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GENERALIZED HÖLDER TYPE SPACES OF HARMONIC FUNCTIONS IN THE UNIT BALL AND HALF SPACE

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Abstract. We study spaces of Hölder type functions harmonic in the unit ball and half space with some smoothness conditions up to the boundary. The first type is the Hölder type space of harmonic functions with prescribed modulus of continuity $\omega = \omega(h)$ and the second is the variable exponent harmonic Hölder space with the continuity modulus $|h|^{\lambda(\cdot)}$. We give a characterization of functions in these spaces in terms of the behavior of their derivatives near the boundary.

Keywords: Hölder space; harmonic function; variable exponent space; modulus of continuity

MSC 2010: 42B35, 46E30, 46E15

1. Introduction

Studies of classical Lipschitz (Hölder) spaces of holomorphic functions are well known. We refer, for instance, to the books, see [20], [21] (see also [8], [9]). In the present paper we study the spaces of Hölder type functions harmonic in the unit ball and in the half space with prescribed modulus of continuity and with variable Hölder exponent. This research is a continuation of the results of the paper, see [16], devoted to the study of Hölder type spaces of holomorphic functions in the unit disc and half plain.

The spaces of functions of such a type are generally referred to as nonstandard growth spaces. The real analysis theory of nonstandard function spaces of measurable

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and smooth functions has been developed intensively during the last two decades. We refer to the books, see [6], [7], [18], [19]. Studies of nonstandard spaces of holomorphic (harmonic) functions are in fact at the very beginning. We refer to the study of variable exponent spaces of holomorphic functions, Orlicz-holomorphic spaces, and Morrey-holomorphic spaces, including some their mixed norm versions, see [4], [5], [10], [11], [12], [13], [14], [15], [17]. A major interest in such spaces is due to the fact that in this way we include into consideration the spaces of functions with a general and nonstandard behavior near the boundary. The behavior of a function in a variable exponent Hölder space when approaching the boundary depends on the boundary point and is different, in general, when approaching different boundary points.

In the case of constant λ and holomorphic functions considered on the ball in $\mathbb{C}^n$ such results are known as well as many other facts on Lipschitz (Hölder) spaces, see [20] (see also [21] for $n = 2$). We follow some ideas of the proofs there.

The paper is organized as follows. In Section 2 we collect definitions and auxiliary statements. Sections 3 and 4 are devoted to the main results of the paper. In Section 3 we provide characterization of the spaces of harmonic functions in the ball $\mathbb{B}^n$ with prescribed modulus of continuity. This characterization is given in terms of growth of gradient of a function near the boundary $\mathbb{S}^{n-1}$ of the ball $\mathbb{B}^n$. In Section 4 we similarly treat the variable exponent space of harmonic Hölder functions in the ball $\mathbb{B}^n$. In Section 5 we extend the results of Sections 3 and 4 to the case of Hölder type spaces of harmonic functions in the half space $\mathbb{R}^n_+$.

2. Preliminaries

A function $\omega\colon\ [0,2] \to \mathbb{R}_+$ is called the modulus of continuity if

(1) ω is continuous in a neighborhood of the origin and $\omega(0) = 0$,

(2) ω is almost increasing on $[0,2]$,

(3) $\omega(h)/h$ is almost decreasing on $[0,2]$.

Note that from the assumption that $\omega(h)/h$ is almost decreasing on $[0,2]$, the semi-additivity property: $\omega(t+s) \leqslant C(\omega(t)+\omega(s))$, $t, s \in [0,2]$, and the so-called doubling property: $\omega(2t) \leqslant C_{(2)}\omega(t)$, $t \in [0,2]$, follow. Here we assume $\omega(h) = \omega(2)$ for $h > 2$ by definition.

In what follows we use the following Zygmund type conditions:

$$\int_0^t \frac{\omega(s)}{s}\,\mathrm{d}s \leqslant C\omega(t), \quad 0 < t < 2, \tag{2.1}$$

$$\int_t^2 \frac{\omega(s)}{s^2}\,\mathrm{d}s \leqslant C\frac{\omega(t)}{t}, \quad 0 < t < 2, \tag{2.2}$$

where C does not depend on t.

