, $\xi\neq 0$ (or $\hat{f}(0,\eta)$, $\eta\neq 0$) in terms of ξ (or η), even assuming that f is of bounded variation over $(\mathbb{R}^+)^2$ in the sense of Hardy (see [17] for definition)?

Problem 3.2. What can be said about the exactness of the constant in (3.18)?

4. Extension of the result to $(\mathbb{R}^+)^N$, $N\in\mathbb{N}$

We start by defining the concept of bounded variation for functions on $(\mathbb{R}^+)^N$, $N\in\mathbb{N}$ in the sense of Vitali.

For a function $f\colon (\mathbb{R}^+)^N\to\mathbb{C}$ and for any rectangle $R=[\alpha_1,\beta_1]\times\ldots\times[\alpha_N,\beta_N]$ in $(\mathbb{R}^+)^N$ with $0\leqslant\alpha_i<\beta_i<\infty$ for all $i=1,2,\ldots,N$, we define $\Delta f(R)$ as follows:

When $N = 2$, we put

$$\Delta f(R) := \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]) = f(\beta_1, \beta_2) - f(\beta_1, \alpha_2) - f(\alpha_1, \beta_2) + f(\alpha_1, \alpha_2);$$

for $N = 3$

$$\begin{aligned}\Delta f(R) &:= \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_3, \beta_3]) \\ &= (f(\beta_1, \beta_2, \beta_3) - f(\beta_1, \alpha_2, \beta_3) - f(\alpha_1, \beta_2, \beta_3) + f(\alpha_1, \alpha_2, \beta_3)) \\ &\quad - (f(\beta_1, \beta_2, \alpha_3) - f(\beta_1, \alpha_2, \alpha_3) - f(\alpha_1, \beta_2, \alpha_3) + f(\alpha_1, \alpha_2, \alpha_3)) \\ &= \Delta_{[\alpha_3, \beta_3]} \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2])\end{aligned}$$

and successively for any $N \geqslant 3$

$$\begin{aligned}\Delta f(R) &:= \Delta f([\alpha_1, \beta_1] \times \ldots \times [\alpha_N, \beta_N]) \\ &= \Delta_{[\alpha_N, \beta_N]} \Delta f([\alpha_1, \beta_1] \times \ldots \times [\alpha_{N-1}, \beta_{N-1}]).\end{aligned}$$

A collection of points $(x_1^0, \ldots, x_N^0)$, $(x_1^0, \ldots, x_{N-1}^0, x_N^1)$, $\ldots$, $(x_1^{s_1}, \ldots, x_N^{s_N})$ of $(\mathbb{R}^+)^N$ satisfying

$$0 \leqslant x_j^0 \leqslant x_j^1 \leqslant \ldots \leqslant x_j^{s_j} < \infty, \quad s_j \in \mathbb{N},\ j = 1, 2, \ldots, N,$$

is called a collection of grid points of $(\mathbb{R}^+)^N$. If P is any such collection of grid points of $(\mathbb{R}^+)^N$ and $f\colon (\mathbb{R}^+)^N \to \mathbb{C}$ is any function, we put

$$S(P, f) = \sum_{i_1=1}^{s_1} \ldots \sum_{i_N=1}^{s_N} |\Delta f([x_1^{i_1-1}, x_1^{i_1}] \times \ldots \times [x_N^{i_N-1}, x_N^{i_N}])|.$$

Now, a function $f\colon (\mathbb{R}^+)^N \to \mathbb{C}$ is said to be of bounded variation over the set $(\mathbb{R}^+)^N$ in the sense of Vitali, in symbol $f \in \mathrm{BV}_V((\mathbb{R}^+)^N)$, if

$$V(f) = V(f, (\mathbb{R}^+)^N) := \sup S(P, f) < \infty, \tag{4.1}$$

where the supremum is extended over all collections P of grid points of $(\mathbb{R}^+)^N$, while $V(f)$ defined in (4.1) is called the total variation of f over $(\mathbb{R}^+)^N$.

