Czechoslovak Mathematical Journal, 70 (145) (2020), 743–755

A VARIATION OF THOMPSON'S CONJECTURE FOR THE SYMMETRIC GROUPS

Mahdi Abedei, Ali Iranmanesh, Farrokh Shirjian, Tehran

Received November 9, 2018. Published online January 29, 2020.

Abstract. Let G be a finite group and let $N(G)$ denote the set of conjugacy class sizes of G. Thompson's conjecture states that if G is a centerless group and S is a non-abelian simple group satisfying $N(G)=N(S)$, then $G\cong S$. In this paper, we investigate a variation of this conjecture for some symmetric groups under a weaker assumption. In particular, it is shown that $G\cong \mathrm{Sym}(p+1)$ if and only if $|G|=(p+1)!$ and G has a special conjugacy class of size $(p+1)!/p$, where $p>5$ is a prime number. Consequently, if G is a centerless group with $N(G)=N(\mathrm{Sym}(p+1))$, then $G\cong \mathrm{Sym}(p+1)$.

Keywords: Thompson's conjecture; conjugacy class size; symmetric groups; prime graph

MSC 2020: 20D08, 20D60

1. Introduction and main results

All groups considered in this paper are finite. Let G be a group and let $N(G)$ denote the set of all conjugacy class sizes of G. A fundamental question in group theory is how the structure of G reflects and is reflected by $N(G)$.

It is clear that the set $N(G)$ does not determine the structure of G up to isomorphism. However, the situation is different when it comes to non-abelian simple groups. Indeed, Thompson's conjecture, see [16], Question 12.38, proposes that non-abelian simple groups are characterized by the set of their conjugacy class sizes.

Thompson's conjecture. If G is a centerless group and S is a non-abelian simple group such that $N(G)=N(S)$, then $G\cong S$.

The conjecture has been confirmed for many families of simple groups so far, see for instance [1], [6], [7], [9], [10], [18]. Inspired by these results, there has been recent growing interest in investigating some variations of Thompson's conjecture under

DOI: 10.21136/CMJ.2020.0501-18

a weaker condition. For instance, Li in [15] characterized simple sporadic groups and simple K_3-groups by using the group order and one or two special conjugacy class sizes of the simple groups. Also, Chen et al. in [11] verified Thompson's conjecture for simple K_4-groups by using the group order and a few conjugacy class sizes. Recall that a finite simple group is called a simple K_n-group if its order is divisible by exactly n distinct primes.

Asboei and Mohammadyari used the group order and just one special conjugacy class size to verify Thompson's conjecture for alternating simple groups of degrees p, $p+1$ and $p+2$, where p is a prime number, see [3] and [4]. They also extended their result to symmetric groups of prime degrees, see [5]. Furthermore, Asboei et al. in [2] recently showed that simple symplectic groups $\mathrm{PSp}_{2n}(2)$ are determined uniquely up to isomorphism by their order and one conjugacy class of size $|\mathrm{PSp}_{2n}(2)|/(2^n+1)$.

In this paper, we characterize the structure of finite groups with the same order and one special conjugacy class size as the symmetric group $\mathrm{Sym}(p+1)$, where $p>5$ is a prime number. The following theorem is the main result of this paper.

Main Theorem. *Let G be a group. Then $G \cong \mathrm{Sym}(p+1)$ if and only if $|G|=(p+1)!$ and G has a special conjugacy class size of $(p+1)!/p$, where $p>5$ is a prime number.*

As a consequence of the Main Theorem, we prove an extension of Thompson's conjecture for the almost simple groups under study.

Corollary. *Let G be a centerless group satisfying $N(G)=N(\mathrm{Sym}(p+1))$, where $p>5$ is a prime number. Then $G \cong \mathrm{Sym}(p+1)$.*

In the sequel, we describe some notations and concepts we use to prove our main results. We write $\pi(G)$ for the set of all prime divisors of the order of group G. The prime graph of group G, denoted by $\Gamma(G)$, is a simple undirected graph whose vertex set is $\pi(G)$, and two vertices p and p' are adjacent if and only if G contains an element of order pp'. Let $t(G)$ denote the number of connected components of $\Gamma(G)$ and $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ denote the connected components of $\Gamma(G)$. Also, let $T(G)$ be the set of connected components of $\Gamma(G)$, i.e. $T(G)=\{\pi_i(G)\colon 1 \leqslant i \leqslant t(G)\}$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1$. Note that we can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_r$, where $\pi(m_i)=\pi_i$ for each i. The numbers m_i are then called the order components of G. We will frequently use the list of finite non-abelian simple groups with disconnected prime graphs which is available in [13].

2. Preliminaries

The aim of this section is to collect some facts and results that will be applied in the next section of the paper.

Lemma 2.1 ([8], Theorem 1). *If G is a Frobenius group of even order with the Frobenius kernel K and the Frobenius complement H, then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(N)\}$.*

Lemma 2.2 ([12], Theorem 10.3.1). *Let G be a Frobenius group with the Frobenius kernel H and the Frobenius complement K. Then $|K| \mid |H| - 1$.*

Recall that a 2-Frobenius group is a group G which has proper normal subgroups K and L such that L is a Frobenius group with kernel K and G/K is a Frobenius group with kernel L/K.

