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BOUNDARY EXACT CONTROLLABILITY FOR A POROUS ELASTIC TIMOSHENKO SYSTEM

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Cordially dedicated to Professor Jaime E. Muñoz Rivera on his 60th birthday.

Abstract. In this paper, we consider a one-dimensional system governed by two partial differential equations. Such a system models phenomena in engineering, such as vibrations in beams or deformation of elastic bodies with porosity. By using the HUM method, we prove that the system is boundary exactly controllable in the usual energy space. We will also determine the minimum time allowed by the method for the controllability to occur.

Keywords: boundary exact controllability; Timoshenko beam; porous elasticity

MSC 2020: 93C20, 93B05

1. Introduction

In this work, we study the boundary exact controllability problem for the following evolution system in one dimension:

$$\begin{aligned} \varrho u_{tt} - \kappa(\mu u_x + b\phi)_x &= 0 \\ J\phi_{tt} - \delta\phi_{xx} + \kappa(bu_x + \xi\phi) &= 0 \end{aligned} \quad \text{in }]0,L[\times]0,T[, \tag{1.1}$$

$$\begin{aligned} u(0,t) = \omega_1(t),\ u(L,t) &= 0 \\ \phi(0,t) = \omega_2(t),\ \phi(L,t) &= 0 \end{aligned} \quad \text{in }]0,T[, \tag{1.2}$$

$$\begin{aligned} u(x,0) = u^0(x),\ u_t(x,0) &= u^1(x) \\ \phi(x,0) = \phi^0(x),\ \phi_t(x,0) &= \phi^1(x) \end{aligned} \quad \text{in }]0,L[, \tag{1.3}$$

where ϱ, μ, J, δ, ξ, and κ are positive constants and b is a constant satisfying $b^2 = \mu\xi$, the functions ω_1 and ω_2 are the controls and u^0, u^1, ϕ^0, and ϕ^1 are the initial data. This type of system models various kinds of phenomena in engineering.

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When $\mu = \xi = 1$, system (1.1) models the transverse vibrations in a beam, taking into account the effect of rotatory inertia and shearing deformations, in this case, u represents the tranverse displacement of the beam and ϕ is the rotation angle of a filament of the beam, cf. Timoshenko [29]. There are many works on stabilization for Timoshenko systems subject to various types of damping and thermal effects [3], [4], [6], [8], [24], [25]. One of the first studies concerning stabilization for the Timoshenko system was carried out by Soufyane in [28]. In this work the author obtained stabilization for the following system:

$$\varrho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \tag{1.4}$$

$$\varrho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + a(x)\psi_t = 0, \tag{1.5}$$

subject to Dirichlet boundary conditions, where $a = a(x)$ is a positive bounded function. Physically, $a(x)\psi_t$ represents a frictional term.

Recently, Mercier and Régnier in [19] studied the issue of stabilization for the following Timoshenko system:

$$\varphi_{tt} - (\varphi_x + \psi)_x = 0, \tag{1.6}$$

$$\psi_{tt} - a\psi_{xx} + b(\varphi_x + \psi) = 0 \tag{1.7}$$

in $(0,1) \times (0,\infty)$, where a and b are positive constants, with boundary conditions

$$\varphi_x(0,t) = \psi(0,t) = \psi(1,t) = 0 \tag{1.8}$$

and the boundary dissipation law given by

$$b^{-1}a\psi_x(1,t) = -\beta\psi_t(1,t). \tag{1.9}$$

Regarding controllability of Timoshenko systems, there are few studies, among which we can mention [5], [11], [12], [17], [27], [30] and more recently [1].

When $\kappa = 1$, system (1.1) is a model for the study of elastic solids with voids, which is one of simple extensions of the classical theory of elasticity. In this case, the variable u represents the displacement of a solid elastic material and ϕ represents the volume fraction, cf. [7]. Although there is a good amount of literature dealing with stabilization of solutions for these systems (see for example [9], [16], [20], [22], [23], [26]), we did not find any works considering controllability for porous elastic systems.

Despite the similarity, there are serious difficulties when we move from the Timoshenko system to the porous elastic system, this can be verified when studying the stability exponents for the two systems (see for example [26] and its references).

