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EQUICONTINUITY, SHADOWING AND DISTALITY IN GENERAL TOPOLOGICAL SPACES

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Abstract. We consider the notions of equicontinuity point, sensitivity point and so on from a topological point of view. Many of these notions can be sensibly defined either in terms of (finite) open covers or uniformities. We show that for the notions of equicontinuity point and sensitivity point, Hausdorff or uniform versions coincide in compact Hausdorff spaces and are equivalent to the standard definitions stated in terms of a metric in compact metric spaces. We prove that a uniformly chain transitive map with uniform shadowing property on a compact Hausdorff uniform space is either uniformly equicontinuous or it has no uniform equicontinuity points.

Keywords: shadowing; chain transitive; equicontinuity; uniform space

MSC 2020: 37B20, 37B05, 54H20

1. Introduction

Throughout this paper (X,T) denotes a topological dynamical system, where X is a Hausdorff space and $T\colon X \to X$ is a continuous map. A number of properties of interest in such systems are defined in purely topological terms, for example transitivity, minimality and proximality. Others are defined in terms of the particular metric, for example sensitivity, equicontinuity, chain transitivity and shadowing. When the phase space X is a compact metric space, the sensitivity and equicontinuity were studied, see [4], [13]. For example, the well-known Auslander-Yorke dichotomy theorem stated that a minimal dynamical system is either sensitive or all points are equicontinuous (see [4] and also [13]), which was further refined in [1]: a transitive system is either sensitive or has a dense set of equicontinuity points; it

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was shown that if a chain transitive system has shadowing property then it is either sensitive or equicontinuous, see [10], Corollary 2.4.

However, in fact it shows that many dynamical properties can be defined in a natural way on Hausdorff (but not necessarily compact or metric) spaces. It turns out that there are two sensible ways to do this, either in terms of finite open covers or in terms of uniformities (compatible with the topology). The uniform approach has been studied in a number of cases: Hood in [16] defined entropy for uniform spaces; Devaney chaos for uniform spaces was considered in [8] (for group actions); Auslander, Greschonig and Nagar in [3] generalized many known results about equicontinuity to the uniform spaces; P. Das and D. Das in [9] extended some results about shadowing to the uniform spaces, see also [14]. Recently, the open cover approach has been studied in a number of cases: Brian in [7] considered chain transitivity in terms of finite open covers in compact Hausdorff spaces; Good and Macias in [14] considered sensitivity and shadowing in terms of finite open covers in compact Hausdorff spaces.

The present work is inspired by the results from the papers mentioned above and is organized as follows. In Section 2, we introduce some notions and results to be used in the article. In Section 3, we define the notions of Hausdorff equicontinuity, Hausdorff equicontinuity point and Hausdorff sensitivity point in terms of finite open covers. We show that in the presence of compactness, Hausdorff equicontinuity (Hausdorff equicontinuity point, Hausdorff sensitivity point) and uniform equicontinuity (or uniform equicontinuity point, uniform sensitivity point) are equivalent, and they coincide exactly with the standard definitions in compact metric spaces, see Theorems 3.7, 3.4 and 3.12. We also point out that the notion of Hausdorff equicontinuity point (or Hausdorff sensitivity point) may not be equivalent to the standard definition in a noncompact space, see Example 3.6 and Remark 3.13. Finally, we prove that a uniformly chain transitive map with uniform shadowing property on a compact Hausdorff uniform space is either uniformly equicontinuous or it has no uniform equicontinuity points, see Corollary 4.12.

2. Preliminaries

2.1. IP-set and IP*-set. Let $\mathbb{N} = \{0, 1, \ldots, n, \ldots\}$ be the set of non-negative integers. Endowing the semigroup $(\mathbb{N}, +)$ with the discrete topology, we take the points of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to be the ultrafilter on $\mathbb{N}$. Since $(\mathbb{N}, +)$ is a semigroup, we extend the operation $+$ to $\beta\mathbb{N}$ so that $(\beta\mathbb{N}, +)$ is a compact Hausdorff right topological semigroup. If $A \subset \mathbb{N}$, then $\widehat{A} = \mathrm{Cl}_{\beta\mathbb{N}} A = \{p \in \beta\mathbb{N} \colon A \in p\}$ is a basic clopen subset of $\beta\mathbb{N}$, see [15].

Recall that a subset $A \subset \mathbb{N}$ is called an IP-*set* if it contains all finite sums of a subsequence of itself. That is, there exists a subsequence $\{s_i\}_{i=1}^{\infty} \subset \mathbb{N}$ such that

$$FS(\{s_i\}_{i=1}^{\infty}) = \{s_{i_1} + s_{i_2} + \ldots + s_{i_n} \colon i_1 < i_2 < \ldots < i_n, i_j \in \mathbb{N}, j = 1, \ldots, n\} \subset A.$$

A set $A \subset \mathbb{N}$ is an IP*-set if it has nonempty intersection with any IP-set. An IP*-set is both syndetic and an IP-set, see [15]. A set $A \subset \mathbb{N}$ is *syndetic* if there exists $m \in \mathbb{N}$ such that $A \cap \{n, n+1, \ldots, n+m\} \neq \emptyset$ for each $n \in \mathbb{N}$. A set $A \subset \mathbb{N}$ is *thick* if for any positive integer n there exists $m \in \mathbb{N}$ such that $A \supset \{m, m+1, \ldots, m+n\}$. A thick set has a nonempty intersection with every syndetic set. We say that $p \in \beta\mathbb{N}$ is an *idempotent* if $p = p+p$. Let $p \in \beta\mathbb{N}$ and let $\{x_n\}$ be a sequence in X. By p-$\lim x_n = y$ we mean that for any open set U containing y the set $\{n \in \mathbb{N} \colon x_n \in U\} \in p$, see [15].

