

Mario Petrich

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RELATIONS ON A LATTICE OF VARIETIES OF
COMPLETELY REGULAR SEMIGROUPS

MARIO PETRICH, Bol

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Abstract. Completely regular semigroups \mathcal{CR} are considered here with the unary operation of inversion within the maximal subgroups of the semigroup. This makes \mathcal{CR} a variety; its lattice of subvarieties is denoted by $\mathcal{L}(\mathcal{CR})$. We study here the relations \mathbf{K} , \mathbf{T} , \mathbf{L} and \mathbf{C} relative to a sublattice Ψ of $\mathcal{L}(\mathcal{CR})$ constructed in a previous publication.

For \mathbf{R} being any of these relations, we determine the \mathbf{R} -classes of all varieties in the lattice Ψ as well as the restrictions of \mathbf{R} to Ψ .

Keywords: semigroup; completely regular; variety; lattice; relation; kernel; trace; local relation; core

MSC 2010: 20M07

1. INTRODUCTION AND SUMMARY

Completely regular semigroups S with the unary operation of inversion within the maximal subgroups of S form a variety \mathcal{CR} . Its lattice of subvarieties is denoted by $\mathcal{L}(\mathcal{CR})$.

There are basically two types of approaches to studying the structure of $\mathcal{L}(\mathcal{CR})$.

1. Starting from the bottom of the lattice $\mathcal{L}(\mathcal{CR})$, we always aim to encompass the larger ones. Parallel to this, one tries to describe the structure of the semigroups involved. For example, the lattice may be described as the direct product of “smaller and/or simpler” lattices. This is the local approach.

2. As contrasted to this, the global approach concerns the entire lattice $\mathcal{L}(\mathcal{CR})$ by designing and studying various relations on $\mathcal{L}(\mathcal{CR})$. These are often complete congruences, so their classes are intervals. The ends of their classes induce lower and upper operators on $\mathcal{L}(\mathcal{CR})$.

The **B**-relation, which identifies two varieties if they contain the same bands, provides a canonical example of the global approach.

Next we outline a concrete case, the subject of the present work.

In [16], the \cap -subsemilattice of $\mathcal{L}(\mathcal{CR})$ of upper ends of **B**-classes was determined. As proved in [4], a large part of it is generated by two countably infinite ascending chains of varieties dubbed canonical. In [5], we constructed the sublattice Σ of $\mathcal{L}(\mathcal{CR})$ generated by \mathcal{CS} , the variety of completely simple semigroups, and four initial canonical varieties. In [7], we described the classes of various relations of varieties in Σ . The next step is provided by the lattice Ψ in [8] which has two more canonical varieties for generators, thereby extending the lattice Σ .

The stage is thus set for describing the classes and the restrictions of the relations alluded above to Ψ , the subject of this work. Hence we essentially oscillate between the two approaches outlined above: the upper ends on the **B**-classes providing the generators for a small sublattice of $\mathcal{L}(\mathcal{CR})$, and then considering some global relations relative to this sublattice.

The relations in question are: kernel **K**, trace **T**, local **L**, and core **C**. The first pair of these was studied in [11], the second pair in [12].

Now we give a brief sketch of the content of the paper. Section 2 contains preparation of the material that comes later, in particular the definition of canonical varieties. The lattice Ψ is introduced in Section 3 by a diagram. We start the material proper by a determination of the **K**-classes of the varieties in the lattice Ψ and the restriction of **K** to Ψ in Section 4. This pattern persists in Sections 5–7 for **T**-, **L**- and **C**-relations, respectively. In Section 8, we discuss an unsolved case.

2. PREPARATION

For all the terminology and notation, we follow the book [13], except that we write \mathcal{O}^L instead of $L\mathcal{O}$.

First we construct two countably infinite ascending chains of subvarieties of $\mathcal{L}(\mathcal{CR})$. To this end, we introduce three infinite sequences of words, see [4].

Let $X = \{x_2, x_2, \dots\}$ be a countably infinite set of variables. In the free completely regular semigroup on X , set

$$G_2 = x_2x_1, \quad H_2 = x_2, \quad I_2 = x_2x_1x_2^0$$

and for $n > 2$, define inductively

$$G_n = x_n\overline{G}_{n-1}, \quad P_n = G_n(x_n\overline{P}_{n-1})^0, \quad P \in \{H, I\},$$

where \overline{W} is the mirror image of W (see [7], Section 2) for the justification of this notation. We refer to the varieties

$$\begin{aligned} \mathcal{H}_2 &= \mathcal{LN}\mathcal{O}, & \overline{\mathcal{H}}_3 &= \mathcal{RN}\mathcal{O}, \\ \mathcal{H}_n &= [G_n = H_n], & \overline{\mathcal{H}}_n &= [\overline{G}_n = \overline{H}_n], & n > 2, \\ \mathcal{I}_n &= [G_n = I_n], & \overline{\mathcal{I}}_n &= [\overline{G}_n = \overline{I}_n], & n \geq 2 \end{aligned}$$

as *canonical*. See [5], Diagram 1. Note that

$$SG = \mathcal{I}_2 \cap \overline{\mathcal{I}}_2, \quad \mathcal{NBG} = \mathcal{H}_3 \cap \overline{\mathcal{H}}_3.$$

We also use the notation

$$\mathcal{R} = \mathcal{I}_3 \cap \overline{\mathcal{I}}_3, \quad \mathcal{F} = \mathcal{H}_4 \cap \overline{\mathcal{H}}_4.$$

Recall that the acronym \mathcal{RO} stands for the variety of regular orthogroups. In addition, it holds

Lemma 2.1. *We have $\mathcal{RO} = \mathcal{R} \cap \mathcal{O}$.*

Proof. By [13], Theorem V.3.3, we have $\mathcal{RO} = \mathcal{I}_2 \vee \overline{\mathcal{I}}_2$. In terms of ladders in Polák's theorem in [15], we get

$$\begin{array}{ccc} \mathcal{G} & \vee & \mathcal{G} & = & \mathcal{G} \\ \mathcal{G} \swarrow & & \mathcal{G} \swarrow & & \mathcal{G} \swarrow \\ & & T^* & & T^* \\ & & \swarrow & & \swarrow \\ T^* & & \mathcal{G} & & \mathcal{G} \\ & & \swarrow & & \swarrow \\ & & T^* & & T^* \\ & & \swarrow & & \swarrow \\ \dots & & \dots & & \dots \end{array}$$

and evaluating

$$\mathcal{I}_2 \vee \overline{\mathcal{I}}_2 = \mathcal{G}^K \cap \mathcal{S}^{T_r T_l} \cap \mathcal{S}^{T_l T_r} = \mathcal{O} \cap \mathcal{I}_3 \cap \overline{\mathcal{I}}_3 = \mathcal{O} \cap \mathcal{R}$$

we arrive at $\mathcal{RO} = \mathcal{R} \cap \mathcal{O}$. (For the notation \mathcal{V}^K , \mathcal{V}^{T_l} , \mathcal{V}^{T_r} , see [11].) □

Beside \mathcal{RO} , we sometimes (when no ambiguity occurs) write the meet of a finite number of varieties by juxtaposition.

If \mathbf{R} is an equivalence relation on $\mathcal{L}(\mathcal{CR})$ all of whose classes are intervals for $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we write the \mathbf{R} -class of \mathcal{V} as $\mathcal{V}\mathbf{R} = [\mathcal{V}\mathbf{R}, \mathcal{V}\mathbf{R}]$.