The goal of this paper is to prove, using the HUM method, the boundary exact controllability for system (1.1) when controls are located only on one end of the domain of each of component-functions of the solution, in particular, as in the case (1.2).

The main contribution of this work is the proof of the following inequality of observability:

$$E(0) \leqslant C_{\alpha,T}\left[\kappa\mu \int_0^T u_x^2(0,t)\,\mathrm{d}t + \delta \int_0^T \phi_x^2(0,t)\,\mathrm{d}t\right], \tag{1.10}$$

where $C_{\alpha,T}$ is a positive constant depending on α and on T. This inequality results in the exact controllability of the system (1.1)–(1.3).

The exact controllability for the system (1.1)–(1.3) is formulated as follows: given $T > 0$, large enough, and initial data $(u^0, u^1, \phi^0, \phi^1)$ in an appropriate space, find a pair of controls (ω_1, ω_2) such that the solution (u, ϕ) of the system (1.1)–(1.3) satisfies

$$u(\cdot,T) = u_t(\cdot,T) = \varphi(\cdot,T) = \varphi_t(\cdot,T) = 0. \tag{1.11}$$

The paper is structured as follows: in Section 2, we study by the semigroup method the existence and uniqueness of the solution for the system (1.1)–(1.3), as well as for its nonhomogeneous counterpart, we also define the solutions by transposition. In Section 3, we establish the direct inequality and the inequality known as observability inequality or inverse inequality. In (1.10), $E(0)$ is the energy of system (1.1)–(1.3) at time $t = 0$ and $C_{T,\alpha}$ is a positive constant that depends on T. Inequalities of this nature have great relevance in mathematics, because they allow the total energy in a system to be estimated from the partial measure in a sub-region of the domain or boundary, moreover, this type of inequality plays an important role in questions of controllability (see [2], [10], [32]). Finally, in Section 4 we use the HUM method (cf. [14], [13]) to prove the boundary exact controllability for the system (1.1)–(1.3).

Throughout the paper, we will use $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|$ to represent the inner product and the norm in space $L^2(0,L)$, respectively.

2. Existence and uniqueness of solutions

In this section, we will study, but without proof, some results of existence and uniqueness of solutions for the system

$$\begin{aligned} \varrho u_{tt} - \kappa(\mu u_x + b\phi)_x &= 0 \\ J\phi_{tt} - \delta\phi_{xx} + \kappa(bu_x + \xi\phi) &= 0 \end{aligned} \qquad \text{in }]0,L[\times]0,T[, \tag{2.1}$$

where ϱ, μ, J, δ, ξ, and κ are positive constants and $b^2 = \mu\xi$. The boundary conditions are

$$\begin{aligned} u(0,t) &= 0, \quad u(L,t) = 0 \\ \phi(0,t) &= 0, \quad \phi(L,t) = 0 \end{aligned} \qquad \text{in }]0,T[\tag{2.2}$$

and initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x) \\ \phi(x,0) &= \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) \end{aligned} \qquad \text{in }]0,L[. \tag{2.3}$$

In this case, the energy of the solution (u, ϕ) at time t is given by

$$2E(t) := \varrho\|u_t(t)\|^2 + J\|\phi_t(t)\|^2 + \delta\|\phi_x(t)\|^2 + \kappa\|(\sqrt{\mu}u_x \pm \sqrt{\xi}\phi)(t)\|^2, \tag{2.4}$$

where $\pm$ depends on the sign of b. By multiplicative techniques, it is possible to establish

$$E(t) = E(0), \quad t > 0. \tag{2.5}$$

2.1. Homogeneous system. The initial and boundary value problem (2.1)–(2.3), can be written as a Cauchy problem for $\Psi = (u, u_t, \phi, \phi_t)'$ as follows:

$$\begin{cases} \Psi_t = \mathcal{A}\Psi, \quad t > 0, \\ \Psi(0) = \Psi_0, \end{cases} \tag{2.6}$$

where $\Psi_0 = (u_0, u_1, \phi_0, \phi_1)'$ and $\mathcal{A}$ is the differential operator