2.2. Transitivity and minimality. Let X be a topological space, and let $A \subset X$. The symbol $\overline{A}$ or $\operatorname{Cl} A$ denotes the closure of A. Let (X, T) be a dynamical system, and let $U, V \subset X$. We denote $N(U, V) = \{n \in \mathbb{N} \colon T^n(U) \cap V \neq \emptyset\}$. The system (X, T) is *(topologically) transitive* if $N(U, V) \neq \emptyset$ for any two nonempty open subsets $U, V \subset X$. A point $x \in X$ is called *transitive* if its orbit $\operatorname{orb}(x, T) = \{T^n(x) \colon n \in \mathbb{N}\}$ is dense in X. We say that $\operatorname{Tran}(X, T)$ denotes the set of all transitive points of (X, T). Moreover (X, T) is *point transitive* if $\operatorname{Tran}(X, T) \neq \emptyset$. The system (X, T) is minimal if $\operatorname{Tran}(X, T) = X$. In general, a subset A of X is invariant if $TA \subset A$. If A is a closed, nonempty invariant subset, then $(A, T|A)$ is called the associated subsystem. *A minimal subset* of X is a closed, nonempty invariant subset such that the associated subsystem is minimal. Clearly, (X, T) is minimal if and only if it admits no proper, closed, nonempty invariant subset. A point $x \in X$ is called *minimal* if it lies in some minimal subset. Zorn's Lemma implies that every closed, nonempty invariant set contains a minimal set. A point $x \in X$ is minimal if and only if $N(x, U) = \{n \in \mathbb{N} \colon T^n(x) \in U\}$ is syndetic for any neighborhood U of x. A dynamical system (X, T) is IP*-*central* (see [5]) if for any nonempty open subset U of X, $N(U, U) = \{n \in \mathbb{N} \colon T^n(U) \cap U \neq \emptyset\}$ is an IP*-set. A point $x \in X$ is IP*-*recurrent* (see [5]) if for any neighborhood U of x, $N(x, U) = \{n \in \mathbb{N} \colon T^n(x) \in U\}$ is an IP*-set.

2.3. Uniform spaces. Let X be a set and let Θ be an entourage of X. Then Θ is called *symmetric* if $\Theta^{-1} = \Theta$, where $\Theta^{-1} = \{(y, x) \colon (x, y) \in \Theta\}$. For $x \in X$, let $\Theta[x] = \{y \in X \colon (x, y) \in \Theta\}$. Regarding a subset $A \subset X$, let $\Theta[A] = \bigcup\limits_{a \in A} \Theta[a]$. The composite $\Theta_1 \circ \Theta_2$ of two entourages Θ_1 and Θ_2 of X is defined as $\Theta_1 \circ \Theta_2 = \{(x, z) \colon \text{there is an element } y \in X \text{ such that } (x, y) \in \Theta_1 \text{ and } (y, z) \in \Theta_2\}$. Note that for every compact Hausdorff space X, there exists a unique uniformity $\mathcal{U}$ on X that induces the original topology of X, see e.g. [12], Theorem 8.3.13.

We need the following lemma (see [14], Lemma 2.6).

Lemma 2.1. *Let X be a compact Hausdorff space, and let $\mathcal{U}$ be the unique uniformity on X that induces the original topology of X. If $\mathcal{A}$ is an open cover of X, then* $\bigcup\limits_{A\in\mathcal{A}} A\times A\in\mathcal{U}$.

We also need the following lemma.

Lemma 2.2. *Let X be a compact Hausdorff space and let $\mathcal{U}$ be the unique uniformity on X that induces its topology. Then for every open cover $\mathcal{A}$ of X, there exists a symmetric entourage $\Theta\in\mathcal{U}$ such that $\{\Theta[x]\colon\ x\in X\}$ refines $\mathcal{A}$.*

Proof. By Exercise 8.3. G. in [12], there exists an entourage $\Theta_1\in\mathcal{U}$ such that $\{\Theta_1[x]\colon\ x\in X\}$ refines $\mathcal{A}$. Let $\Theta=\Theta_1\cap\Theta_1^{-1}$. Then Θ satisfies the desired condition. $\square$

2.4. Chain transitivity and shadowing. Let (X,T) be a dynamical system, where (X,d) is a metric space. Assume that $x_0=x,x_1,\dots,x_n=y\in X$ and $\delta>0$. If $d(T(x_i),x_{i+1})<\delta$ for any $i\in\{0,1,\dots,n-1\}$, the sequence $x_0,x_1,\dots,x_n$ is called a *δ-chain* of T from x to y with length n. A map T is called *chain transitive* if for any two points $x,y\in X$ and any $\delta>0$ there exists a δ-chain from x to y. Given $\delta>0$ we say that the sequence $\xi=\{x_i\}_{i=0}^{\infty}$ in X is a *δ-pseudo-orbit* of T, if $d(T(x_i),x_{i+1})<\delta$ for all $i\in\mathbb{N}$. Given $\varepsilon>0$ we say that the sequence $\xi=\{x_i\}_{i=0}^{\infty}$ in X is ε-shadowed by the point $z\in X$, if $d(T^i(z),x_i)<\varepsilon$ for any $i\in\mathbb{N}$. We say that a map T has the *shadowing property* if for any $\varepsilon>0$, there is $\delta>0$ such that any δ-pseudo-orbit ξ of T is ε-shadowed by some point in X.

Let (X,T) be a dynamical system, where $(X,\mathcal{U})$ is a Hausdorff uniform space. Assume that $x_0=x,x_1,\dots,x_n=y\in X$. Let $\Theta\in\mathcal{U}$ be a symmetric entourage. If $(T(x_i),x_{i+1})\in\Theta$ for any $i\in\{0,1,\dots,n-1\}$, the sequence $x_0,x_1,\dots,x_n$ is called a *Θ-chain* of T from x to y with length n. A map T is called *uniformly chain transitive* if for any two points $x,y\in X$ and any symmetric entourage Θ, there exists a Θ-chain from x to y. Given a symmetric entourage $\Theta\in\mathcal{U}$, we say that the sequence $\xi=\{x_i\}_{i=0}^{\infty}$ in X is a *Θ-pseudo-orbit* of T, if $(T(x_i),x_{i+1})\in\Theta$ for all $i\in\mathbb{N}$. Given a symmetric entourage Θ, we say that the sequence $\xi=\{x_i\}_{i=0}^{\infty}$ in X is Θ-shadowed by the point $z\in X$, if $(T^i(z),x_i)\in\Theta$ for any $i\in\mathbb{N}$. We say that a map T has the *uniform shadowing property* if for any symmetric entourage Θ_1, there is a symmetric entourage Θ_2 such that any Θ_2-pseudo-orbit ξ of T is Θ_1-shadowed by some point in X.

