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EXPONENTIAL STABILITY OF A FLEXIBLE STRUCTURE WITH HISTORY AND THERMAL EFFECT

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Abstract. In this paper we study the asymptotic behavior of a system composed of an integro-partial differential equation that models the longitudinal oscillation of a beam with a memory effect to which a thermal effect has been given by the Green-Naghdi model type III, being physically more accurate than the Fourier and Cattaneo models. To achieve this goal, we will use arguments from spectral theory, considering a suitable hypothesis of smoothness on the integro-partial differential equation.

Keywords: exponential stability; dissipative system; flexible structure; functional analysis

MSC 2020: 35B40, 45N05, 74K10

1. Introduction

The analysis of oscillations in flexible structures (beams, plates, and so on) has received a central treatment in the scientific literature in recent years. This is due to its multiple interesting applications in the field of science and technology. Namely, one of the main problems both from the physical and mathematical point of view corresponds to the question of the stabilization of the vibrations of a flexible structure. As is well known, there are several types of stability, where the most important is the exponential stability. On the other hand, if a flexible structure is given a heat effect, a model is created that is sufficiently precise and realistic from the physical point of view. The effects of heat on a structure are mainly given by the classic Fourier law of heat conduction, but this law has a number of shortcomings. One of the first shortcomings is that the heat propagates infinitely on the body; another

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deficiency is that it is unable to take into account for the effects of memory on certain materials at low temperatures. One way to eliminate these paradoxes is to use another model of heat conduction, such as the Cattaneo model, which predicts the propagation of heat in a structure by means of finite waves [4]. This phenomenon is commonly called second sound. However, but this model generates suspicions among scientists, because it has a subtle deficiency of the analytical type; that is to say, the problem of the non-objectivity of the character of the material derivative of the vector field associated with the heat flow [5]. The model of Green-Naghdi type III [14] does not have the deficiencies presented by the Fourier and Cattaneo models and provides a model of easy analytical manipulation that is free from the paradoxes and controversies of the two previous laws. More details of the above statements can be found in [12], [23].

When a flexible structure oscillates, the moment of the linear balance gives us the relation [1], [17]:

$$mu_{tt} - \sigma_x = f(x),$$

where σ represents the stress defined by the expression

$$\sigma = \sigma(u_x, u_{xt}) = p(x)u_x + 2\delta(x)u_{xt},$$

where $m(x)$ is the mass per unit length of the structure, $\delta(x)$ is the coefficient of internal material damping, and $p(x)$ is a positive function associated with stress acting on the body. From the above, we obtain directly the equation of the longitudinal movement of the beam oscillation when an exterior disturbing force acts on it, namely:

$$mu_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x = f. \tag{1.1}$$

We observe that in the study of this type of problem, a desirable goal is that the semigroup associated with these equations or systems, coupled with some dissipative effect, decays exponentially when t tends to infinity. In this sense we note that in general there are several contributions to the study of the asymptotic behavior of systems associated with thermoviscoelastic problems with memory, namely [16], [7], [9], [20], [3], [10], [22] and references therein. In this direction, the following works can be mentioned where $g(s) \equiv 0$ whose results are closely related to the one presented to this paper:

For example for (1.1), Gorain et al. [13] consider the system

$$\begin{gathered} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \kappa\theta_x = f, \\ \theta_t - \theta_{xx} - \kappa u_{xt} = 0. \end{gathered}$$

and they prove the exponential decay of the semigroup associated with the system, which in this case is composed of the equation (1.1); that is, a flexible non-homogeneous structure which experiences a thermal effect given by the Fourier law.

Recently, Alves et al. [1] considered the system

$$
\begin{aligned}
m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x &= 0,\\
\theta_t + \kappa q_x + \eta u_{xt} &= 0, \qquad (1.2)\\
\tau q_t + \beta q + \kappa\theta_x &= 0.
\end{aligned}
$$

They prove the exponential stability of the associated solutions of a flexible homogeneous structure with a heat effect given by Cattaneo's law.