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 & 0 \\ \frac{\kappa\mu}{\varrho}\partial_x^2 & 0 & \frac{\kappa b}{\varrho}\partial_x & 0 \\ 0 & 0 & 0 & I \\ -\frac{\kappa b}{J}\partial_x & 0 & \frac{\delta}{J}\partial_x^2 - \frac{\kappa\xi}{J}I & 0 \end{bmatrix}$$

with values on the Hilbert space

$$\mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L)$$

and domain

$$D(\mathcal{A}) = \{(u, \varphi, \phi, \psi) \in \mathcal{H}; u \in H^2(0,L), \varphi \in H_0^1(0,L), \phi \in H^2(0,L), \psi \in H_0^1(0,L)\}.$$

The space $\mathcal{H}$ shall be provided with the inner product

$$\begin{aligned}(2.7)\qquad &\langle (u^0,\varphi^0,\phi^0,\psi^0),(u^1,\varphi^1,\phi^1,\psi^1)\rangle_{\mathcal{H}}\\ &\quad= \varrho(\varphi^0,\varphi^1)_{L^2} + J(\psi^0,\psi^1)_{L^2} + \delta(\phi^0_x,\phi^1_x)_{L^2}\\ &\qquad+ \kappa(\sqrt{\mu}u^0_x \pm \sqrt{\xi}\phi^0, \sqrt{\mu}u^1_x \pm \sqrt{\xi}\phi^1)_{L^2},\end{aligned}$$

which induces the norm

$$(2.8)\qquad \|(u,\varphi,\phi,\psi)\|^2_{\mathcal{H}} = \varrho\|\varphi\|^2 + J\|\psi\|^2 + \delta\|\phi_x\|^2 + \kappa\|\sqrt{\mu}u_x \pm \sqrt{\xi}\phi\|^2.$$

If $U = (u,\varphi,\phi,\psi)'$, the above settings allow us to establish

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = 0,$$

implying that $\mathcal{A}$ is a dissipative operator. By using the standard method of the semigroup theory, it is possible to establish that $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions $S(t) = \mathrm{e}^{\mathcal{A}t}$.

2.2. Nonhomogeneous system. Once established that the previous homogeneous system has a solution, if f and g are functions in space $L^1(0,T;L^2(0,L))$, then the nonhomogeneous system

$$(2.9)\qquad \begin{aligned}\varrho u_{tt} - \kappa(\mu u_x + b\phi)_x &= f\\ J\phi_{tt} - \delta\phi_{xx} + \kappa(bu_x + \xi\phi) &= g\end{aligned}\qquad \text{in }]0,L[\times]0,T[,$$

$$(2.10)\qquad \begin{aligned}u(0,t) = 0,\quad u(L,t) &= 0\\ \phi(0,t) = 0,\quad \phi(L,t) &= 0\end{aligned}\qquad \text{in }]0,T[,$$

$$(2.11)\qquad \begin{aligned}u(x,0) = u_0(x),\quad u_t(x,0) &= u_1(x)\\ \phi(x,0) = \phi_0(x),\quad \phi_t(x,0) &= \phi_1(x)\end{aligned}\qquad \text{in }]0,L[$$

admits a single mild solution (see [21]) given by

$$\Psi(t) = S(t)x + \int_0^t S(t-s)F(s)\,\mathrm{d}s,\quad 0\leqslant t\leqslant T,$$

where $F(t) = (0,f(t),0,g(t))'$, $x\in\mathcal{H}$, with regularity $\Psi\in C([0,T];\mathcal{H})$.