Let (X,T) be a dynamical system, where X is a Hausdorff space. Given a finite open cover $\mathcal{A}$ of X, we say that the sequence $\xi=\{x_i\}_{i=0}^{\infty}$ in X is an *$\mathcal{A}$-pseudo-orbit* of T, if for all $i \in \mathbb{N}$ we have $\{T(x_i), x_{i+1}\} \subset A$ for some $A \in \mathcal{A}$. Given a finite open cover $\mathcal{A}$ of X, we say that the sequence $\xi = \{x_i\}_{i=0}^{\infty}$ in X is $\mathcal{A}$-shadowed by the point $z \in X$, if for all $i \in \mathbb{N}$ we have $\{T^i(z), x_i\} \subset A$ for some $A \in \mathcal{A}$. We say that a map T has the *Hausdorff shadowing property* if for any finite open cover $\mathcal{A}$ of X, there is a finite open cover $\mathcal{B}$ of X such that any $\mathcal{B}$-pseudo-orbit ξ of T is $\mathcal{A}$-shadowed by some point in X.

Remark 2.3. The two notions of Hausdorff shadowing and uniform shadowing coincide in compact spaces, and they are equivalent to the standard definition in compact metric spaces (see [14], Theorem 5.1).

3. Equicontinuity points and sensitivity points

Although the notions of equicontinuity and equicontinuity point as defined clearly depend on the specific metric, it turns out that there are two natural definitions of equicontinuity and equicontinuity point for arbitrary topological spaces. In this section, we imposed a constraint on every finite open cover of X such that it does not contain X. Otherwise this may cause an awkward situation (see Remark 3.11).

Definition 3.1. Let (X, T) be a dynamical system.

(1) Assume that X is a Hausdorff space. The system (X, T) is *Hausdorff equicontinuous*, if for any finite open cover $\mathcal{A}$ of X, there is a finite open cover $\mathcal{B}$ of X such that whenever $\{x, y\} \subset B$ for some $B \in \mathcal{B}$, for all $n \in \mathbb{N}$ we have $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$.
(2) Assume that $(X, \mathcal{U})$ is a Hausdorff uniform space. The system (X, T) is *uniformly equicontinuous* if for any symmetric entourage $\Theta \in \mathcal{U}$, there is a symmetric entourage $\Theta_1 \in \mathcal{U}$ such that whenever $(x, y) \in \Theta_1$, we have $(T^n(x), T^n(y)) \in \Theta$ for all $n \in \mathbb{N}$.
(3) Assume that (X, d) is a metric space. The system (X, T) is *equicontinuous* if for any $\varepsilon > 0$, there is $\delta > 0$ such that whenever $d(x, y) < \delta$, we have $d(T^n(x), T^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.

Definition 3.2. Let (X, T) be a dynamical system.

(1) Assume that X is a Hausdorff space. A point $x \in X$ is *Hausdorff equicontinuous* if for any finite open cover $\mathcal{A}$ of X, there is a neighborhood V of x such that whenever $y \in V$ for all $n \in \mathbb{N}$ we have $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$.
(2) Assume that $(X, \mathcal{U})$ is a Hausdorff uniform space. A point $x \in X$ is *uniformly equicontinuous* if for any symmetric entourage $\Theta \in \mathcal{U}$, there is a neighborhood V of x such that whenever $y \in V$, we have $(T^n(x), T^n(y)) \in \Theta$ for all $n \in \mathbb{N}$.

(3) Assume that (X, d) is a metric space. A point $x \in X$ is *equicontinuous* if for any $\varepsilon > 0$, there is a neighborhood V of x such that whenever $y \in V$, we have $d(T^n(x), T^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.

Remark 3.3. A point $x \in X$ is uniformly equicontinuous if and only if for any symmetric entourage Θ_1 of X there is a symmetric entourage Θ_2 of X such that whenever $(x, y) \in \Theta_2$, we have $(T^n(x), T^n(y)) \in \Theta_1$ for all $n \in \mathbb{N}$.

The two notions of Hausdorff equicontinuity point and uniform equicontinuity point coincide in compact spaces, they are equivalent to the standard definition in compact metric spaces.

Theorem 3.4. *Let (X, T) be a dynamical system with $x \in X$, where X is a compact space. Then the following claims are equivalent:*
(1) *x is Hausdorff equicontinuous.*
(2) *x is uniformly equicontinuous.*
If X is metric, then (1) and (2) are equivalent to
(3) *x is equicontinuous.*

Proof. (1) $\Rightarrow$ (2). Assume that x is Hausdorff equicontinuous. Let $\mathcal{U}$ be the unique uniformity on X that induces its topology. Let $\Theta \in \mathcal{U}$ be a symmetric entourage, and let $\Theta' \in \mathcal{U}$ be a symmetric entourage such that $\Theta' \circ \Theta' \subset \Theta$. Then $\{\mathrm{int}_X(\Theta'[z]) \colon z \in X\}$ is an open cover of X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\mathcal{A} = \{\mathrm{int}_X(\Theta'[z_i]) \colon i = 1, 2, \ldots, m\}$ is a finite subcover. As x is Hausdorff equicontinuous, there is a neighborhood V of x such that whenever $y \in V$ for all $n \in \mathbb{N}$ we have $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$. This implies that for each $n \in \mathbb{N}$, there is z_i in X such that $(T^n(x), z_i), (z_i, T^n(y)) \in \Theta'$. Hence $(T^n(x), T^n(y)) \in \Theta$, so (2) holds.

(2) $\Rightarrow$ (1). Let $\mathcal{U}$ be the unique uniformity on X that induces its topology. Assume that x is uniformly equicontinuous. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a finite open cover of X. Let $\Theta = \bigcup\limits_{i=1}^{m} A_i \times A_i$. By Lemma 2.1, we have $\Theta \in \mathcal{U}$. As x is uniformly equicontinuous, for the symmetric entourage Θ there is a neighborhood V of x such that whenever $y \in V$, we have $(T^n(x), T^n(y)) \in \Theta$ for all $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$ we have that $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$, so (1) holds.

For the rest of the proof, we assume that X is metric.

(3) $\Rightarrow$ (1). Suppose that x is equicontinuous. Let $\mathcal{A}$ be a finite open cover of X and let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{A}$. Since x is equicontinuous, there is a neighborhood V of x such that whenever $y \in V$, we have $d(T^n(x), T^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$, we have that $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$, so (1) holds.