In this paper, we will study a variation of the model (1.2) in which a memory effect is considered in the material and a non-classical heat effect given by the model of Green-Naghdi type III which allows us to make an improvement on the deficiencies and controversies given by the models of heat flow named above. Our study system is given by the following equations:

$$
\begin{aligned}
(1.3)\qquad & m(x)u_{tt} - g(0)u_{xx} - \sigma_x(u_x, u_{xt})\\
& \quad - \int_0^\infty g'(s)u_{xx}(t-s)\,\mathrm{d}s - \xi\theta_{xt} = 0 \quad \text{in } \Gamma,\\
(1.4)\qquad & \theta_{tt} - \kappa\theta_{xx} - \beta\theta_{xxt} - \xi u_{xt} = 0 \quad \text{in } \Gamma,
\end{aligned}
$$

where $u = u(x,t)$, $\theta = \theta(x,t)$ are the longitudinal displacement of the beam and the temperature difference between the current state and a referential state. The term $\sigma_x(u_x, u_{xt}) = (p(x)u_x + 2\delta(x)u_{xt})_x$ represents the derivative of the stress operator in the structure, and the set Γ is given by $\Gamma = \Omega \times \mathbb{R}^+ = (0,l) \times (0,\infty)$. The constants β, κ, ξ are assumed to be strictly positive and

$$
(1.5)\qquad m(x), p(x), \delta(x) \in W^{1,\infty}(\Omega), \quad m(x),\ p(x),\ \delta(x) > 0 \quad \forall\, x \in [0,l].
$$

We consider the following boundary conditions:

$$
(1.6)\qquad u(0,t) = u(l,t) = 0, \quad \theta(0,t) = \theta(l,t) = 0 \quad \forall\, t \geqslant 0,
$$

and initial conditions

$$
(1.7)\quad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \varphi_0(x) \quad \text{in } \Omega.
$$

The integral term in (1.3) represents a history term with kernel g satisfying the following hypothesis:

$$
\begin{aligned}
&\mathbb{H}_1\colon\ g(s) \in C^2(0,\infty) \cap C[0,\infty), \quad g' \in L^1(0,\infty),\\
&\mathbb{H}_2\colon\ g(s) > 0,\ g'(s) < 0,\ g''(s) > 0 \quad \text{on } (0,\infty),\\
&\mathbb{H}_3\colon\ g(\infty) > 0,\\
&\mathbb{H}_4\colon\ g''(s) + \varrho g'(s) \geqslant 0 \quad \text{on } (0,\infty) \text{ for some constant } \varrho > 0,
\end{aligned}
$$

and there exist positive constants s_1, K such that for $s \geqslant s_1$, $g''(s) \leqslant K|g'(s)|$.

Remark. According to the hypothesis $\mathbb{H}_3$, we can assume in the rest of this work that $g(\infty) = 1$.

The main goal in this work is to prove the well-posedness and exponential stability of the problem (1.3)–(1.7) under smoothness assumptions on the functions m, p, δ, and g. To prove the well-posedness of the problem, we will use the Lumer-Phillips theorem [21] and for the proof of exponential stability we will use the Gearhart theorem [11], which considers spectral theory arguments, that is:

Theorem 1.1. *Let $\mathrm{e}^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space. Then $T(t) = \mathrm{e}^{\mathcal{A}t}$ is exponentially stable if and only if*

$$
\mathrm{i}\mathbb{R} \subset \varrho(\mathcal{A}) \tag{1.8}
$$

and

$$
\overline{\lim_{|\lambda|\to\infty}}\ \|(\mathrm{i}\lambda I - \mathcal{A})^{-1}\| < \infty \tag{1.9}
$$

hold.

This paper is organized as follows: Section 2 briefly outlines preliminary results and notations. In Section 3, well-posedness of the system is established. In Section 4, we show the exponential stability of the solutions corresponding to the semigroup $T(t)$.

2. Semigroup setting

In this section, we provide the semigroup context and the main tools that will be used to obtain the main result.

The usual spaces that we will use throughout this paper will be the standard Lebesgue and Sobolev spaces; that is to say

$$L^p(\Omega), \quad 1 \leqslant p \leqslant \infty \quad \text{and} \quad H_0^1(\Omega).$$

In the case $p = 2$, we write $\|u\|$ instead of $\|u\|_2$, and according to the Poincaré inequality, we get

$$\|u\| \leqslant C_p \|u_x\| \quad \text{and} \quad \|u\|_{H_0^1(\Omega)} = \|u_x\| \quad \forall\, u \in H_0^1(\Omega).$$