By using reversibility in time, the problem (2.9)–(2.11) is equivalent to

$$(2.12)\qquad \begin{aligned}\varrho u_{tt} - \kappa(\mu u_x + b\phi)_x &= f\\ J\phi_{tt} - \delta\phi_{xx} + \kappa(bu_x + \xi\phi) &= g\end{aligned}\qquad \text{in }]0,L[\times]0,T[,$$

$$(2.13)\qquad \begin{aligned}u(0,t) = 0,\quad u(L,t) &= 0\\ \phi(0,t) = 0,\quad \phi(L,t) &= 0\end{aligned}\qquad \text{in }]0,T[,$$

$$(2.14)\qquad \begin{aligned}u(x,T) = u_0(x),\quad u_t(x,T) &= u_1(x)\\ \phi(x,T) = \phi_0(x),\quad \phi_t(x,T) &= \phi_1(x)\end{aligned}\qquad \text{in }]0,L[.$$

2.3. Solution by transposition. Given

$$(v^0, v^1, \psi^0, \psi^1) \in (L^2(0,L) \times H^{-1}(0,L))^2 \quad \text{and} \quad \omega_1, \omega_2 \in L^2(0,T),$$

a solution by transposition to the system

$$\begin{aligned} \varrho v_{tt} - \kappa(\mu v_x + b\psi)_x &= 0 \\ J\psi_{tt} - \delta\psi_{xx} + \kappa(bv_x + \xi\psi) &= 0 \end{aligned} \qquad \text{in }]0,L[\times]0,T[, \tag{2.15}$$

$$\begin{aligned} v(0,t) = \omega_1(t), &\quad v(L,t) = 0 \\ \psi(0,t) = \omega_2(t), &\quad \psi(L,t) = 0 \end{aligned} \qquad \text{in }]0,T[, \tag{2.16}$$

$$\begin{aligned} v(x,0) = v^0(x), &\quad v_t(x,0) = v^1(x) \\ \psi(x,0) = \psi^0(x), &\quad \psi_t(x,0) = \psi^1(x) \end{aligned} \qquad \text{in }]0,L[\tag{2.17}$$

is a pair $(v,\psi) \in (L^\infty(0,T;L^2(0,L)))^2$ such that

$$\begin{aligned} \int_0^L \int_0^T (fv + g\psi)\,\mathrm{d}t\,\mathrm{d}x = {} & J\langle \psi^1, \phi(0)\rangle_{H^{-1},H_0^1} - J(\psi^0, \phi_t(0))_{L^2} \\ & + \varrho\langle v^1, u(0)\rangle_{H^{-1},H_0^1} - \varrho(v^0, u_t(0))_{L^2} \\ & + \kappa\mu \int_0^T \omega_1(t) u_x(0,t)\,\mathrm{d}t + \delta \int_0^T \omega_2(t)\phi_x(0,t)\,\mathrm{d}t \end{aligned} \tag{2.18}$$

for all $(f,g) \in (L^1(0,T;L^2(0,L)))^2$, where (u,ϕ) is a solution of (2.12)–(2.14). It is possible to show, by using the Riesz representation theorem, that the system (2.15)–(2.17) has a single solution by transposition, cf. [15].

Remark 2.1. By using a method found in [18] it is possible to show that the solution by transposition (v,ψ) above satisfies

$$(v,\psi) \in C^0([0,T], L^2(0,L)) \cap C^1([0,T], H^{-1}(0,L)). \tag{2.19}$$

3. Observability inequality and boundary exact controllability

In this section, we will prove the observability inequality for the system (2.1)–(2.3), which will result in the exact controllability for this system, since it is conservative (see [13]).

Next, we will prove the inverse inequality, also known as observability inequality. In this result we will use the following notation: if $g(t)$ is a function such that t_1 and t_2 belong to the domain, then

$$\sum_{t=t_1}^{t_2} g(t) := g(t_1) + g(t_2). \tag{3.1}$$

Theorem 3.1. *Let* (u,ϕ) *be a solution of* (2.1)–(2.3). *Then for* $T>2\alpha L$, *where*

$$\alpha=\max\left\{\sqrt{\frac{\varrho}{\mu\kappa}},\sqrt{\frac{J}{\delta}}\right\},$$

there exists a positive constant $C_{\alpha,T}=C(\alpha,T)>0$ *such that*

$$E(0)\leqslant C_{\alpha,T}\left[\kappa\mu\int_0^T u_x^2(0,t)\,\mathrm{d}t+\delta\int_0^T\phi_x^2(0,t)\,\mathrm{d}t\right].$$

In this case

$$C_{\alpha,T}=\frac{c}{2(T-2\alpha L)}\quad\text{and}\quad c=\max\left\{\sqrt{\frac{\varrho\xi}{\mu J}},\sqrt{\frac{\kappa\xi}{\delta}}\right\}.$$

Proof. The proof is based on an argument found in [17], which has been motivated by an idea of Zuazua in [31].