Lemma 2.3 ([8], Theorem 2). *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$. Moreover, G/K and K/H are cyclic groups satisfying that $|G/K| \mid |\mathrm{Aut}(K/H)|$, $(|G/K|, |K/H|) = 1$ and $|G/K| < |K/H|$. In particular, G is solvable.*

Lemma 2.4 ([10], Lemma 8). *Let G be a finite group with $t(G) \geqslant 2$ and N a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some of the order components of G but not a π_i-number, then $m_1 m_2 \ldots m_r \mid |N| - 1$.*

The following lemma determines the structure of finite groups with disconnected prime graphs.

Lemma 2.5 ([19], Theorem A). *Suppose that G has more than one prime graph component. Then one of the following holds:*
1. *G is a Frobenius group or a 2-Frobenius group;*
2. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H a non-abelian simple group and $|G/K|$ divides the order of the outer automorphism group of K/H and H is a nilpotent group, and $K/H \leqslant G/H \leqslant \mathrm{Aut}(K/H)$. Besides, $\pi_i \in T(K/H)$ for $i \geqslant 2$.*

Lemma 2.6 ([11], Lemma 2.12). *Let G be a group, N a normal subgroup of G with order p^n, $n \geqslant 1$. If $(r, |\mathrm{Aut}(N)|) = 1$, where $r \in \pi(G)$, then G has an element of order pr. Furthermore, there exists an edge connecting r and p in the prime graph of G.*

Lemma 2.7. *Let p be a prime number and n be a natural number. Then the following holds:*

(1) *If $p \geqslant 6$, then there exists a prime r such that $(p-1)/2 < r < p-1$.*

(2) *If $p \geqslant 13$, then there exist two prime numbers r_1, r_2 such that $(p-1)/2 < r_1 < r_2 < p-1$.*

(3) *If $p \geqslant 19$, then there exist three prime numbers r_1, r_2, r_3 such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$.*

(4) *If $p \geqslant 46$, then there exist four prime numbers r_1, r_2, r_3, r_4 such that $(p-1)/2 < r_1 < r_2 < r_3 < r_4 < p-1$.*

(5) *If $n \geqslant 46$, then there exist two prime numbers r_1, r_2 such that $3n/4 < r_1 < r_2 \leqslant n$.*

Proof. The proof of (1)–(4) goes along exactly the same lines as the proof of Lemma 1 in [14]. Part (5) also follows by the same argument as in [17], page 83. □

3. Proof of the main theorem

It is obvious that if $G \cong \mathrm{Sym}(p+1)$, then $|G| = |\mathrm{Sym}(p+1)|$ and G contains a conjugacy class of size $(p+1)!/p$. Therefore it suffices to prove the sufficiency side of the Main Theorem.

Under the assumption of the Main Theorem, there exists an element x of order p in G such that $\langle x\rangle = C_G(x)$ and $C_G(x)$ is a Sylow p-subgroup of G. Then it follows from the Sylow theorem that $\{p\}$ is a prime graph component of G and $t(G) \geqslant 2$. Furthermore, p is the maximal prime divisor of $|G|$ and an odd-order component of G. In continue, we need to prove the following lemmas.

Lemma 3.1. *With the assumptions of the Main Theorem we have:*

(a) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and $\pi_i \subset \pi(K)$ for $i > 1$. Furthermore, K/H is a non-abelian simple group such that $|G/K|$ divides $|\mathrm{Out}(K/H)|$, H is a nilpotent group and $K/H \leqslant G/H \leqslant \mathrm{Aut}(K/H)$. Besides, $\{p\} \in T(K/H)$.*

(b) $|G/K| \mid p-1$.

(c) *If r is a prime such that $(p-1)/2 < r < p-1$, then $r \mid |K/H|$.*

Proof. (a) First we show that G is not a Frobenius group. By the way of contradiction assume that G is a Frobenius group with kernel H and complement K, and $\{p\}$ is a prime graph component of G. Then, by Lemma 2.1, $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$. If $p \in \pi(H)$, then $|H| = p$ and $|K| = (p+1)(p-1)!$. However, this is impossible since $|K| \mid |H| - 1$ by Lemma 2.2. If $p \in \pi(K)$, then $|K| = p$ and K is a Sylow p-subgroup of G. We then deduce from Lemma 2.7

that there exist a prime r such that $(p-1)/2 < r < p-1$ and $r \in \pi(H)$. Let M be an r-subgroup of H. Then $M \rtimes K$ is a Frobenius group with kernel M and complement K. This in particular implies that $p \mid r-1$, a contradiction.

Next we show that G is not a 2-Frobenius group. On the contrary, assume that G is a 2-Frobenius group. Then, by Lemma 2.3, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \{p\}$, $|G/K| \mid p-1$ and G is solvable. It follows from Lemma 2.7 that there exists a prime r such that $(p-1)/2 < r < p-1$ and $r \in \pi(H)$. Let M be an r-subgroup of H which is normal in G. By Lemma 2.6,

$$(p, |\mathrm{Aut}(M)|) = (p, r-1) = 1,$$

and hence G has an element of order pr. This contradicts the fact that $\{p\}$ is a prime graph component of G. We have thus shown that G is neither a Frobenius nor a 2-Frobenius group. Therefore, (a) follows from Lemma 2.5.

(b) Let P be a Sylow p-subgroup of K. Then $C_G(P) \leqslant K \cap N_G(P)$ and by the Frattini argument, $G = N_G(P)K$. Therefore

$$G/K = N_G(P)K/K \cong N_G(P)/K \cap N_G(P),$$

and also

$$|N_G(P)/C_G(P)| \mid |\mathrm{Aut}(P)| = p-1.$$

Thus $|G/K| \mid p-1$.