3. LATTICE Ψ

In [5] we constructed the sublattice Σ of $\mathcal{L}(\mathcal{CR})$ generated by $\{\mathcal{CS}, \mathcal{I}_2, \bar{\mathcal{I}}_2, \mathcal{H}_3, \bar{\mathcal{H}}_3\}$. We extended this lattice in [9] to the lattice Ψ by adding two more generators \mathcal{I}_3 and $\bar{\mathcal{I}}_3$. This entailed adjoining the interval $[\mathcal{R}, \mathcal{I}_3 \vee \bar{\mathcal{I}}_3]$. See Diagram 1.

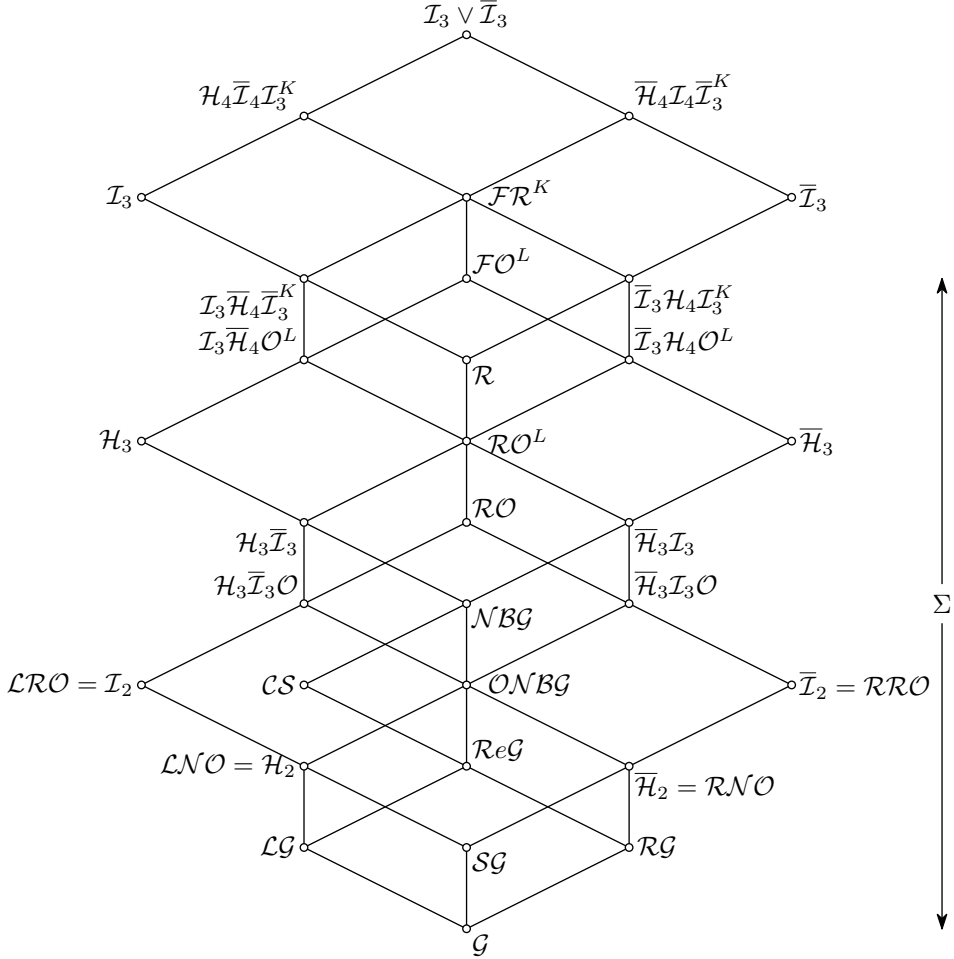


Diagram 1. The lattice Ψ .

On the other hand, in [6] we determined the classes of the kernel, trace, local and core relations relative to Σ . The subject of the present paper is the determination of the classes of these relations for all varieties in Ψ . In view of the results in [5], which contains the information on Σ , it remains to determine the classes of these relations for varieties in $\Psi \setminus \Sigma$.

The essential difference between Σ and Ψ is that the former stays within \mathcal{O}^L while the rest of Ψ is not contained in \mathcal{O}^L . This is a crucial distinction since within \mathcal{O}^L we have enough information needed in the proofs of statements we want to establish, but outside of \mathcal{O}^L , in the present state of the art, this is no longer the case. Nevertheless, all the varieties in $\Psi \setminus \Sigma$, except for \mathcal{R} and $\mathcal{I}_3 \vee \bar{\mathcal{I}}_3$, are joins of a variety in Σ and a variety in $\Psi \setminus \Sigma$. If we extend the lattice Ψ for another block like we extended Σ to Ψ , even this useful property would be lost.

4. KERNEL RELATION

We can say loosely that the kernel relation \mathbf{K} on $\mathcal{L}(\mathcal{CR})$ is the varietal version of the kernel relation on lattices of congruences of (completely) regular semigroups. Yet the kernel relation on $\mathcal{L}(\mathcal{CR})$ admits an intrinsic characterization, see [3], Theorem 1. The results of the papers [11] and [7] for \mathbf{K} indicate that we can expect little help in applying the general theory to special cases.

Theorem 4.1. *The intervals*

$$[\mathcal{G}, \mathcal{O}], \quad [\mathcal{CS}, \mathcal{O}^L], \quad [\mathcal{R}, \mathcal{R}^K], \quad [\mathcal{I}_3, \mathcal{I}_3^K], \quad [\bar{\mathcal{I}}_3, \bar{\mathcal{I}}_3^K], \quad [\mathcal{I}_3 \vee \bar{\mathcal{I}}_3, \mathcal{I}_3^K \vee \bar{\mathcal{I}}_3^K]$$

constitute the complete set of \mathbf{K} -classes of varieties in Ψ .

Proof. The fact that the first two intervals are \mathbf{K} -classes can be deduced from [14], Theorem 2. Recall that $\mathcal{R} = \mathcal{I}_3 \cap \bar{\mathcal{I}}_3$. By [7], Theorem 10.2, we have $\mathcal{I}_3 = \mathcal{I}_3^T$ and $\bar{\mathcal{I}}_3 = \bar{\mathcal{I}}_3^T$, which by [11], Proposition 7.10 implies that $\mathcal{R} = \mathcal{R}^T$. Trivially $\mathcal{R} \supseteq \mathcal{R}e\mathcal{B}$, so that the varietal version of [1], Proposition 8.2 yields $\mathcal{R} = \mathcal{R}_K$. Hence the third interval is a \mathbf{K} -class. Similarly $\mathcal{I}_3 = (\mathcal{I}_3)_K$ and $\bar{\mathcal{I}}_3 = (\bar{\mathcal{I}}_3)_K$ which implies that the fourth and the fifth intervals are \mathbf{K} -classes. Now [2], Theorem 11 implies that also the sixth interval is a \mathbf{K} -class.

From direct inspection of Diagram 1, we draw the following conclusions. Up to $\mathcal{R} \cap \mathcal{O}$, all varieties are contained in the interval $[\mathcal{G}, \mathcal{O}]$, and then up to \mathcal{FO}^L , all the varieties are contained in $[\mathcal{CS}, \mathcal{O}^L]$. The interval $[\mathcal{R}, \mathcal{FR}^K]$ is contained in the interval $[\mathcal{R}, \mathcal{R}^K]$, while $\mathcal{H}_4\bar{\mathcal{I}}_4\mathcal{I}_3^K \in [\mathcal{I}_3, \mathcal{I}_3^K]$, and similarly for its dual. Therefore the above intervals cover all of Ψ . \square

We can now easily derive

Corollary 4.2. *The intervals*

$$[\mathcal{G}, \mathcal{RO}], \quad [\mathcal{CS}, \mathcal{FO}^L], \quad [\mathcal{R}, \mathcal{FR}^K], \quad [\mathcal{I}_3, \mathcal{H}_4\bar{\mathcal{I}}_4\mathcal{I}_3^K], \quad [\bar{\mathcal{I}}_3, \bar{\mathcal{H}}_4\bar{\mathcal{I}}_4\bar{\mathcal{I}}_3^K], \quad \{\mathcal{I}_3 \vee \bar{\mathcal{I}}_3\}$$

constitute the complete set of restrictions of \mathbf{K} -classes to Ψ . See Diagram 2.