We consider, without loss of generality, $b>0$. Let $\mathcal{J}$ be the function given by

$$\begin{aligned}\mathcal{J}(x)=\frac12\int_{\alpha x}^{T-\alpha x}&[\varrho u_t^2(x,t)+J\phi_t^2(x,t)+\delta\phi_x^2(x,t)\\&+\kappa(\sqrt{\mu}u_x(x,t)+\sqrt{\xi}\phi(0,t))^2]\,\mathrm{d}t.\end{aligned}\tag{3.2}$$

We note that

$$\mathcal{J}(0)=\frac12\int_0^T[\delta\phi_x^2(0,t)+\kappa\mu u_x^2(0,t)]\,\mathrm{d}t$$

and

$$\int_0^L\mathcal{J}(x)\,\mathrm{d}x\geqslant(T-2\alpha L)E(0).$$

We have to show that

$$\int_0^L\mathcal{J}(x)\,\mathrm{d}x\leqslant c\mathcal{J}(0)$$

for some constant $c>0$. To do this, it is sufficient to establish that

$$\mathcal{J}'(x)\leqslant c\mathcal{J}(x).$$

Indeed, we write

$$\mathcal{J}(x)\equiv\mathcal{J}_1(x)+\mathcal{J}_2(x)$$

with

$$\mathcal{J}_1(x)=\frac12\int_{\alpha x}^{T-\alpha x}[\varrho u_t^2(x,t)+\kappa(\sqrt{\mu}u_x(x,t)+\sqrt{\xi}\phi(0,t))^2]\,\mathrm{d}t$$

and

$$\mathcal{J}_2(x) = \frac{1}{2}\int_{\alpha x}^{T-\alpha x} [J\phi_t^2(x,t) + \delta\phi_x^2(x,t)]\,\mathrm{d}t.$$

Thus for $\mathcal{J}_1$ we have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}x}(x) = &\int_{\alpha x}^{T-\alpha x} [\varrho u_t u_{tx} + \kappa(\sqrt{\mu}u_x + \sqrt{\xi}\phi)(\sqrt{\mu}u_x + \sqrt{\xi}\phi)_x]\,\mathrm{d}t \\ &- \frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x} (\varrho u_t^2(x,t) + \kappa(\sqrt{\mu}u_x(x,t) + \sqrt{\xi}\phi(0,t))^2)\end{aligned} \tag{3.3}$$

and the fact that $\varrho u_{tt} = \kappa\sqrt{\mu}(\sqrt{\mu}u_x + \sqrt{\xi}\phi)_x$ gives

$$\begin{aligned}\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}x}(x) = &\int_{\alpha x}^{T-\alpha x} \Big[\varrho u_t u_{tx} + \frac{\varrho}{\sqrt{\mu}}u_{tt}(\sqrt{\mu}u_x + \sqrt{\xi}\phi)\Big]\,\mathrm{d}t \\ &- \frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x} (\varrho u_t^2(x,t) + \kappa(\sqrt{\mu}u_x(x,t) + \sqrt{\xi}\phi(0,t))^2).\end{aligned} \tag{3.4}$$

We observe that

$$\int_{\alpha x}^{T-\alpha x} \varrho u_t u_{tx}\,\mathrm{d}t = \varrho u_t u_x\Big|_{t=\alpha x}^{t=T-\alpha x} - \int_{\alpha x}^{T-\alpha x} \varrho u_{tt}u_x\,\mathrm{d}t, \tag{3.5}$$

replacing (3.5) in (3.4) we have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}x}(x) = &\varrho u_t u_x\Big|_{t=\alpha x}^{t=T-\alpha x} + \int_{\alpha x}^{T-\alpha x} \frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}u_{tt}\phi\,\mathrm{d}t \\ &- \frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x} (\varrho u_t^2(x,t) + \kappa(\sqrt{\mu}u_x(x,t) + \sqrt{\xi}\phi(0,t))^2).\end{aligned} \tag{3.6}$$