(c) By the way of contradiction assume that $r \nmid |K/H|$. Then $r \in \pi_1$ or $r \in \pi_i$ for all $i > 1$. If $r \in \pi_i$ for all $i > 1$, we have $\pi_i \in T(K/H)$ for all $i > 1$, and so $r \mid |K/H|$, a contradiction. If $r \in \pi_1$, then $r \nmid |G/K|$ by part (a). Therefore $r \mid |H|$. Let N be a r-subgroup of H which is a normal subgroup of G. Then Lemma 2.6 implies that $(p, |\mathrm{Aut}(N)|) = 1$. This is impossible since $\{p\}$ is a prime graph component of G. Therefore $r \mid |K/H|$. □

The list of order components of finite simple groups with disconnected prime graphs is available in Tables 1–3 of [13]. In the sequel, we use the classification of finite simple groups to eliminate all the possibilities of K/H except for $\mathrm{Alt}(p+1)$.

Lemma 3.2. *K/H is not isomorphic to a sporadic simple group or the Tits group.*

Proof. If $K/H \cong M_{12}$, then $p = 11$. By Lemma 2.7, there exists a prime r such that $(p-1)/2 < r < p-1$, and so $r = 7$. Then Lemma 3.1(c) implies that $7 \mid |K/H| = 2^6{\cdot}3^3{\cdot}5{\cdot}11$, a contradiction.

If $K/H \cong J_2$, then $p = 7$ and 5^2 divides $|J_2|$. Therefore 5^2 divides $|G| = 2^7{\cdot}3^2{\cdot}5{\cdot}7$, a contradiction.

If $K/H \cong Co_1$, then $p = 23$. By Lemma 2.7, there exist two primes r_1, r_2 such that $(p-1)/2 < r_1 < r_2 < p-1$. Therefore $r_1, r_2 \in \{19, 17, 13\}$ and they must divide $|Co_1| = 2^{21}{\cdot}3^9{\cdot}5^4{\cdot}7^2{\cdot}11{\cdot}13{\cdot}23$ by Lemma 3.1(c), which is a contradiction.

If $K/H \cong HS$, then $p = 11$ or $p = 7$. If $p = 11$, then 5^3 divides $|K/H|$ and also divides $|G| = 2^{10}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11$, a contradiction. If $p = 7$, then 11 divides $|G| = 2^7{\cdot}3^2{\cdot}5{\cdot}7$, a contradiction.

If $K/H \cong M_{22}$, then $p \in \{7, 11\}$ and $|K/H| = 2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$. If $p = 11$, then $|G| = 2^{10}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11$. In this case,

$$|G/K| \mid |\mathrm{Out}(K/H)| = 2.$$

This particularly implies that $5 \in \pi(H)$, which contradicts Lemma 2.4. If $p = 7$, then $11 \in \pi(G)$, which is impossible since p is the maximum prime divisor of $|G|$.

The remaining simple sporadic groups can also be eliminated by similar arguments. □

Lemma 3.3. *K/H is not isomorphic to a simple group of Lie type.*

Proof. The list of simple Lie-type groups with prime component has been given in Table 1. Using this list, we consider different possibilities of K/H among simple groups of Lie type and work towards a contradiction.

S	Condition	S	Condition
$A_{p'-1}(q)$	$(p', q) \neq (3, 2), (3, 4)$	$A_{p'}(q)$	$q-1 \mid p'+1$
${}^2A_{p'-1}(q)$		${}^2A_{p'}(q)$	$q+1 \mid p'+1$, $(p', q) \neq (3, 3), (5, 2)$
${}^2A_3(2)$		$B_n(q)$	$n = 2^m \geqslant 4$, q odd
$B_{p'}(3)$		$C_n(q)$	$n = 2^m \geqslant 2$
$C_{p'}(q)$	$q = 2, 3$	$D_{p'}(q)$	$p' \geqslant 5$, $q = 2, 3, 5$
$D_{p'+1}(q)$	$q = 2, 3$	${}^2D_n(q)$	$n = 2^m \geqslant 4$
${}^2D_n(2)$	$n = 2^m + 1$, $m \geqslant 2$	${}^2D_{p'}(3)$	$5 \leqslant p' \neq 2^m + 1$
${}^2D_n(3)$	$n = 2^m + 1 \neq p'$, $m \geqslant 2$	$G_2(q)$	$2 < q \equiv \varepsilon \pmod 3$, $\varepsilon = \pm 1$
${}^3D_4(q)$		$F_4(q)$	q odd
${}^2F_4(2)'$		$E_6(q)$	
${}^2E_6(q)$	$q > 2$	$A_1(q)$	$3 \leqslant q \equiv \varepsilon \pmod 4$, $\varepsilon = \pm$
$A_1(q)$	$2 < q$ even	${}^2A_5(2)$	
${}^2D_{p'}(3)$	$p' = 2^m + 1$, $m \geqslant 1$	$G_2(q)$	$3 \mid q$
${}^2G_2(q)$	$q = 3^{2m+1} > 3$	$F_4(q)$	$2 < q$ even
${}^2F_2(q)$	$q = 2^{2m+1} > 2$	$E_7(q)$	$q = 2, 3$
$A_2(4)$		${}^2B_2(q)$	$q = 2^{2m+1} > 2$
${}^2E_6(2)$		$E_8(q)$	

Table 1. Simple groups of Lie type with prime odd order component.