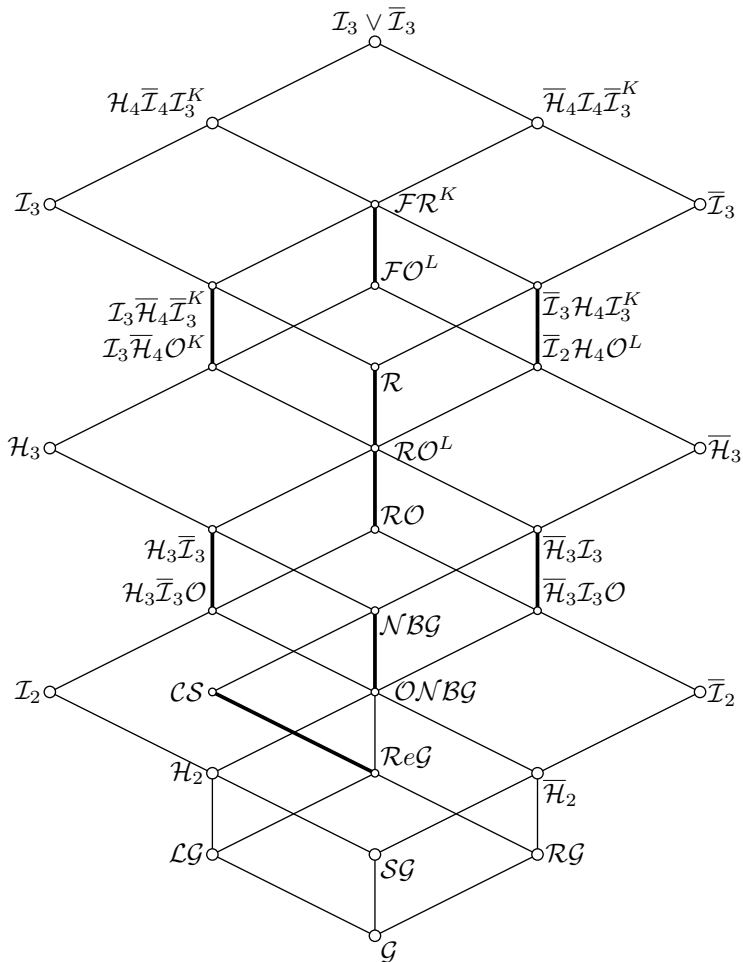


Diagram 2. The relation \mathbf{T} restricted to the lattice Ψ (Corollary 5.2).

Proof. This follows easily from Theorem 4.1 by intersecting \mathbf{K} -classes with Ψ . □

We mention only that given a basis for \mathcal{V} , it is known how to construct a basis for \mathcal{V}^K , but it is not known how to construct a basis for \mathcal{V}_K in general. For information on the \mathbf{K} -relation, consult [11] and [7].

5. TRACE RELATION

We can say loosely that the trace relation \mathbf{T} on $\mathcal{L}(\mathcal{CR})$ is a varietal version of the trace relation on lattices of congruences of (completely) regular semigroups. Yet the trace relation on $\mathcal{L}(\mathcal{CR})$ admits an intrinsic characterization, see [11], Corollary 6.3.

The results of this paper as well as the work [7] make it possible to establish all the results in this section.

Theorem 5.1. *The intervals*

$$(5.1) \quad \{[\mathcal{V}, \mathcal{G} \circ \mathcal{V}]: \mathcal{V} \in \mathcal{L}(\mathcal{NB})\},$$

$$(5.2) \quad [\mathcal{RO}, \mathcal{R}], \quad [\mathcal{FO}^L, \mathcal{F}], \quad [\mathcal{I}_3 \vee \bar{\mathcal{I}}_3, \mathcal{I}_4 \cap \bar{\mathcal{I}}_4],$$

and the intervals

$$(5.3) \quad \{\mathcal{I}_2\}, \quad \{\mathcal{H}_3\}, \quad \{\mathcal{I}_3\},$$

$$(5.4) \quad [\mathcal{H}_3 \bar{\mathcal{I}}_3 \mathcal{O}, \mathcal{H}_3 \bar{\mathcal{I}}_3], \quad [\mathcal{I}_3 \bar{\mathcal{H}}_4 \mathcal{O}^L, \mathcal{I}_3 \bar{\mathcal{H}}_4], \quad [\mathcal{H}_4 \bar{\mathcal{I}}_4 \mathcal{I}_3^K, \mathcal{H}_4 \bar{\mathcal{I}}_4]$$

and their duals constitute the complete set of **T**-classes of varieties in Ψ .

Proof. We will apply [7], Theorem 10.2. Part (5.1) follows from part (i) of this reference, while (5.3) follows from part (ii). The third part of this reference asserts that

$$(5.5) \quad [\mathcal{P}_m \vee \bar{\mathcal{Q}}_n, \bar{\mathcal{P}}_{m+1} \cap \mathcal{Q}_{n+1}], \quad \mathcal{P}_m, \bar{\mathcal{Q}}_n \in \Gamma$$

is a **T**-class. Since Ψ is a lattice, we have $\mathcal{R} \cap \mathcal{O} = \mathcal{I}_2 \vee \bar{\mathcal{I}}_2$ and $\mathcal{R} = \mathcal{I}_3 \cap \bar{\mathcal{I}}_3$, and thus by (5.5), we deduce that the first interval in (5.2) is a **T**-class. Similarly $\mathcal{FO}^L = \mathcal{H}_3 \vee \bar{\mathcal{H}}_3$ and $\mathcal{F} = \mathcal{H}_4 \cap \bar{\mathcal{H}}_4$, which yields that the second interval in (5.2) is a **T**-class. The fact that the third interval in (5.2) is a **T**-class follows directly from (5.5).

Since Ψ is a lattice, we get

$$\mathcal{H}_3 \bar{\mathcal{I}}_3 \mathcal{O} = \bar{\mathcal{H}}_2 \vee \mathcal{I}_2, \quad \mathcal{I}_3 \bar{\mathcal{H}}_4 \mathcal{O}^L = \mathcal{H}_3 \vee \bar{\mathcal{I}}_2, \quad \mathcal{H}_4 \bar{\mathcal{I}}_4 \mathcal{I}_3^K = \bar{\mathcal{H}}_3 \vee \mathcal{I}_3,$$

which in view of (5.5) yields that the three intervals in (5.4) are **T**-classes.

A simple inspection of Diagram 1 will show that we have covered all varieties in Ψ . □

We can now easily derive the restriction of the **T**-relation to Ψ .