In order to write the system (1.3)–(1.7) as a Cauchy problem in a Hilbert space, we introduce a new variable in the form proposed by Dafermos [7]

$$\eta^t(x,s) = \eta(s) = u(x,t) - u(x,t-s), \quad (x,s) \in \Omega \times \mathbb{R}^+,\ t \geqslant 0. \tag{2.1}$$

Substituting the variables $(u, v, \theta, \varphi, \eta)$ in the original system, where $v = u_t$, $\varphi = \theta_t$ have to satisfy the equivalent system:

$$m(x)v_t - u_{xx} - \sigma_x(u_x, v_x) + \int_0^\infty g'(s)\eta_{xx}(s)\,\mathrm{d}s - \xi\varphi_x = 0 \quad \text{in } \Gamma, \tag{2.2}$$

$$\varphi_t - \kappa\theta_{xx} - \beta\varphi_{xx} - \xi v_x = 0 \quad \text{in } \Gamma, \tag{2.3}$$

$$\eta_t - v + \eta_s = 0 \quad \text{in } \Gamma \times \mathbb{R}^+, \tag{2.4}$$

where $\sigma_x(u_x, v_x) = (p(x)u_x + 2\delta(x)v_x)_x$ and the equation (2.4) is obtained by differentiating (2.1). Thus, the boundary conditions become

$$u(0,t) = u(l,t) = \theta_x(0,t) = \theta_x(l,t) = \eta(0,s) = \eta(l,s) = 0 \tag{2.5}$$

for $t \geqslant 0$, $s > 0$ and where the initial conditions are given by

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \varphi(x,0) = \varphi_0(x) \\ \eta^0(x,s) = u(x,0) - u(x,-s) = \eta_0(s), \quad x \in \Omega,\ s > 0. \end{gathered} \tag{2.6}$$

In view of the assumptions $\mathbb{H}_1$–$\mathbb{H}_4$, we define $\mathcal{W} = L^2_{g'}(\mathbb{R}^+, H_0^1)$ being the Hilbert space of all H_0^1-valued, square integrable functions defined on the measure space $(\mathbb{R}^+, |g'|\,\mathrm{d}s)$ equipped with norm

$$\|\eta\|^2_{\mathcal{W}} = \int_\Omega \int_0^\infty |g'(s)|\,|\eta_x(s)|^2\,\mathrm{d}s\,\mathrm{d}x.$$

Let us now introduce the phase space

$$\mathcal{H} = H_0^1 \times L^2 \times H_0^1 \times L^2 \times \mathcal{W}.$$

We define the inner product on $\mathcal{H}$ by

$$\begin{aligned}(2.7)\qquad \langle U, U_1\rangle_{\mathcal{H}} = &\int_\Omega \mathcal{P}(x) u_x \overline{u}_{1x}\,\mathrm{d}x + \int_\Omega m(x) v\overline{v}_1\,\mathrm{d}x + \int_\Omega \varphi\overline{\varphi}_1\,\mathrm{d}x \\ &+ \kappa\int_\Omega \theta_x\overline{\theta}_{1x}\,\mathrm{d}x + \int_\Omega\int_0^\infty |g'(s)|\eta_x\overline{\eta}_{1x}\,\mathrm{d}s\,\mathrm{d}x,\end{aligned}$$

where $U(t) = U = (u, v, \theta, \varphi, \eta)^\top$, $U_1(t) = U_1 = (u_1, v_1, \theta_1, \varphi_1, \eta_1)^\top$ and $\mathcal{P}(x) = p(x) + 1$.

The norm induced in $\mathcal{H}$ is given by

$$\|U\|_{\mathcal{H}}^2 = \|\sqrt{\mathcal{P}(x)}u_x\|^2 + \|\sqrt{m(x)}v\|^2 + \|\varphi\|^2 + \kappa\|\theta_x\|^2 + \|\eta\|_{\mathcal{W}}^2.$$

Note that $\|\cdot\|_{\mathcal{H}}^2$ is equivalent to the usual norm of $\mathcal{H}$