(1) $\Rightarrow$ (3). Suppose that x is Hausdorff equicontinuous. Let $\varepsilon > 0$. Then $\{B(z,\varepsilon/2)\colon z \in X\}$ is an open cover of X, where $B(z,\varepsilon/2)$ denotes the open ball of radius $\varepsilon/2$ centred at z in X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\mathcal{A} = \{B(z_i,\varepsilon/2)\colon i = 1,2,\ldots,m\}$ is a finite subcover. As x is Hausdorff equicontinuous, there is an open neighborhood V of x such that whenever $y \in V$ for all $n \in \mathbb{N}$ we have $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$. Hence we have that $d(T^n(x), T^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$, so (3) holds. $\square$

Example 3.5. Let $X = (0,\infty)$ with the relative topology of the Euclidean space $\mathbb{R}$, and let $T(x) = x + 1$, $x \in X$. Then (X,T) is a dynamical system. Every point of X is both a Hausdorff equicontinuity point and an equicontinuity point.

Proof. It is easy to see that every point of X is equicontinuous. We next show that every point of X is also Hausdorff equicontinuous. Let $x \in X$. Choose $a \in X$ and $a < x$. Suppose that $\mathcal{A}$ is a finite open cover of X. Then there are $b > 1$ and $A \in \mathcal{A}$ such that $(b,\infty) \subset A$. As $\lim\limits_{n\to\infty} T^n(a) = \lim\limits_{n\to\infty} a + n = \infty$, there is $l > 0$ such that whenever $n \geqslant l$, we have $T^n(a) > b$. Note that for every $n \in \{0,1,\ldots,l-1\}$, $T^n(x) \in A_n$ for some $A_n \in \mathcal{A}$. Take an open neighborhood V of x such that $V \subset (a, x+1)$, and whenever $y \in V$ for every $n \in \{0,1,\ldots,l-1\}$, we have $\{T^n(y), T^n(x)\} \subset A_n$ for some $A_n \in \mathcal{A}$. Note that when $n \geqslant l$ for each $y \in V$ we have $\{T^n(y), T^n(x)\} \subset A$. Hence x is Hausdorff equicontinuous. $\square$

Example 3.6. The notion of Hausdorff equicontinuity point may not be equivalent to the standard definition in a noncompact space.

Proof. Let $X = [0,\infty)$ with the relative topology of the Euclidean space $\mathbb{R}$. Then X is a noncompact metric space. Let $T(x) = 2x$, $x \in X$. Then (X,T) is a dynamical system.

It is easy to see that 3 is not equicontinuous. We next show that 3 is Hausdorff equicontinuous. Suppose that $\mathcal{A}$ is a finite open cover of X. Then there are $b > 3$ and $A \in \mathcal{A}$ such that $(b,\infty) \subset A$. As $\lim\limits_{n\to\infty} T^n(2) = \lim\limits_{n\to\infty} 2^{n+1} = \infty$, there is $l > 0$ such that whenever $n \geqslant l$, we have $T^n(2) > b$. Note that for every $n \in \{0,1,\ldots,l-1\}$, $T^n(3) \in A_n$ for some $A_n \in \mathcal{A}$. Take an open neighborhood V of 3 such that $V \subset (2,4)$, and whenever $y \in V$, for every $n \in \{0,1,\ldots,l-1\}$ we have $\{T^n(y), T^n(3)\} \subset A_n$ for some $A_n \in \mathcal{A}$. Note that when $n \geqslant l$ for each $y \in V$ we have $\{T^n(y), T^n(3)\} \subset A$. Hence 3 is Hausdorff equicontinuous. $\square$

Theorem 3.7. *Let (X,T) be a dynamical system, where X is a compact space. Then the following claims are equivalent:*
(1) *(X,T) is Hausdorff equicontinuous.*
(2) *(X,T) is uniformly equicontinuous.*

If X is metric, then (1) *and* (2) *are equivalent to*
(3) (X,T) *is equicontinuous.*

Proof. (1) $\Rightarrow$ (2). Assume that (X,T) is Hausdorff equicontinuous. Let $\mathcal{U}$ be the unique uniformity on X that induces its topology. Let $\Theta \in \mathcal{U}$ be a symmetric entourage, and let $\Theta' \in \mathcal{U}$ be a symmetric entourage such that $\Theta' \circ \Theta' \subset \Theta$. Then $\{\mathrm{int}_X(\Theta'[z])\colon z \in X\}$ is an open cover of X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\mathcal{A} = \{\mathrm{int}_X(\Theta'[z_i])\colon i = 1,2,\ldots,m\}$ is a finite subcover. As (X,T) is Hausdorff equicontinuous, there is a finite open cover $\mathcal{B}$ of X such that whenever $\{x,y\} \subset B$ for some $B \in \mathcal{B}$, for all $n \in \mathbb{N}$ we have $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$. Let $\Theta_1 = \bigcup\limits_{B\in\mathcal{B}} B \times B$. By Lemma 2.1, $\Theta_1 \in \mathcal{U}$. Then whenever $\{x,y\} \in \Theta_1$, for all $n \in \mathbb{N}$ we have $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$. This implies that $(z_i, T^n(x)), (z_i, T^n(y)) \in \Theta'$ for some z_i. Hence $(T^n(x), T^n(y)) \in \Theta$, so (2) holds.

(2) $\Rightarrow$ (1). Let $\mathcal{U}$ be the unique uniformity of X that induces its topology. Assume that (X,T) is uniformly equicontinuous. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a finite open cover of X. Let $\Theta = \bigcup\limits_{i=1}^{m} A_i \times A_i$. By Lemma 2.1, we have $\Theta \in \mathcal{U}$. As (X,T) is uniformly equicontinuous for the symmetric entourage Θ there is a symmetric entourage Θ_1 such that whenever $(x,y) \in \Theta_1$, we have $(T^n(x), T^n(y)) \in \Theta$ for all $n \in \mathbb{N}$. Let $\Theta_2 \in \mathcal{U}$ be a symmetric entourage such that $\Theta_2 \circ \Theta_2 \subset \Theta_1$. Then $\{\mathrm{int}_X(\Theta_2[z])\colon z \in X\}$ is an open cover of X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\mathcal{B} = \{\mathrm{int}_X(\Theta_2[z_i])\colon i = 1,2,\ldots,m\}$ is a finite subcover. If $\{x,y\} \subset \mathrm{int}_X(\Theta_2[z_i])$ for some z_i, then $(x,y) \in \Theta_1$. Hence whenever $\{x,y\} \subset \mathrm{int}_X(\Theta_2[z_i])$ for some z_i, for all $n \in \mathbb{N}$ we have that $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$, so (1) holds.