Corollary 5.2. *The intervals*

$$\{\mathcal{G}\}, [\mathcal{ReG}, \mathcal{CS}], \{\mathcal{SG}\}, [\mathcal{ONBG}, \mathcal{NBG}], [\mathcal{RO}, \mathcal{R}], [\mathcal{FO}^L, \mathcal{FR}^K], \{\mathcal{I}_3 \vee \bar{\mathcal{I}}_3\},$$

and the intervals

$$\{\mathcal{LG}\}, \{\mathcal{H}_2\}, \{\mathcal{I}_2\}, [\mathcal{H}_3 \bar{\mathcal{I}}_3 \mathcal{O}, \mathcal{H}_3 \bar{\mathcal{I}}_3], \{\mathcal{H}_3\}, [\mathcal{I}_3 \bar{\mathcal{H}}_4 \mathcal{O}^L, \mathcal{I}_3 \bar{\mathcal{H}}_4 \bar{\mathcal{I}}_3^K], \{\mathcal{I}_3\}, \{\mathcal{H}_4 \bar{\mathcal{I}}_4 \mathcal{I}_3^K\}$$

and their duals constitute the complete set of restrictions of **T**-classes to Ψ . See Diagram 2.

P r o o f. This follows easily by intersecting the \mathbf{T} -classes of Theorem 5.1 with Ψ . □

For information on the \mathbf{T} -relation, see [11] and [7].

6. LOCAL RELATION

The relation \mathbf{L} is defined directly on $\mathcal{L}(\mathcal{CR})$ by

$$\mathcal{U} \mathbf{L} \mathcal{V} \Leftrightarrow \mathcal{U} \cap \mathcal{M} = \mathcal{V} \cap \mathcal{M},$$

where \mathcal{M} denotes the class of all completely regular monoids. It is an equivalence relation all of whose classes are intervals. For any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, its \mathbf{L} -class is denoted by $\mathcal{V} \mathbf{L} = [\mathcal{V} \mathbf{L}, \mathcal{V} \mathbf{L}]$.

There is enough knowledge about the \mathbf{L} -relation so that, in the present case, we can answer all but one relevant question. The main result here has the following form:

Theorem 6.1. *The intervals*

$$(6.1) \quad [\mathcal{G}, \mathcal{CS}], [\mathcal{SG}, \mathcal{NBG}], [\mathcal{I}_2, \mathcal{H}_3], [\mathcal{RO}, \mathcal{FO}^L], [\bar{\mathcal{I}}_2, \bar{\mathcal{H}}_3], \\ [\mathcal{R}, \mathcal{F}], [\mathcal{I}_3, \mathcal{H}_4] [\bar{\mathcal{I}}_3, \bar{\mathcal{H}}_4], [\mathcal{I}_3 \vee \bar{\mathcal{I}}_3, (\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^L]$$

constitute the complete set of \mathbf{L} -classes of the varieties in Ψ .

P r o o f. By [6], Theorem 4.1, the intervals in (6.1) form the complete set of \mathbf{L} -classes of the varieties in Σ . Next we consider the varieties in $\Psi \setminus \Sigma$ by showing first that the above intervals are \mathbf{L} -classes.

Case $[\mathcal{R}, \mathcal{F}]$: By [12], Theorems 5.1 and 5.3 and [7], Theorem 12.3, we have

$$\mathcal{F}_L = (\mathcal{H}_4 \cap \bar{\mathcal{H}}_4)_L = (\mathcal{H}_4)_L \cap (\bar{\mathcal{H}}_4)_L = \mathcal{I}_3 \cap \bar{\mathcal{I}}_3 = \mathcal{R}.$$

By [13], Proposition II.7.3 (ii) and the last reference, we get

$$\mathcal{R}^L = (\mathcal{I}_3 \cap \bar{\mathcal{I}}_3)^L = (\mathcal{I}_3)^L \cap (\bar{\mathcal{I}}_3)^L = \mathcal{H}_4 \cap \bar{\mathcal{H}}_4 = \mathcal{F}.$$

Notice that this interval contains the varieties $\mathcal{I}_3 \bar{\mathcal{H}}_4$, \mathcal{FR}^K , $\bar{\mathcal{I}}_3 \mathcal{H}_4$.

Case $[\mathcal{I}_3, \mathcal{H}_4]$: Using the same references, we obtain $(\mathcal{H}_4)_L = \mathcal{I}_3$ and $\mathcal{I}_3^L = \mathcal{H}_4$. Note that this interval contains the variety $\mathcal{I}_3 \bar{\mathcal{H}}_4$. The case $[\bar{\mathcal{I}}_3, \bar{\mathcal{H}}_4]$ is dual.

Case $[\mathcal{I}_3 \vee \bar{\mathcal{I}}_3, (\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^L]$: Again using the same references, we get

$$(\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)_L = (\mathcal{I}_3)_L \vee (\bar{\mathcal{I}}_3)_L = \mathcal{I}_3 \vee \bar{\mathcal{I}}_3.$$

This exhausts all the possibilities. □

Observe that we have not determined $(\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^L$. We now easily derive

Corollary 6.2. *The intervals in (6.1) and*

$$[\mathcal{R}, \mathcal{FR}^K], [\mathcal{I}_3, \mathcal{H}_4\bar{\mathcal{I}}_4\mathcal{I}_3^K], [\bar{\mathcal{I}}_3, \bar{\mathcal{H}}_4\mathcal{I}_4\bar{\mathcal{I}}_3^K], \{\mathcal{I}_3 \vee \bar{\mathcal{I}}_3\}$$

constitute the complete set of restrictions of \mathbf{L} -classes to Ψ . See Diagram 3.

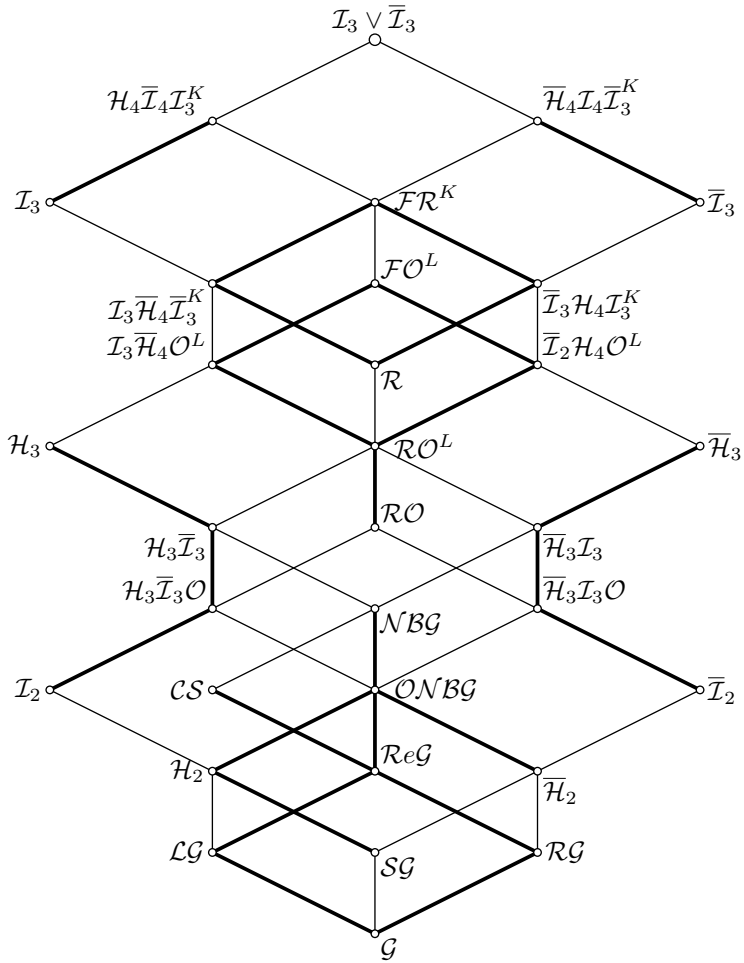


Diagram 3. The relation \mathbf{L} restricted to the lattice Ψ (Corollary 6.2).