Let $\mathbb{B}^n := \{x \in \mathbb{R}^n \colon |x| < 1\}$ be the unit ball in $\mathbb{R}^n$, where $|\cdot|$ is the Euclidean norm, $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, and $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n > 0\}$. Put $S = (0, \ldots, 0, -1)$. For the particular case $n = 2$ we also use the more convenient notation: $\mathbb{D}$ and $\mathbb{T}$, respectively.

Let $\lambda \colon \mathbb{B}^n \to [0, 1]$ be a continuous function. We say that λ satisfies the log-condition (log-Hölder condition) on $\mathbb{B}^n$ if

$$(2.3) \qquad |\lambda(x) - \lambda(y)| \leqslant \frac{C}{\ln(1/|x-y|)}, \quad x, y \in \mathbb{B}^n,\ |x-y| < \frac{1}{2},$$

where C does not depend on $x, y \in \mathbb{B}^n$.

Note that the log-condition imposed on the function λ implies that it is bounded and uniformly continuous on $\mathbb{B}^n$. Hence, it extends to a continuous function on $\overline{\mathbb{B}^n} := \{x \in \mathbb{R}^n \colon |x| \leqslant 1\}$. We use the same notation λ for the so extended function. The log-condition also implies the property:

$$C_1 R^{\lambda(x)} \leqslant R^{\lambda(y)} \leqslant C_2 R^{\lambda(x)}$$

for all $x, y \in \mathbb{B}^n$ such that $|x - y| \leqslant R$, where C_1, C_2 do not depend on x, y.

Let $\lambda \colon \mathbb{R}^n_+ \to [0, 1]$ be a continuous function and let

$$\alpha(x, y) = \frac{|x-y|}{|x-S||y-S|}, \quad x, y \in \mathbb{R}^n_+.$$

We say that λ satisfies the global log-condition on $\mathbb{R}^n_+$ if

$$(2.4) \qquad |\lambda(x) - \lambda(y)| \leqslant \frac{C}{\ln(1/\alpha(x,y))}, \quad x, y \in \mathbb{R}^n_+,\ \alpha(x, y) < \frac{1}{2},$$

where C does not depend on x, y.

We use a Funk-Hecke type formula (see [1], [3]). Let $x = \xi_1 e_1 + \ldots + \xi_n e_n$ in $\mathbb{B}^n$ where $e_1, \ldots, e_n$ is a base in $\mathbb{R}^n$, then $\xi_1 = \cos\phi_1$; $\xi_2 = \cos\phi_2 \sin\phi_1; \ldots;$ $\xi_{n-2} = \cos\phi_{n-2} \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-3}$; $\xi_{n-1} = \sin\theta \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2}$; $\xi_n = \cos\theta \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2}$; where ϕ_i is the angle between x and e_i, $0 \leqslant \phi_i \leqslant \pi$, $j = 1, \ldots, n-2$ and $0 \leqslant \theta < 2\pi$.

If $f(x)$ is a continuous real-valued function defined in $\mathbb{B}^n$ which may be written in the form

$$f(\xi_1, \ldots, \xi_n) = g(\alpha_1 \xi_1 + \ldots + \alpha_n \xi_n,\ \xi_1^2 + \ldots + \xi_n^2),$$

where the constants α_i are independent of $\xi_1, \ldots, \xi_n$, then

$$(2.5) \qquad \int_{\mathbb{S}^{n-1}} f(x)\,\mathrm{d}\sigma(x) = \int_{\mathbb{S}^{n-1}} g(a \cdot x, |x|^2)\,\mathrm{d}\sigma(x)$$
$$= \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^{\pi} g(|a| \cos\phi_1, 1) \sin^{n-2}\phi_1\,\mathrm{d}\phi_1,$$

where $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, "$\cdot$" denotes the scalar product in $\mathbb{R}^n$, and σ is the surface measure on $\mathbb{S}^{n-1}$.