▷ K/H is isomorphic to $A_{p'-1}(q)$, where $(p',q) \neq (3,2),(3,4)$. Then

$$p = \frac{q^{p'}-1}{(q-1)(p',q-1)}.$$

First, assume that $p \geqslant 19$. By Lemma 2.7, there exist three primes r_1, r_2, r_3 such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$. Thus, for any $1 \leqslant i \leqslant 3$, r_i divides $|K/H| = q^{p'(p'-1)/2} \prod_{i=1}^{p'-1} p(q^i-1)$. Therefore

$$r_i > \frac{p-1}{2} > q^{p'-3}-1,$$

and also

$$r_i^3 > (q^{p'-2}-1)(q^{p'-1}-1).$$

By Lemma 3.1 (c), we have $r_1 \cdot r_2 \cdot r_3 \mid |K/H|$, and so $r_1 \cdot r_2 \cdot r_3 \mid (q^{p'-2}-1)(q^{p'-1}-1)$, a contradiction.

Next, assume that $p = 17$ or $p = 11$. Then there are no p' and q satisfying the equation

$$p = \frac{q^{p'}-1}{(q-1)(p',q-1)}.$$

If $p = 13$, then $p' = 3$ and $q = 3$. By Lemma 2.7, $11 \in \pi(A_2(3))$, which is a contradiction. If $p = 7$, then we have that $q = 2$ or 4, and $p' = 3$. If $q = 2$, then by Lemma 2.7, $5 \in \pi(A_2(2))$, which is impossible. If $q = 4$, then K/H isomorphic to $\mathrm{Alt}(8)$. Now Since $\mathrm{Aut}(\mathrm{Alt}(8)) = \mathrm{Sym}(8)$, we have $\mathrm{Alt}(8) \leqslant G/H \leqslant \mathrm{Sym}(8)$ which in turn implies that $G \cong \mathrm{Sym}(p+1)$ for $p = 7$, as desired.

▷ K/H is isomorphic to $A_{p'}(q)$, where $q-1 \mid p'+1$. Then $p = (q^{p'}-1)/(q-1)$. Let $p \geqslant 19$. By Lemma 2.7, there exist three primes r_1, r_2, r_3 such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$. Then for $1 \leqslant i \leqslant 3$ we have that r_i divides

$$|K/H| = q^{p'(p'+1)/2}(q^{p'+1}-1) \prod_{i=1}^{p'-1} p(q^i-1).$$

Therefore $r_i > (p-1)/2 > q^{p'-1}-1$ and also $r_i^3 > q^{p'+1}-1$. By Lemma 3.1 (c), we have $r_1 \cdot r_2 \cdot r_3 \mid |K/H|$, and so $r_1 \cdot r_2 \cdot r_3 \mid q^{p'+1}-1$, which is a contradiction.

If $p = 17$ or $p = 11$, then there are no p' and q satisfying the equation $p = (q^{p'}-1)/(q-1)$.

If $p = 13$, then by Lemma 2.7, there exist a prime r such that $(p-1)/2 < r < p-1$. Thus $r = 7$. By Lemma 3.1 (c), $7 \mid |K/H| = 2^8 \cdot 3^6 \cdot 5 \cdot 13$, which is a contradiction. If $p = 7$, then $p' = 3$ and $q = 2$. Then K/H isomorphic to $\mathrm{Alt}(8)$.

Now since $\mathrm{Aut}(\mathrm{Alt}(8)) = \mathrm{Sym}(8)$, we have $\mathrm{Alt}(8) \leqslant G/H \leqslant \mathrm{Sym}(8)$, which in turn implies that $G \cong \mathrm{Sym}(p+1)$ for $p = 7$, as desired.

- ▷ K/H is isomorphic to $B_n(q)$, where $n = 2^m \geqslant 4$ and q is odd. Then $p = (q^n+1)/2$. First, assume that $q = 3$ and $m = 2$. Then $p = 41$. By Lemma 2.7, there exist a prime r such that $(p-1)/2 < r < p-1$, and so $r = 23$. Now Lemma 3.1 (c) implies that $23 \mid |K/H| = 2^{14} \cdot 3^{16} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$, which is impossible. Next, assume that $q > 3$ and $n > 8$. Therefore $p \geqslant 70$. By Lemma 2.7, there exist four primes r_1, r_2, r_3, r_4 such that $(p-1)/2 < r_1 < r_2 < r_3 < r_4 < p-1$. Then for any $1 \leqslant i \leqslant 4$ we have that r_i divides

$$|K/H| = q^{n^2}(q^n-1)\prod_{i=1}^{n-1} p(q^{2i}-1).$$

Therefore $r_i > p - 1/2 > q^{n-1}-1$ and also $r_i^4 > (q^{n-1}+1)(q^n-1)$. Lemma 3.1 (c) yields that $r_1 \cdot r_2 \cdot r_3 \cdot r_4 \mid |K/H|$, which implies $r_1 \cdot r_2 \cdot r_3 \cdot r_4 \mid (q^{n-1}+1)(q^n-1)$, a contradiction.