Proof. This follows easily from Theorem 6.1. □

For information on the \mathbf{L} -relation, consult [12].

7. CORE RELATION

The relation \mathbf{C} is defined directly on $\mathcal{L}(\mathcal{CR})$ by

$$\mathcal{U} \mathbf{C} \mathcal{V} \Leftrightarrow \mathcal{U} \cap \mathcal{I} = \mathcal{V} \cap \mathcal{I},$$

where \mathcal{I} denotes the class of all idempotent generated completely regular semigroups.

We start with some citations from literature.

Fact 7.1. The following statements hold:

- (i) The mappings $\mathcal{V} \mapsto \mathcal{V}_C$ and $\mathcal{V} \mapsto \mathcal{V}^C$ are \vee - and \cap -endomorphisms of $\mathcal{L}(\mathcal{CR})$, respectively.
- (ii) For any canonical variety \mathcal{V} , we have $\mathcal{V}^C = \mathcal{V}$.
- (iii) For the upper operators, we have $KC = CK$ and $TC = CT$.

Proof. (i) The first assertion follows from [13], Lemma I.2.2 and the second is proved in [13], Proposition II.7.6 (ii).

(ii) This was proved in [7], Theorem 13.2.

(iii) This forms part of [10], Lemmas 5.3 and 5.5. □

Similarly like in Lemma 2.1, we have

Lemma 7.2. $\mathcal{RBG} = \mathcal{R} \cap \mathcal{BG}$.

Proof. By [5], Theorem 5.1 (iv), we obtain $\mathcal{R} = [(axya)^0 = (axa^0ya)^0]$, and by [13], Proposition V.5.4, that $\mathcal{RBG} = [(axya)^0 = (axaya)^0]$. Hence

$$\mathcal{RBG} \subseteq \mathcal{R} \cap \mathcal{BG} \subseteq \mathcal{RBG}$$

and the equality prevails. □

Lemma 7.3. $\mathcal{R}_C \subseteq \mathcal{RBG} \subseteq \mathcal{R} = \mathcal{R}^C$.

Proof. By Fact 7.1 (i) and (ii), we get

$$\mathcal{R}^C = (\mathcal{I}_3 \cap \overline{\mathcal{I}}_3)^C = \mathcal{I}_3^C \cap \overline{\mathcal{I}}_3^C = \mathcal{I}_3 \cap \overline{\mathcal{I}}_3 = \mathcal{R}.$$

By Lemma 7.2 and Fact 7.1 (i), we obtain

$$\mathcal{RBG}^C = (\mathcal{R} \cap \mathcal{BG})^C = \mathcal{R}^C \cap \mathcal{BG}^C = \mathcal{R} \cap \mathcal{BG}^C$$

where by Fact 7.1 (iii), we have

$$\mathcal{BG}^C = \mathcal{B}^{TC} = \mathcal{B}^{CT} = \mathcal{O}^T$$

so that $\mathcal{RBG}^C = \mathcal{R} \cap \mathcal{O}^T$. Now using the notation of [4], and the varietal version of [13], Corollary VII.4.4 (ii) and [4], Proposition 4.1, we obtain

$$\mathcal{R} \cap \mathcal{O}^T = \mathcal{I}_3 \cap \overline{\mathcal{I}}_3 \cap \mathcal{O}^T = \overline{\mathcal{I}}_2^{T_l} \cap \mathcal{I}_2^{T_r} \cap \mathcal{O}^{T_l} \cap \mathcal{O}^{T_r} = \overline{\mathcal{I}}_2^{T_l} \cap \mathcal{I}_2^{T_r} = \mathcal{I}_3 \cap \overline{\mathcal{I}}_3 = \mathcal{R}$$

since $\mathcal{I}_2, \overline{\mathcal{I}}_2 \subseteq \mathcal{O}$. Therefore $\mathcal{RBG}^C = \mathcal{R} = \mathcal{R}^C$ whence \mathcal{RBGCR} , and consequently $\mathcal{R}_C \subseteq \mathcal{RBG}$. Trivially $\mathcal{RBG} \subseteq \mathcal{R}$. \square

Recall [13], Notation V.4.1 and Lemma V.4.2 concerning \mathcal{L}^* .

Lemma 7.4. $\mathcal{RBG}_C = \mathcal{RBG}$.

Proof. By [13], Proposition V.4.4, \mathcal{RBG} consists precisely of completely regular semigroups in which both Green's relations \mathcal{L} and \mathcal{R} are congruences. In [8], Theorem 6.1, it was shown that $\mathcal{L}_C^* = \mathcal{L}^*$, where \mathcal{L}^* is the variety consisting of all completely regular semigroups in which \mathcal{L} is a congruence. Letting \mathcal{R}^* be its dual, we get $\mathcal{R}_C^* = \mathcal{R}^*$. The combined proof of these two statements implies that $(\mathcal{L}^* \cap \mathcal{R}^*)_C = \mathcal{L}^* \cap \mathcal{R}^*$. By the first reference, we conclude that $\mathcal{L}^* \cap \mathcal{R}^* = \mathcal{RBG}$ and thus $\mathcal{RBG}_C = \mathcal{RBG}$. \square

Corollary 7.5. We have that $[\mathcal{RBG}, \mathcal{R}]$ is a C-class.

Proof. By Lemma 7.3, we have $\mathcal{RBGCR} = \mathcal{R}_C$, which by Lemma 7.4 yields that $\mathcal{RBG} = \mathcal{R}_C$. \square

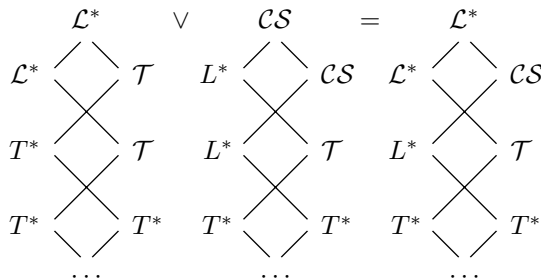
Next using the notation of [4], we have

Lemma 7.6. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be canonical varieties. Then $(\mathcal{UVW}^K)^C = \mathcal{UVW}^K$, $(\mathcal{FR}^K)^C = \mathcal{FR}^K$.

Proof. By Fact 7.1, we have $(\mathcal{UVW}^K)^C = \mathcal{U}^C \mathcal{V}^C \mathcal{W}^{CK} = \mathcal{UVW}^{CK} = \mathcal{UVW}^K$. Similarly $(\mathcal{FR}^K)^C = \mathcal{F}^C \mathcal{R}^{CK} = (\mathcal{H}_4 \cap \overline{\mathcal{H}}_4)^C \mathcal{R}^{CK} = (\mathcal{H}_4^C \cap \overline{\mathcal{H}}_4^C) \mathcal{R}^K$. \square

Next we use Polák's theorem to compute some joins, see [15].