We use the conformal transformation that maps the unit ball $\mathbb{B}^n$ onto the half space $\mathbb{R}^n_+$. The continuous map

$$x \to x^*, \quad \text{where } x^* = \begin{cases} \dfrac{x}{|x|^2} & \text{if } x \neq 0, \infty, \\ 0 & \text{if } x = \infty, \\ \infty & \text{if } x = 0, \end{cases}$$

is called the inversion of $\mathbb{R}^n \cup \{\infty\}$ relative to the unit sphere. Here 0 denotes the origin in $\mathbb{R}^n$. This inversion is conformal on $\mathbb{R}^n \setminus \{0\}$. It maps spheres containing 0 onto hyperplanes and the interiors of such spheres onto open half-spaces.

Let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$. Consider $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(x) = 2(x - S)^* + S.$$

Some properties of Φ are (see [2], Proposition 7.18):

(1) $\Phi(\Phi(x)) = x$ for all $x \in \mathbb{R}^n \cup \{\infty\}$,

(2) Φ is conformal one-to-one map of $\mathbb{R}^n \setminus \{S\}$ onto $\mathbb{R}^n \setminus \{S\}$,

(3) Φ maps $\mathbb{B}^n$ onto $\mathbb{R}^n_+$ and $\mathbb{R}^n_+$ onto $\mathbb{B}^n$ whereas $\Phi(S) = \infty$, $\Phi(N) = 0$.

The modified Kelvin transform $\mathcal{K}$ that maps harmonic functions on $\mathbb{B}^n$ to harmonic functions on $\mathbb{R}^n_+$ and vice versa is a linear transform defined as:

$$\mathcal{K}f(x) = 2^{(n-2)/2}|x - S|^{2-n} f(\Phi(x)).$$

See [2], Proposition 7.19 for details.

3. Generalized Hölder spaces of harmonic functions in the unit ball $\mathbb{B}^n$ with prescribed modulus of continuity

Let $\omega\colon [0, 2] \to \mathbb{R}_+$ be a modulus of continuity. Here we consider the spaces $A^\omega(\mathbb{B}^n)$ and $B^\omega(\mathbb{B}^n)$ of complex-valued harmonic functions in $\mathbb{B}^n$.

By $A^\omega(\mathbb{B}^n)$ we denote the space of functions harmonic in $\mathbb{B}^n$ such that

$$|f(x) - f(y)| \leqslant C\omega(|x - y|), \quad x, y \in \mathbb{B}^n,$$

where C does not depend on x, y. The semi-norm and norm of a function $f \in A^\omega(\mathbb{B}^n)$ are given by

$$\|f\|_{\#, A^\omega(\mathbb{B}^n)} = \sup_{x, y \in \mathbb{B}^n} \frac{|f(x) - f(y)|}{\omega(|x - y|)} \quad \text{and} \quad \|f\|_{A^\omega(\mathbb{B}^n)} = \|f\|_{\#, A^\omega(\mathbb{B}^n)} + \|f\|_{L^\infty(\mathbb{B}^n)},$$

respectively.

Since ω is a modulus of continuity, it follows that any $f \in A^\omega(\mathbb{B}^n)$ is continuous in $\overline{\mathbb{B}^n}$. This implies that

$$|f(u) - f(v)| \leqslant C\omega(|u - v|), \quad u, v \in \mathbb{S}^{n-1},$$

where C does not depend on u, v.

By $B^\omega(\mathbb{B}^n)$ we denote the space of functions harmonic in $\mathbb{B}^n$ such that

$$|\nabla f(x)| \leqslant C\frac{\omega(1 - |x|)}{1 - |x|}, \quad x \in \mathbb{B}^n,$$

where C does not depend on x and

$$\nabla f = \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big).$$

The semi-norm and norm of a function $f \in B^\omega(\mathbb{B}^n)$ are given by

$$\|f\|_{\#,B^\omega(\mathbb{B}^n)} = \sup_{x \in \mathbb{B}^n} |\nabla f(x)| \frac{1 - |x|}{\omega(1 - |x|)}, \quad \|f\|_{B^\omega(\mathbb{B}^n)} = \|f\|_{\#,B^\omega(\mathbb{B}^n)} + \|f\|_{L^\infty(\mathbb{B}^n)}.$$

Our first result provides the relation between spaces $A^\omega(\mathbb{B}^n)$ and $B^\omega(\mathbb{B}^n)$. The symbol $\hookrightarrow$ is used for the continuous inclusion between spaces. Recall that the Poisson kernel for the unit ball $\mathbb{B}^n$ is given by

$$P(x, t) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{1 - |x|^2}{|x - t|^n}, \quad x \in \mathbb{B}^n, \ t \in \mathbb{S}^{n-1},$$

see, e.g., formula 1.15 of Chapter 1 in [2].