On the other hand, using (2.1) and $v = u_t$, $\varphi = \theta_t$ the initial value problem (2.2)–(2.6) can be reduced to the following abstract Cauchy problem for a first-order evolution equation

$$(2.8)\qquad \frac{\mathrm{d}}{\mathrm{d}t}U(t) = \mathcal{A}U(t), \quad U(0) = U_0,$$

where

$$U_0 = (u_0, v_0, \theta_0, \varphi_0, \eta_0)^\top.$$

The linear operator $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$(2.9)\qquad AU = \begin{pmatrix} v \\ \dfrac{1}{m(x)}\Big[u_{xx} + \sigma_x(u_x, v_x) - \displaystyle\int_0^\infty g'(s)\eta_{xx}(s)\,\mathrm{d}s + \xi\varphi_x\Big] \\ \varphi \\ \kappa\theta_{xx} + \beta\varphi_{xx} + \xi v_x \\ v - \eta_s \end{pmatrix}$$

with the domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$ defined by

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \Big\{&(u, v, \theta, \varphi, \eta) \in \mathcal{H}\colon\ v \in H_0^1,\ \kappa\theta + \beta\varphi \in H^2,\ \eta \in \mathcal{W},\ \eta_s \in \mathcal{W} \\ &\eta(0) = 0,\ \mathcal{P}(x)u_x + 2\delta(x)v_x - \int_0^\infty g'(s)\eta_x(s)\,\mathrm{d}s \in H^1\Big\}.\end{aligned}$$

3. Well-posedness

Theorem 3.1. *The operator $\mathcal{A}$ generates a C_0-semigroup $T(t) = \mathrm{e}^{\mathcal{A}t}$ of contractions on the space $\mathcal{H}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and 0 belongs to the resolvent set of $\mathcal{A}$, denoted by $\varrho(\mathcal{A})$. We observe firstly that $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}$, by using (2.7) and we have for any $U \in \mathcal{D}(\mathcal{A})$ that

$$
\begin{aligned}
(3.1)\qquad \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = &\int_\Omega \mathcal{P}(x) v_x \overline{u}_x \,\mathrm{d}x + \int_\Omega [u_{xx} + (p(x)u_x + 2\delta(x)v_x)_x]\overline{v}\,\mathrm{d}x \\
&- \int_\Omega \Big[\int_0^\infty g'(s)\eta_{xx}\,\mathrm{d}s - \xi\varphi_x\Big]\overline{v}\,\mathrm{d}x + \kappa\int_\Omega \varphi_x\overline{\theta}_x\,\mathrm{d}x \\
&+ \int_\Omega [\kappa\theta_{xx} + \beta\varphi_{xx} + \xi v_x]\overline{\varphi}\,\mathrm{d}x \\
&+ \int_\Omega\int_0^\infty |g'(s)|(v-\eta_s)_x\overline{\eta}_x\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

Integrating by parts in (3.1), we easily see that

$$
(3.2)\quad \mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -2\int_\Omega \delta(x)|v_x|^2\,\mathrm{d}x - \beta\int_\Omega |\varphi_x|^2\,\mathrm{d}x - \frac{1}{2}\int_0^\infty g''(s)\|\eta_x\|^2\,\mathrm{d}s \leqslant 0,
$$

thus $\mathcal{A}$ is a dissipative operator.

On the other hand, to prove that $0 \in \varrho(\mathcal{A})$, we will use similar arguments to those given in [18], [2], [8] and references therein. In fact, given $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$, we must show that there exists a unique $U = (u, v, \theta, \varphi, \eta)^\top$ in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = F$. Indeed

$$
\begin{aligned}
&(3.3) && v = f_1 \in H_0^1,\\
&(3.4) && u_{xx} + \sigma_x(u_x, v_x) - \int_0^\infty g'(s)\eta_{xx}(s)\,\mathrm{d}s + \xi\varphi_x = m(x)f_2 \in L^2,\\
&(3.5) && \varphi = f_3 \in H_0^1,\\
&(3.6) && \kappa\theta_{xx} + \beta\varphi_{xx} + \xi v_x = f_4 \in L^2,\\
&(3.7) && v - \eta_s = f_5 \in \mathcal{W}.
\end{aligned}
$$

We can get a unique $v \in H_0^1$ from (3.3), and then from (3.7) we get

$$
(3.8)\qquad \eta(s) = \int_0^s (v - f_5(\tau))\,\mathrm{d}\tau = sf_1 - \int_0^s f_5(\tau)\,\mathrm{d}\tau
$$

It is clear that $\eta(0)=0$ and $\eta_s \in \mathcal{W}$. We want to prove that $\eta \in \mathcal{W}$. For any $T>0$, $\varepsilon>0$, by $\mathbb{H}_4$ and the Cauchy-Schwarz inequality, we have

$$\int_\varepsilon^T |g'(s)||\eta_x(s)|^2\,\mathrm{d}s \leqslant -\int_\varepsilon^T g'(s)|\eta_x(s)|^2\,\mathrm{d}s \leqslant \frac{1}{\varrho}\int_\varepsilon^T g''(s)|\eta_x(s)|^2\,\mathrm{d}s.$$