For the rest of the proof, we assume that X is metric.

(3) $\Rightarrow$ (1). Suppose that (X,T) is equicontinuous. Let $\mathcal{A}$ be a finite open cover of X and let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{A}$. Since (X,T) is equicontinuous, there is $\delta > 0$ such that whenever $d(x,y) < \delta$, we have $d(T^n(x), T^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$. Then $\{B(z,\delta/2)\colon z \in X\}$ is an open cover of X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\{B(z_i,\delta/2)\colon i = 1,2,\ldots,m\}$ is a finite subcover. If $\{x,y\} \subset B(z_i,\delta/2)$ for some z_i, then $d(x,y) < \delta$. Hence whenever $\{x,y\} \subset B(z_i,\delta/2)$ for some z_i, for all $n \in \mathbb{N}$ we have that $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$, so (1) holds.

(1) $\Rightarrow$ (3). Suppose that (X,T) is Hausdorff equicontinuous. Let $\varepsilon > 0$. Then $\{B(z,\varepsilon/2)\colon z \in X\}$ is an open cover of X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\mathcal{A} = \{B(z_i,\varepsilon/2)\colon i = 1,2,\ldots,m\}$ is a finite subcover. As (X,T) is Hausdorff equicontinuous, for $\mathcal{A}$ there is a finite open cover $\mathcal{B}$

of X such that whenever $\{x,y\} \subset B$ for some $B \in \mathcal{B}$, for all $n \in \mathbb{N}$ we have $\{T^n(x), T^n(y)\} \subset A_n$ for some $A_n \in \mathcal{A}$. Let $\delta > 0$ be a Lebesgue number for $\mathcal{B}$. Hence whenever $d(x,y) < \delta$, we have that $d(T^n(x), T^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$, so (3) holds. $\square$

Proposition 3.8. *Let (X,T) be a dynamical system.*

(1) *Assume that $(X,\mathcal{U})$ is a compact Hausdorff uniform space. The system (X,T) is uniformly equicontinuous if and only if each point x of X is uniformly equicontinuous.*

(2) *Assume that X is a compact Hausdorff space. The system (X,T) is Hausdorff equicontinuous if and only if each point x of X is Hausdorff equicontinuous.*

Proof. (1) Suppose that each point of X is uniformly equicontinuous, where (X,T) is not uniformly equicontinuous. Then there is a symmetric entourage $\Theta_0 \in \mathcal{U}$ such that for any symmetric entourage $\Theta \in \mathcal{U}$ there are $(x_\Theta, y_\Theta) \in \Theta$ and $n_\Theta \geqslant 1$ such that $(T^{n_\Theta}(x_\Theta), T^{n_\Theta}(y_\Theta)) \notin \Theta_0$. Let $\mathcal{V}$ be the collection of all symmetric entourages of X. Define a relation $\geqslant$ on $\mathcal{V}$ as follows: $\alpha \geqslant \beta \Leftrightarrow \alpha \subset \beta$. Then $(\mathcal{V}, \geqslant)$ is a directed set. Hence $\{x_\Theta\colon \Theta \in \mathcal{V}\}$ is a net. By compactness, it has a subnet $\{x_\Theta\colon \Theta \in \mathcal{W}\}$ such that $\lim\limits_{\Theta\in\mathcal{W}} x_\Theta = \lim\limits_{\Theta\in\mathcal{W}} y_\Theta = z$. Let $\Theta_1 \in \mathcal{V}$ be such that $\Theta_1 \circ \Theta_1 \subset \Theta_0$, and let $\Theta_2 \in \mathcal{V}$ be such that $\Theta_2 \circ \Theta_2 \subset \Theta_1$. As z is uniformly equicontinuous, for Θ_1 there is an open neighborhood U of z such that whenever $x \in U$, $(T^n(x), T^n(z)) \in \Theta_1$ for all $n \in \mathbb{N}$. We choose $x_\Theta, y_\Theta \in U \cap \Theta_2[z]$ such that $(T^{n_\Theta}(x_\Theta), T^{n_\Theta}(y_\Theta)) \notin \Theta_0$ for some $n_\Theta \geqslant 1$. Then either $(T^{n_\Theta}(x_\Theta), T^{n_\Theta}(z)) \notin \Theta_1$ or $(T^{n_\Theta}(y_\Theta), T^{n_\Theta}(z)) \notin \Theta_1$. This shows that z is not uniformly equicontinuous, a contradiction.

(2) By (1), Theorems 3.4 and 3.7, (2) holds. $\square$

Although the notion of sensitivity (sensitivity point) as defined clearly depend on the specific metric, it turns out that there are two natural definitions of sensitivity (or sensitivity point) for arbitrary topological spaces.

Definition 3.9. Let (X,T) be a dynamical system.

(1) Assume that X is a Hausdorff space. The system (X,T) is *Hausdorff sensitive* (see [14]) if there is a finite open cover $\mathcal{A}$ of X such that for any nonempty open subset U of X, there are distinct points $x,y \in U$ and $n \geqslant 1$ such that $|\{T^n(x), T^n(y)\} \cap A| \leqslant 1$ for all $A \in \mathcal{A}$, where $|C|$ denotes the cardinality of C.

(2) Assume that $(X,\mathcal{U})$ is a Hausdorff uniform space. The system (X,T) is *uniformly sensitive* (see [8]) if there is a symmetric entourage $\Theta \in \mathcal{U}$ such that for any nonempty open subset U of X there are $x,y \in U$ and $n \geqslant 1$ with $(T^n(x), T^n(y)) \notin \Theta$.

(3) Assume that (X,d) is a metric space. The system (X,T) is *sensitive* (see [4] and [13]) if there is $\varepsilon > 0$ such that for any nonempty open subset U of X there are $x, y \in U$ and $n \geqslant 1$ with $d(T^n(x), T^n(y)) > \varepsilon$.

Definition 3.10. Let (X,T) be a dynamical system.