Similarly to the case of functions $f \in \mathrm{BV}(\mathbb{R}^+)$ and $f \in \mathrm{BV}_V((\mathbb{R}^+)^2)$, the above definition can also be equivalently reformulated as follows. A function $f\colon (\mathbb{R}^+)^N \to \mathbb{C}$ is of bounded variation over $(\mathbb{R}^+)^N$ in the sense of Vitali if and only if f is of bounded variation over all closed and bounded N-rectangles

$$[a_1, b_1] \times \ldots \times [a_N, b_N], \quad 0 \leqslant a_i < b_i < \infty,\ i = 1, 2, \ldots, N$$

in the sense of Vitali (see e.g. [14] or [5] for definition), and in addition, the set of total variations of f over all such closed and bounded N-rectangles $[a_1, b_1] \times \ldots \times [a_N, b_N]$ is bounded. Furthermore, if this is the case, then the supremum of the set of these total variations over all such closed and bounded N-rectangles $[a_1, b_1] \times \ldots \times [a_N, b_N]$ is equal to $V(f)$ defined in (4.1).

Next, for a complex-valued Lebesgue integrable function f on $(\mathbb{R}^+)^N$, in symbol $f \in L^1((\mathbb{R}^+)^N)$, we consider its Walsh-Fourier transform defined as

$$\hat{f}(\xi_1, \ldots, \xi_N) := \int_{(\mathbb{R}^+)^N} f(x_1, \ldots, x_N)\psi_{\xi_1}(x_1) \ldots \psi_{\xi_N}(x_N)\, \mathrm{d}x_1 \ldots \mathrm{d}x_N,$$

where $(\xi_1, \ldots, \xi_N) \in (\mathbb{R}^+)^N$.

In this case also, similar to Lemma 3.1, a version of Riemann-Lebesgue lemma holds, that is, $\hat{f}(\xi_1, \ldots, \xi_N) \to 0$ as $\xi_1, \ldots, \xi_N \to \infty$. But in general, there is no definite rate at which the Walsh-Fourier transform tends to zero. In fact, the Walsh-Fourier transform of an integrable function on $(\mathbb{R}^+)^N$ can tend to zero as slowly as we wish (see e.g. [13], 32.47 (b)). Therefore, as in one and two dimensional cases, it is interesting to know for functions of which subclasses of $L^1((\mathbb{R}^+)^N)$ there is a definite rate at which the Walsh-Fourier transform tends to zero. In this section, we state the result for functions of bounded variation on $(\mathbb{R}^+)^N$ in the sense of Vitali, which is an extension of our theorems in Sections 2 and 3. The proof of this theorem is similar to that of Theorem 3.1.

Theorem 4.1. *If* $f \in L^1((\mathbb{R}^+)^N) \cap \mathrm{BV}_V((\mathbb{R}^+)^N)$ *and* $(\xi_1, \ldots, \xi_N) \in (\mathbb{R}^+)^N$ *is such that* $\prod_{i=1}^{N} \xi_i \neq 0$, *then*

$$\hat{f}(\xi_1, \ldots, \xi_N) = O\Big(1 \Big/ \prod_{i=1}^{N} \xi_i\Big), \quad \xi_1, \ldots, \xi_N \to \infty.$$

More precisely,

$$|\hat{f}(\xi_1, \ldots, \xi_N)| \leqslant 4^N V(f) \Big/ \prod_{i=1}^{N} \xi_i, \quad \xi_1, \ldots, \xi_N \geqslant 1. \tag{4.2}$$

P r o b l e m 4.1. How to estimate $\hat{f}(\xi_1, \ldots, \xi_N)$ if $(\xi_1, \ldots, \xi_N) \neq (0, \ldots, 0)$, but $\xi_j = 0$ for some $j \in \{1, \ldots, N\}$, even assuming that f is of bounded variation over $(\mathbb{R}^+)^N$ in the sense of Hardy (which can be defined similarly as in the case of $(\mathbb{R}^+)^2$)?

P r o b l e m 4.2. What can be said about the exactness of the constant in (4.2)?

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