Integrating by parts and straightforward calculations, we have

$$\int_\varepsilon^T |g'(s)||\eta_x(s)|^2\,\mathrm{d}s \leqslant -\frac{2}{\varrho}g'(\varepsilon)|\eta_x(\varepsilon)|^2 + \frac{4}{\varrho^2}\int_\varepsilon^T |g'(s)|\;|\eta_{xs}(s)|^2\,\mathrm{d}s. \tag{3.9}$$

Let us notice that $-2\varrho^{-1}g'(\varepsilon)\|\eta_x(\varepsilon)\|^2 \to 0$ as $\varepsilon \to 0$. As a result from (3.9) by letting $T \to \infty$ and $\varepsilon \to 0$ that $\eta \in \mathcal{W}$ and

$$\|\eta\|^2_{\mathcal{W}} \leqslant \frac{4}{\varrho^2}\int_0^\infty |g'(s)|\,\|\eta_{xs}(s)\|^2\,\mathrm{d}s.$$

On the other hand, from (3.6) and (3.3) we obtain

$$(\kappa\theta+\beta\varphi)_{xx} \in L^2.$$

From the regularity theory for the linear elliptic equations we find that

$$\kappa\theta+\beta\varphi \in H^2.$$

Moreover, from (3.4) we get

$$\left(\mathcal{P}(x)u_x + 2\delta(x)v_x - \int_0^\infty g'(s)\eta_x(s)\,\mathrm{d}s\right)_x \in L^2. \tag{3.10}$$

Hence,

$$\mathcal{P}(x)u_x + 2\delta(x)v_x - \int_0^\infty g'(s)\eta_x(s)\,\mathrm{d}s \in H^1.$$

Moreover, it is obvious that there is a positive constant C, being independent of U, such that $\|U\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}$. Therefore, we conclude that $0 \in \varrho(\mathcal{A})$, and so $\mathcal{A}$ becomes the infinitesimal generator for a contraction semigroup in $\mathcal{H}$. □

From this theorem it follows the well-posedness for the abstract Cauchy problem (2.8) thanks to the semigroup theory, specifically to the Lumer-Phillips theorem. In particular, the following theorem [21] is obtained immediately.

Theorem 3.2. *For every initial condition $U_0 \in \mathcal{D}(\mathcal{A})$, problem (2.2)–(2.6) has a unique solution satisfying*

$$U \in C^1((0,\infty);\mathcal{H}) \cap C((0,\infty);\mathcal{D}(\mathcal{A})).$$

4. Exponential stability

In this section, we focus on proving the exponential stability for $T(t) = \mathrm{e}^{\mathcal{A}t}$, a semigroup associated with the operator $\mathcal{A}$, given by (2.9). Firstly we need to consider the resolvent equation, i.e., for any $F \in \mathcal{H}$ and $U \in \mathcal{D}(\mathcal{A})$ the following holds

$$(\mathrm{i}\lambda I - \mathcal{A})U = F,$$

i.e.,

$$\mathrm{i}\lambda u - v = f_1, \tag{4.1}$$

$$\mathrm{i}\lambda v - \frac{1}{m(x)}\Big(u_{xx} + \sigma_x(u_x, v_x) - \int_0^\infty g'(s)\eta_{xx}(s)\,\mathrm{d}s + \xi\varphi_x\Big) = f_2, \tag{4.2}$$

$$\mathrm{i}\lambda\theta - \varphi = f_3, \tag{4.3}$$

$$\mathrm{i}\lambda\varphi - \kappa\theta_{xx} - \beta\varphi_{xx} - \xi v_x = f_4, \tag{4.4}$$

$$\mathrm{i}\lambda\eta - v + \eta_s = f_5. \tag{4.5}$$

Theorem 4.1. *The semigroup $T(t) = \mathrm{e}^{\mathcal{A}t}$, generated by the operator $\mathcal{A}$ given in* (2.9) *is exponentially stable, i.e., there exist constants M, $\gamma > 0$ such that*

$$\|T(t)\| \leqslant M\mathrm{e}^{-\gamma t} \quad \forall\, t > 0$$

To prove this proposition we will use Theorem 1.1. Firstly we will prove (1.8) i.e., $\mathrm{i}\mathbb{R} \subset \varrho(\mathcal{A})$. We will follow similar ideas to those given in [18], [2], [19], which consist of the following steps:

(i) Since $0 \in \varrho(\mathcal{A})$, for any real number λ with $\|\lambda\mathcal{A}^{-1}\| < 1$, the linear bounded operator $\mathrm{i}\lambda\mathcal{A}^{-1} - I$ is invertible, therefore $\mathrm{i}\lambda I - \mathcal{A} = \mathcal{A}(\mathrm{i}\lambda\mathcal{A}^{-1} - I)$ is invertible and its inverse belongs to $\mathcal{L}(\mathcal{H})$; that is, $\mathrm{i}\lambda \in \varrho(\mathcal{A})$. Moreover, $\|(\mathrm{i}\lambda I - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}^{-1}\|^{-1}, \|\mathcal{A}^{-1}\|^{-1})$.

(ii) If $\sup\{\|(\mathrm{i}\lambda I - \mathcal{A})^{-1}\|\colon |\lambda| < \|\mathcal{A}^{-1}\|^{-1}\} = M < \infty$, then for $|\lambda_0| < \|\mathcal{A}^{-1}\|^{-1}$ and $\lambda \in \mathbb{R}$ such that $|\lambda - \lambda_0| < M^{-1}$, we have $\|(\lambda - \lambda_0)(\mathrm{i}\lambda_0 I - \mathcal{A})^{-1}\| < 1$. Therefore, the operator

$$\mathrm{i}\lambda I - \mathcal{A} = (\mathrm{i}\lambda_0 I - \mathcal{A})(I + \mathrm{i}(\lambda - \lambda_0)(\mathrm{i}\lambda_0 I - \mathcal{A})^{-1})$$

is invertible with inverse in $\mathcal{L}(\mathcal{H})$; that is, $\mathrm{i}\lambda \in \varrho(\mathcal{A})$. Since λ_0 is arbitrary, we can conclude that $\{\mathrm{i}\lambda\colon |\lambda| < \|\mathcal{A}^{-1}\|^{-1} + M^{-1} \subset \varrho(\mathcal{A})\}$ and the function $\|(\mathrm{i}\lambda I - \mathcal{A})^{-1}\|$ is continuous in the interval $(-\|\mathcal{A}^{-1}\|^{-1} - M^{-1}, \|\mathcal{A}^{-1}\|^{-1} + M^{-1})$.

(iii) Thus, it follows by item (ii) that if $\mathrm{i}\mathbb{R} \subset \varrho(\mathcal{A})$ is not true, then there exists $\omega \in \mathbb{R}$ with $\|\mathcal{A}^{-1}\|^{-1} < |\omega|$ such that $\{\mathrm{i}\lambda\colon |\lambda| < |\omega|\} \subset \varrho(\mathcal{A})$ and

$\sup\{\|(\mathrm{i}\lambda I-\mathcal{A})^{-1}\|\colon |\lambda|<|\omega|\}=\infty$. Therefore, there exists a sequence of real numbers (λ_n) with $\lambda_n\to\omega$, $|\lambda_n|<|\omega|$ and sequences of vector functions $U_n=(u_n,v_n,\theta_n,\varphi_n,\eta_n)^\top\in\mathcal{D}(\mathcal{A}), F_n=(f_{1n},f_{2n},f_{3n},f_{4n},f_{5n})^\top\in\mathcal{H}$, such that $(\mathrm{i}\lambda I-\mathcal{A})U_n=F_n$ and $\|U_n\|_{\mathcal{H}}=1$ and $F_n\to 0$ in $\mathcal{H}$ when $n\to\infty$, that is:

$$\mathrm{i}\lambda_n u_n - v_n = f_{1n}\to 0 \quad \text{in } H_0^1, \tag{4.6}$$

$$\begin{aligned}\mathrm{i}\lambda_n v_n &- \frac{1}{m(x)}(u_{nxx}+\sigma_x(u_x,v_x))\\ &- \frac{1}{m(x)}\Big(\int_0^\infty g'(s)\eta_{nxx}(s)\,\mathrm{d}s-\xi\varphi_{nx}\Big)=f_{2n}\to 0 \quad \text{in } L^2,\end{aligned} \tag{4.7}$$

$$\mathrm{i}\lambda_n\theta_n-\varphi_n=f_{3n}\to 0 \quad \text{in } H_0^1, \tag{4.8}$$

$$\mathrm{i}\lambda_n\varphi_n-\kappa\theta_{nxx}-\beta\varphi_{nxx}-\xi v_{nx}=f_{4n}\to 0 \quad \text{in } L^2, \tag{4.9}$$

$$\mathrm{i}\lambda_n\eta_n-v_n+\eta_{ns}=f_{5n}\to 0 \quad \text{in } \mathcal{W}. \tag{4.10}$$