(1) Assume that X is a Hausdorff space. A point $x \in X$ is *Hausdorff sensitive* if there is a finite open cover $\mathcal{A}$ of X such that for any neighborhood V of x there are $y \in V$ and $n \geqslant 1$ such that $|\{T^n(x), T^n(y)\} \cap A| \leqslant 1$ for all $A \in \mathcal{A}$.

(2) Assume that $(X, \mathcal{U})$ is a Hausdorff uniform space. A point $x \in X$ is *uniformly sensitive* if there is a symmetric entourage $\Theta \in \mathcal{U}$ such that for any neighborhood V of x there are $y \in V$ and $n \geqslant 1$ with $(T^n(x), T^n(y)) \notin \Theta$.

(3) Assume that (X,d) is a metric space. A point $x \in X$ is *sensitive* (see [4] and [13]) if there is $\varepsilon > 0$ such that for any neighborhood V of x there are $y \in V$ and $n \geqslant 1$ with $d(T^n(x), T^n(y)) > \varepsilon$.

Remark 3.11. In this section, we imposed a constraint on every finite open cover of X such that it does not contain X. Otherwise this may cause an awkward situation. For instance, let $T\colon X \to X$ be a constant map. Let $\mathcal{A} = \{X\}$. Then for every nonempty open subset U of X and all points $x, y \in U$, we have $|\{Tx, Ty\} \cap A| \leqslant 1$ for all $A \in \mathcal{A}$.

The two notions of Hausdorff sensitivity and uniform sensitivity coincide in compact spaces, they are equivalent to the standard definition in compact metric spaces, see [14], Theorem 3.2. Next, we point out that for sensitivity points, there is a similar result.

Theorem 3.12. *Let (X,T) be a dynamical system with $x \in X$, where X is a compact space. Then the following claims are equivalent:*

(1) *x is Hausdorff sensitive.*

(2) *x is uniformly sensitive.*

If X is metric, then (1) *and* (2) *are equivalent to*

(3) *x is sensitive.*

P r o o f. (1) $\Rightarrow$ (2). Assume that x is Hausdorff sensitive. Let $\mathcal{U}$ be the unique uniformity on X that induces its topology. Let $\mathcal{A}$ be a finite open cover of X given by the definition of a Hausdorff sensitivity point. Since X is compact, by Lemma 2.2 there exists a symmetric entourage $\Theta \in \mathcal{U}$ such that $\{\Theta[z]\colon z \in X\}$ refines $\mathcal{A}$. As x is Hausdorff sensitive, there are $y \in \Theta[x]$ and $n \geqslant 1$ such that $|\{T^n(x), T^n(y)\} \cap A| \leqslant 1$ for all $A \in \mathcal{A}$. This implies $(T^n(x), T^n(y)) \notin \Theta$. Otherwise, $(T^n(x), T^n(y)) \in \Theta$. Hence, $\{T^n(x), T^n(y)\} \subset \Theta[T^n(x)] \subset A$ for some $A \in \mathcal{A}$, a contradiction. So (2) holds.

(2) $\Rightarrow$ (1). Let $\mathcal{U}$ be the unique uniformity on X that induces its topology. Assume that x is uniformly sensitive. Let $\Theta \in \mathcal{U}$ be a symmetric entourage given by the definition of a uniform sensitivity point. Let $\Theta_1 \in \mathcal{U}$ be a symmetric entourage such that $\Theta_1 \circ \Theta_1 \subset \Theta$. Since $\{\Theta_1[z] \colon z \in X\}$ covers X, we have that $\{\mathrm{int}_X(\Theta_1[z]) \colon z \in X\}$ also covers X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\{\mathrm{int}_X(\Theta_1[z_i]) \colon i = 1, 2, \ldots, m\}$ is a finite subcover. We show that $\mathcal{B} = \{\mathrm{int}_X(\Theta_1[z_i]) \colon i = 1, 2, \ldots, m\}$ satisfies the definition of a Hausdorff sensitivity point. Let V be a neighborhood of x. As x is uniformly sensitive, there are $y \in V$ and $n \geqslant 1$ with $(T^n(x), T^n(y)) \notin \Theta$. Hence, $|\{T^n(x), T^n(y)\} \cap B| \leqslant 1$ for all $B \in \mathcal{B}$. Otherwise, there is $i \in \{1, 2, \ldots, m\}$ such that $\{T^n(x), T^n(y)\} \subset \Theta_1[z_i]$. Then $(T^n(x), z_i) \in \Theta_1$ and $(T^n(y), z_i) \in \Theta_1$, thus $(T^n(x), T^n(y)) \in \Theta$, a contradiction. So (1) holds.

For the rest of the proof, we assume that X is metric.

(3) $\Rightarrow$ (1). Assume that x is sensitive. Let $\varepsilon > 0$ be the constant given by the definition of a sensitivity point. Then $\mathcal{A} = \{B(z, \varepsilon/2) \colon z \in X\}$ be an open cover of X. Since X is compact, there are $z_1, z_2, \ldots, z_m$ in X such that $\{B(z_i, \varepsilon/2) \colon i = 1, 2, \ldots, m\}$ is a finite subcover. We show that $\mathcal{B} = \{B(z_i, \varepsilon/2) \colon i = 1, 2, \ldots, m\}$ satisfies the definition of a Hausdorff sensitivity point. Since x is sensitive, for any neighborhood V of x, we can take $y \in V$ and $n \geqslant 1$ with $d(T^n(x), T^n(y)) > \varepsilon$. Hence, $|\{T^n(x), T^n(y)\} \cap B| \leqslant 1$ for all $B \in \mathcal{B}$. Otherwise, there is $i \in \{1, 2, \ldots, m\}$ such that $\{T^n(x), T^n(y)\} \subset B(z_i, \varepsilon/2)$. This implies that $d(T^n(x), T^n(y)) \leqslant d(T^n(x), z_i) + d(z_i, T^n(y)) < \varepsilon$, a contradiction. So (1) holds.

(1) $\Rightarrow$ (3). Assume that x is Hausdorff sensitive. Let $\mathcal{A}$ be a finite open cover of X given by the definition of a Hausdorff sensitivity point. Let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{A}$. Since x is Hausdorff sensitive for any neighborhood V of x there are $y \in V$ and $n \geqslant 1$ such that $|\{T^n(x), T^n(y)\} \cap A| \leqslant 1$ for all $A \in \mathcal{A}$. Since ε is a Lebesgue number for $\mathcal{A}$, this implies that $d(T^n(x), T^n(y)) > \varepsilon$. Therefore, (3) holds. $\square$

Remark 3.13. The Hausdorff sensitivity point may not be equivalent to the standard definition in a noncompact space.