Taking into account that

$$\int_{\alpha x}^{T-\alpha x} \frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}u_{tt}\phi\,\mathrm{d}t = \frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}u_t\phi\Big|_{\alpha x}^{T-\alpha x} - \frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}\int_{\alpha x}^{T-\alpha x} u_t\phi_t\,\mathrm{d}t \tag{3.7}$$

and replacing (3.7) in (3.6), we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}x}(x) = &\frac{\varrho}{\sqrt{\mu}}u_t(\sqrt{\mu}u_x + \sqrt{\xi}\phi)\Big|_{t=\alpha x}^{t=T-\alpha x} - \frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}\int_{\alpha x}^{T-\alpha x} u_t\phi_t\,\mathrm{d}t \\ &- \frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x} (\varrho u_t^2(x,t) + \kappa(\sqrt{\mu}u_x(x,t) + \sqrt{\xi}\phi(0,t))^2).\end{aligned} \tag{3.8}$$

The first term on the right-hand side of (3.8) can be increased using Young's inequality as follows:

$$\frac{\varrho}{\sqrt{\mu}}u_t(\sqrt{\mu}u_x+\sqrt{\xi}\phi)\leqslant\frac{\varrho\beta}{2\sqrt{\mu}}u_t^2+\frac{\varrho}{2\sqrt{\mu}\beta}(\sqrt{\mu}u_x+\sqrt{\xi}\phi)^2 \tag{3.9}$$
$$=\frac{\beta}{2\sqrt{\mu}}\varrho u_t^2+\frac{\varrho}{2\sqrt{\mu}\beta\kappa}\kappa(\sqrt{\mu}u_x+\sqrt{\xi}\phi)^2$$

If we choose $\beta=\sqrt{\varrho/\kappa}$, we obtain

$$\frac{\varrho}{\sqrt{\mu}}u_t(\sqrt{\mu}u_x+\sqrt{\xi}\phi)\leqslant\frac{1}{2}\sqrt{\frac{\varrho}{\mu\kappa}}(\varrho u_t^2+\kappa(\sqrt{\mu}u_x+\sqrt{\xi}\phi)^2). \tag{3.10}$$

Putting $\alpha=\max\{\sqrt{\varrho/(\mu\kappa)},\sqrt{J/\delta}\}$, we have

$$\frac{\varrho}{\sqrt{\mu}}u_t(\sqrt{\mu}u_x+\sqrt{\xi}\phi)\leqslant\frac{\alpha}{2}(\varrho u_t^2+\kappa(\sqrt{\mu}u_x+\sqrt{\xi}\phi)^2).$$

Then

(3.11)
$$\frac{\varrho}{\sqrt{\mu}}u_t(\sqrt{\mu}u_x+\sqrt{\xi}\phi)\Big|_{t=\alpha x}^{t=T-\alpha x}\leqslant\frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x}[\varrho u_t^2(x,t)+\kappa(\sqrt{\mu}u_x(x,t)+\sqrt{\xi}\phi(x,t))^2],$$

from the inequality above and from (3.8), we obtain

$$\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}x}(x)\leqslant-\frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}\int_{\alpha x}^{T-\alpha x}u_t\phi_t\,\mathrm{d}t. \tag{3.12}$$

On the other hand,

$$\frac{\mathrm{d}\mathcal{J}_2}{\mathrm{d}x}(x)=\int_{\alpha x}^{T-\alpha x}[J\phi_t\phi_{tx}+\delta\phi_x\phi_{xx}]\,\mathrm{d}t-\frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x}[J\phi_t^2(x,t)+\delta\phi_x^2(x,t)]\,\mathrm{d}t. \tag{3.13}$$

We note that

$$\int_{\alpha x}^{T-\alpha x}J\phi_t\phi_{tx}\,\mathrm{d}t=J\phi_t\phi_x\Big|_{t=\alpha x}^{t=T-\alpha x}-\int_{\alpha x}^{T-\alpha x}J\phi_{tt}\phi_x\,\mathrm{d}t. \tag{3.14}$$