We observe that

$$\operatorname{Re}\langle \mathrm{i}\lambda_n U_n-\mathcal{A}U_n,U_n\rangle_{\mathcal{H}}\to 0 \quad \text{as } n\to\infty. \tag{4.11}$$

Thus from (3.2) we have

$$2\int_\Omega \delta(x)|v_{nx}|^2\,\mathrm{d}x+\beta\int_\Omega|\varphi_{nx}|^2\,\mathrm{d}x+\frac{1}{2}\int_0^\infty g''(s)\|\eta_{nx}\|^2\,\mathrm{d}s\to 0. \tag{4.12}$$

From $\mathbb{H}_4$ we obtain

$$\frac{\varrho}{2}\|\eta\|_{\mathcal{W}}^2\leqslant\frac{1}{2}\int_0^\infty g''(s)\|\eta_{nx}\|^2\,\mathrm{d}s. \tag{4.13}$$

Using (4.12) and (4.13), it follows

$$\eta_n\to 0 \quad \text{in } \mathcal{W} \quad \text{and} \quad v_{nx}\to 0,\ \varphi_{nx}\to 0 \quad \text{in } L^2 \text{ as } n\to\infty. \tag{4.14}$$

From the Poincaré inequality and (1.5) we get

$$v_n\to 0 \quad \text{and} \quad \sqrt{m(x)}v_n\to 0 \quad \text{in } L^2 \text{ as } n\to\infty. \tag{4.15}$$

We note that $v_n\to 0$ in H_0^1. Thus from (4.6) we have $u_n\to 0$ in H_0^1, and therefore from (1.5) we see that

$$\sqrt{\mathcal{P}(x)}u_{nx}\to 0 \quad \text{in } L^2 \text{ as } n\to\infty. \tag{4.16}$$

On the other hand, taking the inner product in (4.9) with φ_n in L^2 and integrating by parts, it follows that

$$\mathrm{i}\lambda_n\langle\varphi_n,\varphi_n\rangle+\kappa\langle\theta_{nx},\varphi_{nx}\rangle+\beta\langle\varphi_{nx},\varphi_{nx}\rangle+\xi\langle v_n,\varphi_{nx}\rangle\to 0 \text{ as } n\to\infty, \tag{4.17}$$

therefore by (4.14) and (4.17) it follows that

$$\varphi_n\to 0 \quad \text{in } L^2 \text{ as } n\to\infty. \tag{4.18}$$

Using (4.14) and (4.18) in (4.8), we have

$$\theta_n\to 0 \quad \text{and} \quad \theta_{nx}\to 0 \quad \text{in } L^2 \text{ as } n\to\infty. \tag{4.19}$$

From (4.14), (4.15), (4.16), (4.18), and (4.19) we have that

$$\lim_{n\to\infty}\|U_n\|_{\mathcal{H}}=0. \tag{4.20}$$

Hence, U_n cannot be of unit $\mathcal{H}$-norm.

For the proof of (1.9) we will use contradiction arguments. Suppose that (1.9) is not true. Then there exists a sequence λ_n with $|\lambda_n|\to\infty$ and a sequence $U_n=(u_n,v_n,\theta_n,\varphi_n,\eta_n)^\top$ in $\mathcal{D}(\mathcal{A})$ with unit norm in $\mathcal{H}$ such that $\|(\mathrm{i}\lambda_nI-\mathcal{A})U_n\|\to 0$ as $n\to\infty$, i.e.,

$$\mathrm{i}\lambda_nu_n-v_n\to 0 \quad \text{in } H_0^1, \tag{4.21}$$

$$\begin{aligned}\mathrm{i}\lambda_nv_n&-\frac{1}{m(x)}(u_{nxx}+\sigma_x(u_x,v_x))\\&-\frac{1}{m(x)}\left(\int_0^\infty g'(s)\eta_{nxx}(s)\,\mathrm{d}s-\xi\varphi_{nx}\right)\to 0 \quad \text{in } L^2,\end{aligned} \tag{4.22}$$