For instance, for the system (X, T) in Example 3.6, 3 is a sensitivity point, but not a Hausdorff sensitivity point.

4. Distality and uniform shadowing

Let (X, T) be a dynamical system and $x, y \in X$. The points x and y are *proximal*, if there is a sequence $\{n_i\} \subset \mathbb{N}$ such that $\lim\limits_{i\to\infty} T^{n_i}(x) = \lim\limits_{i\to\infty} T^{n_i}(y)$. A *point* $x \in X$ is *distal* if it is not proximal to any point in the orbit closure other than itself. A *system* (X, T) is *distal* if every point of X is distal.

We need the following result (see [11], Proposition 5.2.7 and Corollary 5.2.8).

Lemma 4.1. *Let (X,T) be a dynamical system and $x \in X$, where X is a compact Hausdorff space. Then the following statements are equivalent:*
(1) *x is a distal point.*
(2) *For any neighborhood U of x, the set $N(x,U)$ is an* IP^**-set.*
(3) *p-$\lim T^n x = x$ for all idempotents $p \in \beta\mathbb{N}$.*

We also need the following lemma, see e.g. [15], Theorem 16.4.

Lemma 4.2. *A subset $A \subset \mathbb{N}$ is an* IP*-set if and only if there is an idempotent $p \in \beta\mathbb{N}$ such that $A \in p$.*

Lemma 4.3. *Let (X,T) be an* IP^**-central dynamical system, where X is a compact Hausdorff uniform space. Then any uniform equicontinuity point of (X,T) is distal.*

P r o o f. Let (X,T) be an IP^*-central dynamical system. Let $x \in X$ be a uniform equicontinuity point. We will show that x is distal. This is equivalent to showing that p-$\lim T^n x = x$ for all idempotents $p \in \beta\mathbb{N}$ by Lemma 4.1. In contrast, suppose that there exists a point $y \neq x$ such that p-$\lim T^n x = y$ for some idempotent $p \in \beta\mathbb{N}$.

Let Θ_1 be a symmetric entourage such that $\Theta_1[x] \cap \Theta_1[y] = \emptyset$. Let Θ_2 be a symmetric entourage such that $\Theta_2 \circ \Theta_2 \subset \Theta_1$. Let $\Theta_3 \subset \Theta_2$ be a symmetric entourage corresponding to Θ_2 in the definition of uniform equicontinuity of x, see Remark 3.3. Since p-$\lim T^n x = y$, we have $N(x,\Theta_2[y]) \in p$. This implies that $N(x,\Theta_2[y])$ is an IP-set by Lemma 4.2.

Claim. $\Theta_3[x] \cap \Big(\bigcup_{n \in N(x,\Theta_2[y])} \Theta_2[T^n(x)]\Big) = \emptyset.$

In contrast, suppose that there exists a point $z \in \Theta_3[x]$, $n \in N(x,\Theta_2[y])$ such that $z \in \Theta_2[T^n(x)]$. Then $(T^n(x),y) \in \Theta_2, (z,T^n(x)) \in \Theta_2$. Hence $(z,y) \in \Theta_2 \circ \Theta_2 \subset \Theta_1$. This implies that $z \in \Theta_3[x] \cap \Theta_1[y]$. This contradicts $\Theta_1[x] \cap \Theta_1[y] = \emptyset$. Hence the claim holds.

Since x is uniformly equicontinuous, we have

$$\bigcup_{n \in N(x,\Theta_2[y])} T^n(\Theta_3[x]) \subset \bigcup_{n \in N(x,\Theta_2[y])} \Theta_2[T^n(x)].$$

Hence, by claim we have

$$\Theta_3[x] \cap \Bigg(\bigcup_{n \in N(x,\Theta_2[y])} T^n(\Theta_3[x])\Bigg) = \emptyset.$$

This means that for any n in the IP-set $N(x,\Theta_2[y])$, $\Theta_3[x]\cap T^n(\Theta_3[x])=\emptyset$. Note that (X,T) is an IP*-central dynamical system, which contradicts the fact that $\{n\in\mathbb{N}\colon\ \Theta_3[x]\cap T^n(\Theta_3[x])\neq\emptyset\}$ is an IP*-set. $\square$

Lemma 4.4. *Let (X,T) be a transitive system and $x\in X$, where X is a Hausdorff uniform space. If x is uniformly equicontinuous, then x is transitive.*

Proof. Let x be a uniform equicontinuity point. We will show that $\overline{\operatorname{orb}(x,T)}=X$. Let $y\in X$ and let Θ be an entourage. We choose a symmetric entourage Θ_1 such that $\Theta_1\circ\Theta_1\subset\Theta$. Let $\Theta_2\subset\Theta_1$ be a symmetric entourage corresponding to Θ_1 in the definition of a uniform equicontinuity point of x. Since (X,T) is topologically transitive, there is $m\geqslant 1$ such that $T^m(\Theta_2[x])\cap\Theta_1[y]\neq\emptyset$. Then there is $z\in\Theta_2[x]$ such that $(T^m(z),y)\in\Theta_1$. Since x is uniformly equicontinuous and $z\in\Theta_2[x]$, we have $(T^m(z),T^m(x))\in\Theta_1$. Hence $(T^m(x),y)\in\Theta$, that is, $T^m(x)\in\Theta[y]$. $\square$

Lemma 4.5. *Let (X,T) be a dynamical system with a uniform equicontinuity point $x\in X$, where X is a Hausdorff uniform space. If $y\in X$ is a transitive point, then y is uniformly equicontinuous.*

Proof. Let Θ_0 be a symmetric entourage. Let Θ_1 be a symmetric entourage such that $\Theta_1\circ\Theta_1\subset\Theta_0$. Let $\Theta_2\subset\Theta_1$ be a symmetric entourage corresponding to Θ_1 in the definition of uniform equicontinuity of x.