By multiplying the second equation in (2.1) by ϕ_x and integrating on $[\alpha x, T-\alpha x]$, we obtain

$$\int_{\alpha x}^{T-\alpha x}J\phi_{tt}\phi_x\,\mathrm{d}t=\int_{\alpha x}^{T-\alpha x}[\delta\phi_{xx}\phi_x-\kappa\sqrt{\xi}(\sqrt{\mu}u_x+\sqrt{\xi}\phi)\phi_x]\,\mathrm{d}t \tag{3.15}$$

and combining (3.13), (3.14), and (3.15), we have

$$\frac{\mathrm{d}\mathcal{J}_2}{\mathrm{d}x}(x) = J\phi_t\phi_x\Big|_{t=\alpha x}^{t=T-\alpha x} + \kappa\sqrt{\xi}\int_{\alpha x}^{T-\alpha x}(\sqrt{\mu}u_x + \sqrt{\xi}\phi)\phi_x\,\mathrm{d}t \\ - \frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x}[J\phi_t^2(x,t) + \delta\phi_x^2(x,t)]\,\mathrm{d}t. \tag{3.16}$$

Making a calculation similar to the one made for $\mathcal{J}_1$, we obtain

$$J\phi_t\phi_x\Big|_{\alpha x}^{T-\alpha x} \leqslant \frac{\alpha}{2}\sum_{t=\alpha x}^{T-\alpha x}(J\phi_t^2(x,t) + \delta\phi_x^2(x,T)), \tag{3.17}$$

where $\alpha = \max\{\sqrt{\varrho/(\mu\kappa)}, \sqrt{J/\delta}\}$. By replacing (3.17) in (3.16), we arrive at

$$\frac{\mathrm{d}\mathcal{J}_2}{\mathrm{d}x}(x) \leqslant \kappa\sqrt{\xi}\int_{\alpha x}^{T-\alpha x}(\sqrt{\mu}u_x + \sqrt{\xi}\phi)\phi_x\,\mathrm{d}t. \tag{3.18}$$

From (3.12) and (3.18), we get

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}x}(x) \leqslant -\frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}\int_{\alpha x}^{T-\alpha x}u_t\phi_t\,\mathrm{d}t + \kappa\sqrt{\xi}\int_{\alpha x}^{T-\alpha x}(\sqrt{\mu}u_x + \sqrt{\xi}\phi)\phi_x\,\mathrm{d}t. \tag{3.19}$$

By Young's inequality, we have

$$\frac{\varrho\sqrt{\xi}}{\sqrt{\mu}}u_t\phi_t \leqslant \sqrt{\frac{\varrho\xi}{\mu J}}\cdot\frac{1}{2}(\varrho u_t^2 + J\phi_t^2),$$
$$\kappa\sqrt{\xi}(\sqrt{\mu}u_x + \sqrt{\xi}\phi)\phi_x \leqslant \sqrt{\frac{\kappa\xi}{\delta}}\cdot\frac{1}{2}(\delta\phi_x^2 + \kappa(\sqrt{\mu}u_x + \sqrt{\xi}\phi)^2).$$

By setting

$$c = \max\left\{\sqrt{\frac{\varrho\xi}{\mu J}}, \sqrt{\frac{\kappa\xi}{\delta}}\right\},$$

we obtain

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}x}(x) \leqslant \frac{c}{2}\int_{\alpha x}^{T-\alpha x}[\varrho|u_t|^2 + J|\phi_t|^2 + \delta|\phi_x|^2 + \kappa|\sqrt{\mu}u_x + \sqrt{\xi}\phi|^2]\,\mathrm{d}t = c\mathcal{J}(x).$$

Then, we conclude that

$$(T - 2\alpha L)E(0) \leqslant \frac{c}{2}\int_0^T[\kappa\mu u_x^2(0,t) + \delta\phi_x^2(0,t)]\,\mathrm{d}t,$$

and the result follows for $T > 2\alpha L$. □

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