$$\mathrm{i}\lambda_n\theta_n-\varphi_n\to 0 \quad \text{in } H_0^1, \tag{4.23}$$

$$\mathrm{i}\lambda_n\varphi_n-\kappa\theta_{nxx}-\beta\varphi_{nxx}-\xi v_x\to 0 \quad \text{in } L^2, \tag{4.24}$$

$$\mathrm{i}\lambda_n\eta_n-v_n+\eta_{ns}\to 0 \quad \text{in } \mathcal{W}. \tag{4.25}$$

Again we have (4.11), i.e.,

$$\mathrm{Re}\langle\mathrm{i}\lambda_nU_n-\mathcal{A}U_n,U_n\rangle_{\mathcal{H}}\to 0 \quad \text{as } n\to\infty.$$

Thus

$$2\int_\Omega\delta(x)|v_{nx}|^2\,\mathrm{d}x+\beta\int_\Omega|\varphi_{nx}|^2\,\mathrm{d}x+\frac{1}{2}\int_0^\infty g''(s)\|\eta_{nx}\|^2\,\mathrm{d}s\to 0.$$

Using similar steps to those given in the above proof and straightforward calculations, we have

$$\eta_n \to 0 \quad \text{in } \mathcal{W} \text{ as } n \to \infty$$

and

$$\sqrt{m(x)}v_n \to 0, \ \ \sqrt{\mathcal{P}(x)}u_{nx} \to 0 \quad \text{in } L^2 \text{ as } n \to \infty,$$
$$\theta_{nx} \to 0, \ \ \varphi_n \to 0 \quad \text{in } L^2 \text{ as } n \to \infty.$$

Therefore, we conclude

$$\lim_{n\to\infty} \|U_n\|_{\mathcal{H}} = 0. \tag{4.26}$$

Hence, U_n cannot be of unit $\mathcal{H}$-norm. In this way our main theorem is proven.

Further remarks:

(i) Note that in this work we have considered $g(\infty) = 1$. Otherwise, the functional $-g(\infty)u_{xx}$ can be changed in (2.2) for the functional $-u_{xx}$ and we can redefine the integral kernel $g(\cdot) = g(\cdot)/g(\infty)$, thus obtaining a similar problem to (2.2)–(2.6).

(ii) An example of a function g satisfying the hypothesis $\mathbb{H}_1$–$\mathbb{H}_4$, is given by the called Maxwell type kernel, namely

$$g(s) = 1 + M\mathrm{e}^{-ks}, \quad k, M > 0.$$

(iii) The proof of $-2\varrho^{-1}g'(\varepsilon)\|\eta_x(\varepsilon)\|^2 \to 0$ as $\varepsilon \to 0$ is proved with similar arguments to those given in [18], [20] and references therein.

(iv) Exploiting (4.23), and with similar steps to those given in [18], [19], it follows that $\mathrm{i}\lambda_n\theta_n - \varphi_n \to 0$ in L^2 and $\mathrm{i}\lambda_n\theta_{nx} - \varphi_{nx} \to 0$ in L^2. Then with the assumptions about φ_n and φ_{nx} we obtain (4.19).

(v) The condition (1.5) guarantees the equivalence of the norms $\|v\|$ and $\|\sqrt{m}v\|$, as well as the equivalence $\|u_x\|$ and $\|\sqrt{\mathcal{P}}u_x\|$. In this way we can show (4.15) and (4.16).

5. Conclusions

As we know, Fourier's law essentially tells us that any thermal disturbance at one point has an instantaneous effect on any other part of the body and at the same time does not consider the memory effects at low temperatures. On the other hand, the Cattaneo model has the defect of the derivative of the vector field associated with the heat flow. When we prove the well-posedness and the exponential decay and not another weaker type of decay (for example, the polynomial decay) we have made an improvement of the works named in the Section 1, offering with it a much more realistic model from the physical point of view. Let us say that these results can be improved furthermore by considering the Coleman-Gurtin [6] or Gurtin-Pipkin [15] law instead of equation (1.4), obtaining an effective prediction for heating propagation on the structure. All these statements constitute a promising set of new questions to be addressed in further research.

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