- ▷ K/H is isomorphic to $C_n(q)$, where $n = 2^m \geqslant 2$. Then $p = (q^n+1)/(2, q-1)$. First, assume that $q = 2$, $m = 2$. Then $p = 17$. By Lemma 2.7, there exists a prime r such that $(p-1)/2 < r < p-1$, and so $r = 13$. Lemma 3.1(c) then implies that $13 \mid |K/H| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction. Next, assume that $q = 2$ and $n > 4$. Then $p \geqslant 19$ and $p = (2^n+1)/(2,1)$. By Lemma 2.7, there exist three primes r_1, r_2, r_3 such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$. Then for any $1 \leqslant i \leqslant 3$, r_i divides

$$|K/H| = 2^{n^2}(2^n-1)\prod_{i=1}^{n-1} p(2^{2i}-1).$$

Therefore $r_i > (p-1)/2 > 2^{n-1}-1$ and also $r_i^3 > (2^n-1)(2^{n-1}+1)$. Now Lemma 3.1 (c) yields that $r_1 \cdot r_2 \cdot r_3 \mid |K/H|$, which implies $r_1 \cdot r_2 \cdot r_3 \mid (2^{n-1}+1) \times (2^n-1)$, again a contradiction.

If $q = 3, 4$ and $n > 4$, we get a contradiction using similar arguments as before. So we suppose that $q > 4$ and $n > 4$. Then $p \geqslant 46$ and $p = (q^n+1)/(2, q-1)$. By Lemma 2.7, there exist four primes r_1, r_2, r_3, r_4 such that $(p-1)/2 < r_1 < r_2 < r_3 < r_4 < p-1$. Then for any $1 \leqslant i \leqslant 4$ we have that r_i divides

$$|K/H| = q^{n^2}(q^n-1)\prod_{i=1}^{n-1} p(q^{2i}-1).$$

Therefore $r_i > (p-1)/2 > q^{n-1}-1$ and also $r_i^4 > (q^n-1)(q^{n-1}+1)$. By Lemma 3.1 (c), we have $r_1 \cdot r_2 \cdot r_3 \cdot r_4 \mid |K/H|$. This implies that $r_1 \cdot r_2 \cdot r_3 \cdot r_4 \mid (q^{n-1}+1)(q^n-1)$, which is again a contradiction.

▷ K/H is isomorphic to $A_1(q)$, where $3 < q \equiv \varepsilon \pmod 4$. Then $p = (q+\varepsilon)/2$ or $p = q$. Let $\varepsilon = 1$. Then $p = (q+1)/2$ or $p = q$. If $p = q$, then $|K/H| = p(p^2-1)/2$. First assume that $p \geqslant 19$. Then, by Lemma 2.7, there exist three primes r_1, r_2, r_3 such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$. Now Lemma 3.1(c) implies that $r_1{\cdot}r_2{\cdot}r_3 \mid |K/H|$, which in turn yields that $r_1{\cdot}r_2{\cdot}r_3 \mid (p-1)(p+1)/2$. This is while $r_1{\cdot}r_2{\cdot}r_3 > ((p-1)/2)^3 > (p-1)(p+1)/2$, a contradiction. Next assume that $p = 17$. By Lemma 2.7, there exist a prime r such that $(p-1)/2 < r < p-1$. Thus $r = 13$. Now using Lemma 3.1 (c), we deduce that $13 \mid |K/H| = 2^4{\cdot}3^2{\cdot}17$, which is impossible. If $p = 13$, then Lemma 2.7 implies that there exists a prime r such that $(p-1)/2 < r < p-1$, and hence $r = 11$. Therefore $11 \mid |K/H| = 2^2{\cdot}3{\cdot}7{\cdot}13$ by Lemma 3.1 (c), a contradiction. If $p = 11$, then $r = 7$ and we have that $7 \mid |K/H| = 2^2{\cdot}3{\cdot}5{\cdot}11$, a contradiction. Finally, when $p = 7$, one gets a contradiction by using a similar argument.

If $p = (q+1)/2$, then $|K/H| = p(2p-1)(2p-2)$. We proceed as before to reach a contradiction in each case. First let $p \geqslant 19$. By Lemma 2.7, there exist three primes r_1, r_2, r_3 such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$. Since $r_1{\cdot}r_2{\cdot}r_3 \mid |K/H|$, we must have $r_1{\cdot}r_2{\cdot}r_3 \mid (p-1)(2p-1)$, which violates the inequality $r_1{\cdot}r_2{\cdot}r_3 > ((p-1)/2)^3 > (p-1)(2p-1)$. Next assume that $p = 17$. Using Lemma 2.7 again, we deduce that there exists a prime r such that $(p-1)/2 < r < p-1$, and hence $r = 13$. Lemma 3.1(c) then implies that $13 \mid |K/H| = 2^5{\cdot}3{\cdot}11{\cdot}17$, a contradiction. If $p = 13$, then, by Lemma 2.7, there exists a prime $r = 11$ such that $11 \mid |K/H| = 2^3{\cdot}3{\cdot}5^2{\cdot}13$, which is impossible. Finally, if $p = 7$, we get a contradiction using a similar argument.

The case in which $\varepsilon = -1$ can be handled by using the same arguments as in the case $\varepsilon = 1$.