Lemma 7.7. The join $\mathcal{L}^* \vee \overline{\mathcal{H}}_3$. The ladders are



and evaluating with the help of [8], Lemma 5.1 (ii) and (iii) and [4], Proposition 4.1, we obtain

$$\begin{aligned}
\mathcal{L}^* \vee \overline{\mathcal{H}}_3 &= \mathcal{L}^{*K} \cap \mathcal{CS}^{KT_i} \cap \mathcal{LN}\mathcal{B}^{T_r T_i} \cap \mathcal{T}^{KT_i T_r} \cap \mathcal{S}^{T_i T_r T_i} \cap \mathcal{S}^{T_r T_i T_r} \\
&= (\mathcal{I}_3 \cap \mathcal{B}^{T_i})^K \cap \mathcal{O}^{LT_i} \cap \mathcal{H}_4 \cap \mathcal{B}^{T_i T_r} \cap \mathcal{I}_4 \cap \overline{\mathcal{I}}_4 \\
&= \mathcal{I}_3^k \cap \mathcal{B}^{T_i K} \cap \mathcal{O}^{LT_i} \cap \mathcal{H}_4 \cap \mathcal{B}^{T_i T_r} \cap \overline{\mathcal{I}}_4 \\
&= \mathcal{H}_4 \overline{\mathcal{I}}_4 \mathcal{I}_3^K \mathcal{O}^{LT_i} \mathcal{B}^{T_i K} \mathcal{B}^{T_i T_r}.
\end{aligned}$$

Lemma 7.8. *The join $\mathcal{H}_3 \vee \mathcal{RBG}$. The ladders are*

$$\begin{array}{ccc}
\begin{array}{c} \mathcal{CS} \\ \mathcal{CS} \diagdown \quad \diagup R^* \\ T^* \diagdown \quad \diagup R^* \\ T^* \diagdown \quad \diagup T^* \\ \dots \end{array} & \vee & \begin{array}{c} \mathcal{RBG} \\ \mathcal{T} \diagdown \quad \diagup \mathcal{T} \\ T^* \diagdown \quad \diagup T^* \\ \dots \end{array} & = & \begin{array}{c} \mathcal{RBG} \\ \mathcal{CS} \diagdown \quad \diagup \mathcal{T} \\ T^* \diagdown \quad \diagup R^* \\ T^* \diagdown \quad \diagup T^* \\ \dots \end{array}
\end{array}$$

and evaluating with the help of Lemma 7.2, we obtain

$$\begin{aligned}
\mathcal{RBG}^K \cap \mathcal{CS}^{KT_r} \cap \mathcal{T}^{KT_i} \cap \mathcal{S}^{T_r T_i} \cap \mathcal{RN}\mathcal{B}^{T_i T_r} \\
= (\mathcal{R} \cap \mathcal{BG})^K \cap \mathcal{O}^{LT_r} \cap \mathcal{B}^{T_i} \cap \mathcal{I}_3 \cap \overline{\mathcal{H}}_4 = \mathcal{R}^K \mathcal{BG}^K \mathcal{O}^{LT_r} \mathcal{B}^{T_i} \mathcal{I}_3 \overline{\mathcal{H}}_4.
\end{aligned}$$

Lemma 7.9. *The join $\mathcal{H}_3 \vee \overline{\mathcal{H}}_3 \vee \mathcal{RBG}$. This is the join of the two preceding ones. Hence its ladder is*

$$\begin{array}{c}
\mathcal{RBG} \\
\mathcal{CS} \diagdown \quad \diagup \mathcal{CS} \\
L^* \diagdown \quad \diagup R^* \\
T^* \diagdown \quad \diagup T^* \\
\dots
\end{array}$$

and its evaluation is

$$\begin{aligned}
\mathcal{RBG}^K \cap \mathcal{CS}^{KT_r} \cap \mathcal{CS}^{KT_i} \cap \mathcal{LN}\mathcal{B}^{T_r T_i} \cap \mathcal{RN}\mathcal{B}^{T_i T_r} \\
= \mathcal{R}^K \mathcal{BG}^K \mathcal{O}^{LT} \mathcal{H}_4 \overline{\mathcal{H}}_4 = \mathcal{R}^K \mathcal{BG}^K \mathcal{O}^{LT} \mathcal{F}.
\end{aligned}$$

Lemma 7.10. For any $\mathcal{V} \in \Psi$ we have $\mathcal{V}^C = \mathcal{V}$.

Proof. This follows from [6], Lemma 5.8 and Fact 7.1. □

We are finally ready for the main result of this section.

Theorem 7.11. *The intervals*

$$\begin{aligned} & \{[\mathcal{V} \cap \mathcal{B}, \mathcal{V}]: \mathcal{V} \in [\mathcal{G}, \mathcal{R}\mathcal{O}]\}, \{\mathcal{C}\mathcal{S}\}, \{\mathcal{N}\mathcal{B}\mathcal{G}\}, [\mathcal{R}\mathcal{O}^L(\mathcal{B}\mathcal{G}), \mathcal{R}\mathcal{O}^L], \{\mathcal{F}\mathcal{O}^L\}, \\ & \{\mathcal{F}\mathcal{O}^L\}, [\mathcal{R}\mathcal{B}\mathcal{G}, \mathcal{R}], [\mathcal{R}^K \mathcal{R}\mathcal{G}^K \mathcal{O}^{LT_r} \mathcal{F}, \mathcal{R}^K \mathcal{F}], [\mathcal{L}^* \vee \mathcal{R}^*, (\mathcal{I}_3 \vee \overline{\mathcal{I}}_3)^C], \end{aligned}$$

and the intervals

$$\begin{aligned} & [\mathcal{H}_3 \overline{\mathcal{I}}_3 \mathcal{B}^{T_r}, \mathcal{H}_3 \overline{\mathcal{I}}_3], \{\mathcal{H}_3\}, [\mathcal{I}_3 \overline{\mathcal{H}}_4 \mathcal{O}^L \mathcal{B}^{T_l}, \mathcal{I}_3 \overline{\mathcal{H}}_4 \mathcal{O}^L], [\mathcal{L}^*, \overline{\mathcal{I}}_3], \\ & [\mathcal{H}_4 \overline{\mathcal{I}}_4 \mathcal{I}_3^K \mathcal{O}^{LT_l} \mathcal{B}^{T_l K} \mathcal{B}^{T_l T_r}, \mathcal{H}_4 \overline{\mathcal{I}}_4 \mathcal{I}_3^K], [\mathcal{R}^K \mathcal{B}\mathcal{G}^K \mathcal{O}^{LT_r} \mathcal{B}^{T_l}, \mathcal{I}_3 \overline{\mathcal{H}}_4 \overline{\mathcal{I}}_3] \end{aligned}$$

and their duals form the set of **C**-classes of all varieties in Ψ .

Proof. In the statement of the theorem, we have used the notation of the general form \mathcal{V}^{T_l} and \mathcal{V}^{T_r} . See [11] for the treatment of \mathbf{T}_l - and \mathbf{T}_r -relations, and [4], Proposition 4.1.

In view of the results in [6], Section 5, Corollary 7.5 and Lemma 7.10, we may parametrize these **C**-classes by their upper ends as in Diagram 1. By duality, it suffices to treat the following cases.

Case \mathcal{I}_3 : In [8], Theorem 6.1, it is proved that $\mathcal{L}_C^* = \mathcal{L}^*$ and $\mathcal{L}^{*C} = \mathcal{I}_3$. Hence $[\mathcal{L}^*, \mathcal{I}_3]$ is a **C**-class.

Case $\mathcal{H}_4 \overline{\mathcal{I}}_4 \mathcal{I}_3^K$: From Diagram 1, we know that $\mathcal{I}_3 \vee \overline{\mathcal{H}}_3 = \mathcal{H}_4 \overline{\mathcal{I}}_4 \mathcal{I}_3^K$. By Fact 7.1 (i), we know that

$$(\mathcal{I}_3 \vee \overline{\mathcal{H}}_3)_C = (\mathcal{I}_3)_C \vee (\overline{\mathcal{H}}_3)_C = \mathcal{L}^* \vee \overline{\mathcal{H}}_3.$$

This join was computed in Lemma 7.7.