Since y is transitive, there is $n\geqslant 1$ such that $T^n(y)\in\Theta_2[x]$. Then there is a neighborhood U of y such that $T^n(U)\subset\Theta_2[x]$. Hence for any $z\in U$ we have $T^n(z)\in\Theta_2[x]$. Since x is uniformly equicontinuous, whenever $z\in U$ we have $(T^{m+n}(z),T^{m+n}(y))\in\Theta_0$ for all $m\in\mathbb{N}$. Since for every $i\in\{0,1,\ldots,n\}$, T^i is continuous at y, it is easy to see that there is a neighborhood V of y such that whenever $z\in V$, we have $(T^k(z),T^k(y))\in\Theta_0$ for all $k\in\mathbb{N}$. $\square$

The following lemma can be obtained from [6], Theorem 3.10.

Lemma 4.6. *If (X,T) is a point transitive distal system, where X is a compact Hausdorff space, then (X,T) is minimal.*

Theorem 4.7. *Let (X,T) be an* IP**-central and transitive system, where X is a compact Hausdorff uniform space. If (X,T) has a uniform equicontinuity point, then it is distal, minimal, and all points of X are uniformly equicontinuous.*

Proof. Let $x\in X$ be a uniform equicontinuity point. By Lemma 4.4, x is a transitive point. Let Θ be an entourage. We choose a symmetric entourage Θ_1 such that $\Theta_1\circ\Theta_1\circ\Theta_1\subset\Theta$. Let $\Theta_2\subset\Theta_1$ be a symmetric entourage corresponding to Θ_1 in the definition of an equicontinuity point of x. By Lemma 4.3, x is distal. Then $N(x, \Theta_2[x])$ is an IP*-set by Lemma 4.1. We will show that for any $z \in X$ and any $n \in N(x, \Theta_2[x])$ we have $(T^n(z), z) \in \Theta$. Then z is IP*-recurrent, and it is distal by Lemma 4.1.

Choose $z \in X$ and some $n \in N(x, \Theta_2[x])$. Since T^n is continuous, for Θ_1 there is a symmetric entourage $\Theta_3 \subset \Theta_2$ such that for any $v \in \Theta_3[z]$ we have $(T^n(v), T^n(z)) \in \Theta_1$. Note that since $n \in N(x, \Theta_2[x])$ for all $m \in \mathbb{N}$ we have $(T^{n+m}(x), T^m(x)) \in \Theta_1$. As x is transitive, hence there is $k \geqslant 1$ such that $T^k(x) \in \Theta_3[z]$. This implies that $(T^k(x), z) \in \Theta_3$. Then $(T^{n+k}(x), T^n(z)) \in \Theta_1$. Hence, $(T^n(z), z) \in \Theta_1 \circ \Theta_1 \circ \Theta_3 \subset \Theta$. By Lemma 4.6, (X, T) is minimal. By Lemma 4.5, all points of X are uniformly equicontinuous. $\square$

We need the following result, see e.g. [15], Lemma 16.13.

Lemma 4.8. *Let $k \in \mathbb{N}$. Then $k\mathbb{N}$ is an* IP*-*set in* $(\mathbb{N}, +)$.

Lemma 4.9. *Let (X, T) be a dynamical system. If (X, T) is uniformly chain transitive and T has uniform shadowing, then (X, T) is transitive.*

P r o o f. Let U and V be two nonempty open subsets of X. Take $x \in U$ and $y \in V$. Then there is a symmetric entourage Θ_1 of X such that $\Theta_1[x] \subset U$ and $\Theta_1[y] \subset V$. Let Θ_2 be a symmetric entourage corresponding to Θ_1 in the definition of uniform shadowing. As (X, T) is uniformly chain transitive, there is a Θ_2-chain $\xi = \{x_0 = x, x_1, \ldots, x_n = y\}$ from x to y. As T has uniform shadowing, there is $z \in X$ such that $(z, x) \in \Theta_1$ and $(T^n(z), y) \in \Theta_1$. That is, $z \in \Theta_1[x]$ and $T^n(z) \in \Theta_1[y]$. This implies that $T^n(U) \cap V \neq \emptyset$. $\square$

Proposition 4.10. *Let (X, T) be a dynamical system, where X is a compact Hausdorff uniform space. If T is a uniformly chain transitive map with uniform shadowing property, then (X, T) is* IP*-*central and transitive.*

P r o o f. By Lemma 4.9, (X, T) is transitive. Let U be a nonempty open subset of X. Consider a point $x \in U$. Let Θ_1 be a symmetric entourage such that $\Theta_1[x] \subset U$ and let Θ_2 be a symmetric entourage corresponding to Θ_1 in the definition of uniform shadowing. Suppose that $\xi = \{x_0, x_1, \ldots, x_k\}$ is a Θ_2-chain from x to x with length k. Let

$$\zeta := \xi, \overline{\xi \setminus x_0},$$

where $\overline{a}$ represents repetition. Note that ζ is a Θ_2-pseudo orbit. So we can find $z \in X$ which Θ_1-shadows ζ. Since k is fixed, we have

$$(z, x) \in \Theta_1, \quad (T^k(z), x) \in \Theta_1, \quad (T^{ik}(z), x) \in \Theta_1, \quad i = 1, 2, \ldots$$

Therefore, $N(U,U) \supset N(\Theta_1[x],\Theta_1[x]) = k\mathbb{N}$. By Lemma 4.8, $N(U,U)$ is an IP*-set. $\square$

Corollary 4.11. *Let (X,T) be a dynamical system, where X is a compact Hausdorff uniform space, and T is a uniformly chain transitive map with uniform shadowing property. If (X,T) has a uniform equicontinuity point, then it is distal, minimal and uniformly equicontinuous.*

Proof. By Proposition 4.10, (X,T) is IP*-central and transitive. By Theorem 4.7, (X,T) is distal, minimal, and all points are uniformly equicontinuous. By Proposition 3.8, (X,T) is uniformly equicontinuous. $\square$

Corollary 4.12. *Let (X,T) be a dynamical system, where X is a compact Hausdorff uniform space. If T is uniformly chain transitive map with uniform shadowing property, then (X,T) is either uniformly equicontinuous or it has no uniform equicontinuity points.*

We know that if (X,T) is transitive, where X is a compact metric space, and all points of X are sensitive, then (X,T) is sensitive (see [2], Theorem 3.1). We end this section with the following question:

Question 1. If (X,T) is transitive, where $(X,\mathcal{U})$ is a compact Hausdorff uniform space, and all points of X are uniformly sensitive, is (X,T) uniformly sensitive?

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