▷ K/H is isomorphic to $A_1(q)$, where $4 < q$ is even. Then $p = q-1$ or $p = q+1$. If $p = q-1$, then $|K/H| = p(p+1)(p+2)$. Let $p \geqslant 19$. By Lemma 2.7, there exist three primes r_1, r_2, r_3 such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$. Since $r_1{\cdot}r_2{\cdot}r_3 \mid |K/H|$, we get $r_1{\cdot}r_2{\cdot}r_3 \mid (p+1)(p+2)$. This is while $r_1{\cdot}r_2{\cdot}r_3 > ((p-1)/2)^3 > (p+1)(p+2)$, a contradiction. If $p = 17$, then Lemma 2.7 implies that there exists a prime r such that $(p-1)/2 < r < p-1$, and so $r = 13$. Lemma 3.1 (c) then yields that $13 \mid |K/H| = 2{\cdot}3^2{\cdot}17{\cdot}19$, which is impossible. If $p = 13$, then, by Lemma 2.7, there exists a prime r such that $(p-1)/2 < r < p-1$, and hence $r = 11$. By Lemma 3.1 (c), $11 \mid |K/H| = 2{\cdot}3{\cdot}5{\cdot}7{\cdot}13$, which is a contradiction. If $p = 11$, then $r = 7$ and we have $7 \mid |K/H| = 2^2{\cdot}3{\cdot}11{\cdot}13$, a contradiction. If $p = 7$, then $r = 5$ and $5 \mid |K/H| = 2^3{\cdot}3^2{\cdot}7$, a contradiction.

Finally, if $p = q+1$, then $|K/H| = p(p-1)(p-2)$. This case can be handled similarly as before.

- K/H is isomorphic to $E_6(q)$. Then $p = (q^6+q^3+1)/(3, q-1)$ and also $p \geqslant 19$. By Lemma 2.7, there exist three primes r_1, r_2, r_3, such that $(p-1)/2 < r_1 < r_2 < r_3 < p-1$. Then for any $1 \leqslant i \leqslant 3$ we have that r_i divides $|K/H| = q^{36}(q^3-1)^3(q^3+1)^2(q^6+1)(q^2-1)^2(q^2+1)(q^4+1)(q^5-1)$. If $q \neq 4$, then $r_i > (p-1)/2 > q^5$ and also $r_i^3 > (q^6+1)$. By Lemma 3.1 (c), we get $r_1{\cdot}r_2{\cdot}r_3 \mid |K/H|$. Therefore $r_1{\cdot}r_2{\cdot}r_3 \mid q^6+1$, a contradiction.

  If $q = 4$, then $p = 1387 = 19 \cdot 73$ and $|K/H| = 2^{72}{\cdot}3^9{\cdot}5^4{\cdot}7^3{\cdot}11{\cdot}13^2{\cdot}17^2{\cdot}19{\cdot}31{\cdot}73{\cdot}241{\cdot}257$. By Lemma 2.7, we get $r = 809$, which is again a contradiction.
- K/H is isomorphic to $G_2(q)$, where $q \equiv 0 \pmod 3$. Then $p = q^2-q+1$ or $p = q^2+q+1$.

  First assume that $p = q^2+q+1$. If $q = 3$, then $p = 13$. By Lemma 2.7 there exists a prime r such that $(p-1)/2 < r < p-1$, and so $r = 11$. However, this violates Lemma 3.1 (c) by which we have $11 \mid |K/H| = 2^6{\cdot}3^6{\cdot}7{\cdot}13$. If $q > 3$, then $p \geqslant 46$. Using Lemma 2.7 again, we obtain that there exist four primes r_1, r_2, r_3, r_4 such that $(p-1)/2 < r_1 < r_2 < r_3 < r_4 < p-1$. Then for any $1 \leqslant i \leqslant 4$ we have that r_i divides $|K/H| = q^6(q^2-1)^2(q^2-q+1)(q^2+q+1)$. Therefore $r_i^2 > q^2-q+1$ and $r_i^2 > q^2-1$. By Lemma 3.1 (c), we have $r_1 \cdot r_2 \cdot r_3 \cdot r_4 \mid |K/H|$. Therefore we must have

$$r_1 \cdot r_2 \cdot r_3 \cdot r_4 \mid (q^2-1)(q^2-q+1),$$

  since $\gcd(r_i, q^6) = 1 = \gcd(r_i, q^2+q+1) = \gcd(r_i, p) = 1$. However, this violates the former inequalities.

  Next assume that $p = q^2-q+1$. If $q = 3$, then $p = 7$. By Lemma 2.7, there exists a prime r such that $(p-1)/2 < r < p-1$. Then $r = 5$ and by Lemma 3.1 (c), we have that $5 \mid |K/H| = 2^6 \cdot 3^6 \cdot 7 \cdot 13$, a contradiction. Let $q > 3$. Then $p \geqslant 46$. By Lemma 2.7 there exist four primes r_1, r_2, r_3, r_4 such that $(p-1)/2 < r_1 < r_2 < r_3 < r_4 < p-1$. Therefore $r_i > q+1$ and $r_i^4 > q^2+q+1$. For $1 \leqslant i \leqslant 4$ we have that r_i divides $|K/H| = q^6(q^2-1)^2(q^2-q+1)(q^2+q+1)$. By Lemma 3.1(c), we get that $r_1{\cdot}r_2{\cdot}r_3{\cdot}r_4 \mid |K/H|$. This implies that $r_1{\cdot}r_2{\cdot}r_3{\cdot}r_4 \mid (q^2+q+1)$ since $\gcd(r_i, q^6) = \gcd(r_i, q^2-q+1) = \gcd(r_i, p) = 1$ and $r_i > q+1$, which is again a contradiction.
- K/H is isomorphic to ${}^2A_3(2)$. Then $p = 5$, which violates the assumption $p > 5$.
- $K/H \cong {}^2F_4(2)$, then $p = 13$. By Lemma 2.7, there exists a prime r such that $(p-1)/2 < r < p-1$, which in turn yields $r = 11$. Now Lemma 3.1 (c) implies that $r \mid |K/H| = 2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$, which is a contradiction.
- K/H is isomorphic to ${}^2B_2(q)$, where $2 < q = 2^{2m+1}$. Then $p = q - \sqrt{2q}+1$, $p = q+\sqrt{2q}+1$ or $p = q-1$.