Case $\mathcal{I}_3 \overline{\mathcal{H}}_4 \overline{\mathcal{I}}_3^K$: By Fact 7.1 (i) and Corollary 7.5, we obtain

$$(\mathcal{I}_3 \overline{\mathcal{H}}_4 \overline{\mathcal{I}}_3^K)_C = (\mathcal{H}_3 \vee \mathcal{R})_C = (\mathcal{H}_3)_C \vee \mathcal{R}_C = \mathcal{H}_3 \vee \mathcal{R}\mathcal{B}\mathcal{G}.$$

This join was computed in Lemma 7.8.

Case $\mathcal{F}\mathcal{R}^K$: From Diagram 1, we know that $\mathcal{F}\mathcal{R}^K = \mathcal{I}_3 \overline{\mathcal{H}}_4 \overline{\mathcal{I}}_4 \vee \overline{\mathcal{I}}_3 \mathcal{H}_4 \mathcal{I}_4^K$. By Fact 7.1 (i), we get

$$(\mathcal{F}\mathcal{R}^K)_C = (\mathcal{I}_3 \overline{\mathcal{H}}_4 \overline{\mathcal{I}}_4^K)_C \vee (\overline{\mathcal{I}}_3 \mathcal{H}_4 \mathcal{I}_4^K)_C.$$

By the preceding case and its dual, it follows that $(\mathcal{F}\mathcal{R}^K)_C = \mathcal{H}_3 \vee \overline{\mathcal{H}}_3 \vee \mathcal{R}\mathcal{B}\mathcal{G}$. This join was computed in Lemma 7.9.

Case $\mathcal{I}_3 \vee \bar{\mathcal{I}}_3$: By Fact 7.1 (i), and Case \mathcal{I}_3 above and its dual, we have

$$(\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)_C = (\mathcal{I}_3)_C \vee (\bar{\mathcal{I}}_3)_C = \mathcal{L}^* \vee \mathcal{R}^*.$$

□

The values of $\mathcal{I}_3 \vee \bar{\mathcal{I}}_3$, $(\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^C$, $\mathcal{L}^* \vee \mathcal{R}^*$ remain undetermined.

Corollary 7.12. *We have $\mathbf{C}|_\Psi = \varepsilon$.*

Proof. Except for the variety $\mathcal{I}_3 \vee \bar{\mathcal{I}}_3$, this follows easily by intersecting the \mathbf{C} -classes in Theorem 7.11 with Ψ . See Diagram 1. Since $(\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)_C = \mathcal{L}^* \vee \mathcal{R}^*$, which is evidently different from $(\mathcal{H}_4 \bar{\mathcal{I}}_4 \mathcal{I}_3^K)_C$ and its dual, $\mathcal{I}_3 \vee \bar{\mathcal{I}}_3$ cannot be \mathbf{C} -related to any other variety in Diagram 1. □

8. LACUNAE

The completion of Theorems 6.1 and 7.11 depends on the computation of a basis of identities for $\mathcal{I}_3 \vee \bar{\mathcal{I}}_3$ and $\mathcal{L}^* \vee \mathcal{R}^*$ as well as determination of

$$(8.1) \quad (\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^L, \quad (\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^C.$$

The ladders of the first of these are

$$\begin{array}{ccc}
 \mathcal{I}_3 & \vee & \bar{\mathcal{I}}_3 & = & \mathcal{I}_3 \vee \bar{\mathcal{I}}_3 \\
 \mathcal{I}_3 \begin{array}{c} \diagdown \\ \diagup \end{array} \mathcal{G} & & \mathcal{G} \begin{array}{c} \diagdown \\ \diagup \end{array} \bar{\mathcal{I}}_3 & & \mathcal{I}_3 \begin{array}{c} \diagdown \\ \diagup \end{array} \bar{\mathcal{I}}_3 \\
 T^* \begin{array}{c} \diagdown \\ \diagup \end{array} \mathcal{G} & & \mathcal{G} \begin{array}{c} \diagdown \\ \diagup \end{array} T^* & & \mathcal{G} \begin{array}{c} \diagdown \\ \diagup \end{array} \mathcal{G} \\
 T^* \begin{array}{c} \diagdown \\ \diagup \end{array} T^* & & T^* \begin{array}{c} \diagdown \\ \diagup \end{array} T^* & & T^* \begin{array}{c} \diagdown \\ \diagup \end{array} T^* \\
 \dots & & \dots & & \dots
 \end{array}$$

and their evaluation is

$$(\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^K \cap \mathcal{I}_3^{KT_r} \cap \bar{\mathcal{I}}_3^{KT_l} \cap \mathcal{G}^{KT_r T_l} \cap \mathcal{G}^{KT_l T_r} \cap \mathcal{S}^{T_r T_l T_r} \cap \mathcal{S}^{T_l T_r T_l}.$$

First $(\mathcal{I}_3 \vee \bar{\mathcal{I}}_3)^K = \mathcal{I}^K \vee \bar{\mathcal{I}}_3^K$ in which we know nothing about \mathcal{I}_3^K and $\bar{\mathcal{I}}_3^K$.

A similar discussion is valid for the join $\mathcal{L}^* \vee \mathcal{R}^*$. Indeed, using [8], Theorem 6.1, the ladders are

$$\begin{array}{c}
\mathcal{L}^* \quad \vee \quad \mathcal{R}^* \quad = \quad \mathcal{L}^* \vee \mathcal{R}^* \\
\begin{array}{ccc}
\mathcal{L}^* & \begin{array}{c} \diagdown \quad \diagup \\ \mathcal{T} \end{array} & \\
& \diagup \quad \diagdown & \\
T^* & \begin{array}{c} \diagdown \quad \diagup \\ \mathcal{T} \end{array} & \\
& \diagup \quad \diagdown & \\
T^* & \begin{array}{c} \diagdown \quad \diagup \\ \mathcal{T}^* \end{array} & \\
& \diagup \quad \diagdown & \\
\cdots & &
\end{array}
\end{array}$$

and the obstacle begins already with \mathcal{L}^{*K} and \mathcal{R}^{*K} , the same way as for \mathcal{I}_3^K and $\overline{\mathcal{I}}_3^K$. Note that by [8], Lemma 5.1 (iii) and (ii), we have $\mathcal{L}^* = \mathcal{I}_3 \cap \mathcal{B}^{\mathcal{I}_1}$.

However, if we had a basis of identities for $\mathcal{I}_3 \vee \overline{\mathcal{I}}_3$, we could use [13], Proposition II.7.7 (iii) to construct a basis for $(\mathcal{I}_2 \vee \overline{\mathcal{I}}_3)^L$. The formula

$$(\mathcal{I}_3 \vee \overline{\mathcal{I}}_3)^C = \{S \in \mathcal{CS} : C(S) \in \mathcal{I}_3 \vee \overline{\mathcal{I}}_3\}$$

is hardly helpful no matter what the basis for $\mathcal{I}_3 \vee \overline{\mathcal{I}}_3$ may be. We may say that Polák's theorem, at least within the present knowledge of the kernel relation, fails to help.

For the cases $(\mathcal{I}_3 \vee \overline{\mathcal{I}}_3)^L$ and $(\mathcal{I}_3 \vee \overline{\mathcal{I}}_3)^C$, we may examine the instances of $\mathcal{I}_2 \vee \overline{\mathcal{I}}_2$ and $\mathcal{H}_3 \vee \overline{\mathcal{H}}_3$ instead of \mathcal{I}_3 and $\overline{\mathcal{I}}_3$.

Lemma 8.1. *The following statements hold.*

- (i) $(\mathcal{I}_2 \vee \overline{\mathcal{I}}_2)^L = (\mathcal{I}_2)^L \vee (\overline{\mathcal{I}}_2)^L = \mathcal{H}_3 \vee \overline{\mathcal{H}}_3 = \mathcal{FO}^L$.
- (ii) $(\mathcal{I}_2 \vee \overline{\mathcal{I}}_2)^C = (\mathcal{I}_2)^C \vee (\overline{\mathcal{I}}_2)^C = \mathcal{I}_2 \vee \overline{\mathcal{I}}_2 = \mathcal{RO}$.
- (iii) $(\mathcal{H}_3 \vee \overline{\mathcal{H}}_3)^L = (\mathcal{H}_3)^L \vee (\overline{\mathcal{H}}_3)^L = \mathcal{H}_3 \vee \overline{\mathcal{H}}_3 = \mathcal{FO}^L$.
- (iv) $(\mathcal{H}_3 \vee \overline{\mathcal{H}}_3)^C = (\mathcal{H}_3)^C \vee (\overline{\mathcal{H}}_3)^C = \mathcal{H}_3 \vee \overline{\mathcal{H}}_3 = \mathcal{FO}^C$.

Proof. (i) According to [7], Theorem 12.3, we have $(\mathcal{I}_2)^L = \mathcal{H}_3$ and $(\overline{\mathcal{I}}_2)^L = \overline{\mathcal{H}}_3$. This proves the second equality sign. The third one forms part of [5], Theorem 5.4 (v). By [5], Theorem 5.4 (iii), we have $\mathcal{I}_2 \vee \overline{\mathcal{I}}_2 = \mathcal{RO}$. By Lemma 2.1 and [13], Proposition II.7.3 (ii), and the first reference of this proof, we get

$$\begin{aligned}
(\mathcal{RO})^L &= (\mathcal{R} \cap \mathcal{O})^L = \mathcal{R}^L \cap \mathcal{O}^L = (\mathcal{I}_3 \cap \overline{\mathcal{I}}_3)^L \cap \mathcal{O}^L \\
&= \mathcal{I}_3^L \cap \overline{\mathcal{I}}_3^L \cap \mathcal{O}^L = \mathcal{H}_4 \cap \overline{\mathcal{H}}_4 \cap \mathcal{O}^L = \mathcal{FO}^L
\end{aligned}$$

which yields $(\mathcal{I}_2 \vee \overline{\mathcal{I}}_2)^L = \mathcal{FO}^L$.

(ii) By the last references above, Fact 7.1 (i)–(ii), and Lemma 7.2, we get

$$\begin{aligned}
(\mathcal{I}_2 \vee \overline{\mathcal{I}}_2)^C &= (\mathcal{RO})^C = (\mathcal{R} \cap \mathcal{O})^C = \mathcal{R}^C \cap \mathcal{O}^C = \mathcal{R} \cap \mathcal{G}^{KC} \\
&= \mathcal{R} \cap \mathcal{G}^{CK} = \mathcal{R} \cap \mathcal{O} = \mathcal{RO}
\end{aligned}$$

and $(\mathcal{I}_2 \vee \overline{\mathcal{I}}_2)^C = \mathcal{I}_2 \vee \overline{\mathcal{I}}_2 = \mathcal{RO}$. □

The argument here for parts (iii) and (iv) is *mutatis mutandis* essentially the same as in the proof of parts (i) and (ii) and is left to the assiduous reader. The guess what happens in $(\mathcal{I}_3 \vee \overline{\mathcal{I}}_3)^L$ and $(\mathcal{I}_3 \vee \overline{\mathcal{I}}_3)^C$ is now obvious.

If we had a basis of $\mathcal{I}_3 \vee \overline{\mathcal{I}}_3$, we could use [13], Proposition II.7.3 (iii) to get a basis of $(\mathcal{I}_3 \vee \overline{\mathcal{I}}_3)^L$, thereby solving the first part of our problem.

References

- [1] *J. Kad'ourek*: On the word problem for free bands of groups and for free objects in some other varieties of completely regular semigroups. *Semigroup Forum* 38 (1989), 1–55. [zbl](#) [MR](#) [doi](#)
- [2] *F. Pastijn*: The lattice of completely regular semigroup varieties. *J. Aust. Math. Soc., Ser. A* 49 (1990), 24–42. [zbl](#) [MR](#) [doi](#)
- [3] *M. Petrich*: Some relations on the lattice of varieties of completely regular semigroups. *Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat., VIII. Ser.* 5 (2002), 265–278. [zbl](#) [MR](#)
- [4] *M. Petrich*: Canonical varieties of completely regular semigroups. *J. Aust. Math. Soc.* 83 (2007), 87–104. [zbl](#) [MR](#) [doi](#)
- [5] *M. Petrich*: A lattice of varieties of completely regular semigroups. *Commun. Algebra* 42 (2014), 1397–1413. [zbl](#) [MR](#) [doi](#)
- [6] *M. Petrich*: Certain relations on a lattice of varieties of completely regular semigroups. *Commun. Algebra* 43 (2015), 4080–4096. [zbl](#) [MR](#) [doi](#)
- [7] *M. Petrich*: Varieties of completely regular semigroups related to canonical varieties. *Semigroup Forum* 90 (2015), 53–99. [zbl](#) [MR](#) [doi](#)
- [8] *M. Petrich*: Some relations on a semilattice of varieties of completely regular semigroups. *Semigroup Forum* 93 (2016), 607–628. [zbl](#) [MR](#) [doi](#)
- [9] *M. Petrich*: Another lattice of varieties of completely regular semigroups. *Commun. Algebra* 45 (2017), 2783–2794. [zbl](#) [MR](#) [doi](#)
- [10] *M. Petrich, N. R. Reilly*: Semigroups generated by certain operators on varieties of completely regular semigroups. *Pac. J. Math.* 132 (1988), 151–175. [zbl](#) [MR](#) [doi](#)
- [11] *M. Petrich, N. R. Reilly*: Operators related to E -disjunctive and fundamental completely regular semigroups. *J. Algebra* 134 (1990), 1–27. [zbl](#) [MR](#) [doi](#)
- [12] *M. Petrich, N. R. Reilly*: Operators related to idempotent generated and monoid completely regular semigroups. *J. Aust. Math. Soc., Ser. A* 49 (1990), 1–23. [zbl](#) [MR](#) [doi](#)
- [13] *M. Petrich, N. R. Reilly*: Completely Regular Semigroups. Canadian Mathematical Society Series of Monographs and Advanced Texts 23. A Wiley-Interscience Publication. John Wiley & Sons, Chichester, 1999. [zbl](#) [MR](#)
- [14] *L. Polák*: On varieties of completely regular semigroups. I. *Semigroup Forum* 32 (1985), 97–123. [zbl](#) [MR](#) [doi](#)
- [15] *L. Polák*: On varieties of completely regular semigroups. II. *Semigroup Forum* 36 (1988), 253–284. [zbl](#) [MR](#) [doi](#)
- [16] *N. R. Reilly, S. Zhang*: Decomposition of the lattice of pseudovarieties of finite semigroups induced by bands. *Algebra Univers.* 44 (2000), 217–239. [zbl](#) [MR](#) [doi](#)

Author's address: Mario Petrich, 21420 Bol, Brač, Croatia.