  First assume $m = 1$ and $q = 8$. Then $p = 13$ or $p = 7$. If $p = 13$, then Lemma 2.7 implies the existence of a prime r such that $(p-1)/2 < r < p-1$. So

$r = 11$. Using Lemma 3.1 (c), we have $11 \mid |K/H| = 2^6{\cdot}5{\cdot}7{\cdot}13$, a contradiction. If $p = 7$, then $13 \in \pi(|K/H|)$, which is impossible since $|G| = 2^7{\cdot}3^2{\cdot}5{\cdot}7$.

Next assume that $m = 2$ and $q = 32$. Then $p = 31$ or $p = 41$, and by Lemma 2.7, $r = 23$. Then $23 \mid |K/H| = 2^{10}{\cdot}5^2{\cdot}31{\cdot}41$, a contradiction.

Finally assume that $m > 2$ and $q > 120$. Then $p > 46$. By Lemma 2.7, there exist four primes r_1, r_2, r_3, r_4 such that $(p-1)/2 < r_1 < r_2 < r_3 < r_4 < p-1$. Therefore

$$r_i^2 > \Big(\frac{p-1}{2}\Big)^2 > q + \sqrt{2q} + 1 > q - 1 > q - \sqrt{2q} + 1.$$

Then for any $1 \leqslant i \leqslant 4$ we have that r_i divides

$$|K/H| = q^2(q-1)\big(q + \sqrt{2q} + 1\big)\big(q - \sqrt{2q} + 1\big).$$

Now by the possible values of p, we obtain that $r_1{\cdot}r_2{\cdot}r_3{\cdot}r_4 \mid (q + \sqrt{2q} + 1)(q - \sqrt{2q} + 1)$, $r_1{\cdot}r_2{\cdot}r_3{\cdot}r_4 \mid (q-1)(q + \sqrt{2q} + 1)$ or $r_1{\cdot}r_2{\cdot}r_3{\cdot}r_4 \mid (q-1)(q - \sqrt{2q} + 1)$, which is a contradiction.

- ▷ K/H is isomorphic to $E_8(q)$, where $q \equiv 0, 1, 4 \pmod 5$. Then, by [13], Table 3, K/H has four order components. Therefore we have $p = q^8 - q^4 + 1$, $p = q^8 - q^6 + q^4 - q^2 + 1$, $p = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$ or $p = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$ and also $p > q^7 \geqslant 128$. It follows from Lemma 2.7 (5) that there exist four primes r_1, r_2, r_3 and r_4 such that $(p-1)/2 \leqslant 9(p-1)/16 < r_1 < r_2 \leqslant 3(p-1)/4 < r_3 < r_4 < p-1$. Therefore $r_i{\cdot}r_j > q^9$. Then for any $1 \leqslant i \leqslant 4$ we have that r_i divides $|K/H| = |E_8(q)|$. We outline the argument for the case $p = q^8 - q^4 + 1$. The other cases can be handled similarly. Note that $p > q^7 \geqslant 128$. Then it follows from $r_1{\cdot}r_2{\cdot}r_3{\cdot}r_4 \mid |E_8(q)|$ that

$$r_1{\cdot}r_2{\cdot}r_3{\cdot}r_4 \mid (q^8 - q^6 + q^4 - q^2 + 1)(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1)(q^8 + q^7 - q^5 - q^4 - q^3 + q + 1),$$

  which violates the fact that $r_i r_j > q^9$.

The other simple groups given in Table 1 can be eliminated by using a similar method as before. □

Lemma 3.4. *K/H is isomorphic to the alternating group* $\mathrm{Alt}(p+1)$ *and* $G \cong \mathrm{Sym}(p+1)$.

Proof. By Lemmas 3.2 and 3.3, K/H is isomorphic to a simple alternating group. Using [13], Tables 1–3, we obtain that K/H is isomorphic to A_n, where $6 < n = p', p'+1, p'+2$, p' is a prime, and one of the numbers n or $n-2$ is not a prime.

If $n = p'$, then $p = p'$, $K/H \cong \mathrm{Alt}(p)$ and $\mathrm{Alt}(p) \leqslant G/H \leqslant \mathrm{Sym}(p)$. Therefore $|H| = p + 1$ or $|H| = 2p + 2$. This contradicts Lemma 2.4. If $n = p' + 2$, then $p = p'$ and $K/H \cong \mathrm{Alt}(p + 2)$. Since $\mathrm{Alt}(p + 2) \leqslant G/H \leqslant \mathrm{Sym}(p + 2)$, we again get a contradiction according to $|G|$.

If K/H is isomorphic to $\mathrm{Alt}(p')$, where $6 < p'$ and both of p' and $p' - 2$ are primes, then $p = p'$ or $p = p' - 2$. If $p = p'$, then $K/H \cong \mathrm{Alt}(p)$. If $p = p' - 2$, then $K/H \cong \mathrm{Alt}(p + 2)$. In both cases we get a contradiction arguing as before.

Finally, we get $n = p' + 1$. Then $p = p'$ and $K/H \cong \mathrm{Alt}(p + 1)$. Now since $\mathrm{Aut}(\mathrm{Alt}(p + 1)) = \mathrm{Sym}(p + 1)$, $(p > 5)$, we have $\mathrm{Alt}(p + 1) \leqslant G/H \leqslant \mathrm{Sym}(p + 1)$, which in turn implies that $G \cong \mathrm{Sym}(p+1)$ since $|G| = |\mathrm{Sym}(p+1)|$. This completes the proof of the Main Theorem. $\square$

P r o o f of the Corollary. Note that p is a connected component of $\Gamma(G)$ and $\Gamma(\mathrm{Sym}(p + 1))$, and hence we have $t(G) \geqslant 2$ and $t(\mathrm{Sym}(p + 1)) \geqslant 2$. Therefore, a similar argument as in [9], Lemma 1.4 implies that $|G| = |\mathrm{Sym}(p + 1)|$. Now the assertion follows from the Main Theorem. $\square$

Acknowledgement. The authors are thankful to the anonymous referee for careful reading of the manuscript and for his/her invaluable comments. Parts of this research work were completed while the second author was spending his sabbatical leave at the Department of Mathematics of University of California, Berkeley (UCB). This author expresses his thanks for the warm hospitality and facilities provided by Prof. Fraydoun Rezakhanlou and the Department of Mathematics of UCB.

References

[1] *N. Ahanjideh*: On Thompson's conjecture for some finite simple groups. J. Algebra *344* (2011), 205–228. zbl MR doi
[2] *A. K. Asboei, M. R. Darafsheh, R. Mohammadyari*: The influence of order and conjugacy class length on the structure of finite groups. Hokkaido Math. J. *47* (2018), 25–32. zbl MR doi
[3] *A. K. Asboei, R. Mohammadyari*: Characterization of the alternating groups by their order and one conjugacy class length. Czech. Math. J. *66* (2016), 63–70. zbl MR doi
[4] *A. K. Asboei, R. Mohammadyari*: Recognizing alternating groups by their order and one conjugacy class length. J. Algebra Appl. *15* (2016), Article ID 1650021. zbl MR doi
[5] *A. K. Asboei, R. Mohammadyari*: New characterization of symmetric groups of prime degree. Acta Univ. Sapientiae, Math. *9* (2017), 5–12. zbl MR doi
[6] *G. Chen*: On Thompson's conjecture for sporadic groups. Proc. of the First Academic Annual Meeting of Youth. Fujian Science and Technology Publishing House, Fuzhou, 1992, pp. 1–6. (In Chinese.) MR
[7] *G. Chen*: On Thompson's Conjecture, PhD Thesis. Sichuan University, Chengdu, 1994.
[8] *G. Chen*: On the structure of Frobenius group and 2-Frobenius group. J. Southwest China Normal. Univ. *20* (1995), 485–487. (In Chinese.)
[9] *G. Chen*: On Thompson's conjecture. J. Algebra *185* (1996), 184–193. zbl MR doi
[10] *G. Chen*: Further reflections on Thompson's conjecture. J. Algebra *218* (1999), 276–285. zbl MR doi

[11] *Y. Chen, G. Chen, J. Li*: Recognizing simple K_4-groups by few special conjugacy class sizes. Bull. Malays. Math. Sci. Soc. (2) *38* (2015), 51–72. zbl MR doi

[12] *D. Gorenstein*: Finite Groups. Chelsea Publishing, New York, 1980. zbl MR

[13] *A. Iranmanesh, S. H. Alavi, B. Khosravi*: A characterization of $PSL(3,q)$ where q is an odd prime power. J. Pure Appl. Algebra *170* (2002), 243–254. zbl MR doi

[14] *A. S. Kondrat'ev, V. D. Mazurov*: Recognition of alternating groups of prime degree from their element orders. Sib. Math. J. *41* (2000), 294–302 (In English. Russian original.); translation from Sib. Mat. Zh. *41* (2000), 359–369. zbl MR doi

[15] *J. B. Li*: Finite Groups with Special Conjugacy Class Sizes or Generalized Permutable Subgroups, Ph.D. Thesis. Southwest University, Chongqing, 2012.

[16] *V. D. Mazurov, E. I. Khukhro (eds.)*: The Kourovka Notebook: Unsolved Problems in Group Theory. Institute of Mathematics, Russian Academy of Sciences, Siberian Div., Novosibirsk, 2018. zbl MR

[17] *W. J. Shi, J. X. Bi*: A new characterization of the alternating groups. Southeast Asian Bull. Math. 16 (1992), 81–90. zbl MR

[18] *A. V. Vasil'ev*: On Thompson's conjecture. Sib. Elektron. Mat. Izv. *6* (2009), 457–464. zbl MR

[19] *J. S. Williams*: Prime graph components of finite groups. J. Algebra *69* (1981), 487–513. zbl MR doi

Authors' address: Mahdi Abedei, Ali Iranmanesh (corresponding author), Farrokh Shirjian, Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-137, Tehran, Iran, e-mail: mahdi_abedi@modares.ac.ir, iranmanesh@modares.ac.ir, farrokh.shirjian@modares.ac.ir.