Theorem 3.1. *The following statements are true.*

(1) *Let ω satisfy the condition* (2.2), *then $A^\omega(\mathbb{B}^n) \hookrightarrow B^\omega(\mathbb{B}^n)$ and $\|f\|_{\#,B^\omega(\mathbb{B}^n)} \leqslant C\|f\|_{\#,A^\omega(\mathbb{B}^n)}$, where C does not depend on f.*

(2) *Let ω satisfy the condition* (2.1), *then $B^\omega(\mathbb{B}^n) \hookrightarrow A^\omega(\mathbb{B}^n)$ and $\|f\|_{\#,A^\omega(\mathbb{B}^n)} \leqslant C\|f\|_{\#,B^\omega(\mathbb{B}^n)}$, where C does not depend on f.*

Proof. Let us prove the first statement. Let $f \in A^\omega(\mathbb{B}^n)$ and ω satisfy (2.2). By the Poisson integral representation we obtain

$$(3.1) \quad f(x) = \int_{\mathbb{S}^{n-1}} P(x, t) f(t) \, \mathrm{d}\sigma(t) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{\mathbb{S}^{n-1}} \frac{1 - |x|^2}{|x - t|^n} f(t) \, \mathrm{d}\sigma(t), \quad x \in \mathbb{B}^n.$$

Now, if $x = (x_1, \ldots, x_n)$ and $t = (t_1, \ldots, t_n)$, it is easy to see that

$$\frac{\partial P(x,t)}{\partial x_i} = \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{-2x_i|t-x|^n + n(1-|x|^2)|t-x|^{n-2}(t_i - x_i)}{|t-x|^{2n}}, \quad i = 1, \ldots, n.$$

For $t \in \mathbb{S}^{n-1}$ and $x \in \mathbb{B}^n$, we get $1 - |x| \leqslant |t-x|$ and $|t_i - x_i| \leqslant |t-x|$, hence

$$\left|\frac{\partial P(x,t)}{\partial x_i}\right| \leqslant \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{2(1+n)}{|x-t|^n}, \quad i = 1, \ldots, n. \tag{3.2}$$

Since $\int_{\mathbb{S}^{n-1}} P(x,t)\,\mathrm{d}\sigma(t) = 1$, we have

$$\int_{\mathbb{S}^{n-1}} \frac{\partial P(x,t)}{\partial x_i}\,\mathrm{d}\sigma(t) = 0, \quad i = 1, \ldots, n.$$

Hence, from (3.1) we obtain

$$\frac{\partial f(x)}{\partial x_i} = \int_{\mathbb{S}^{n-1}} \frac{\partial P(x,t)}{\partial x_i}(f(t) - f(x'))\,\mathrm{d}\sigma(t), \quad i = 1, \ldots, n, \tag{3.3}$$

where $x' = x/|x| \in \mathbb{S}^{n-1}$. Thus, by (3.2)

$$\begin{aligned} |\nabla f(x)| &\leqslant C \int_{\mathbb{S}^{n-1}} \frac{|f(t) - f(x')|}{|t-x|^n}\,\mathrm{d}\sigma(t) \\ &\leqslant C_1 \|f\|_{\#, A^\omega(\mathbb{B}^n)} \int_{\mathbb{S}^{n-1}} \frac{\omega(|t-x'|)}{|t-x|^n}\,\mathrm{d}\sigma(t), \end{aligned} \tag{3.4}$$

where C, C_1 do not depend on f.

Split $\mathbb{S}^{n-1} = D_{|x|} \cup E_{|x|}$, where $E_{|x|} := \{t \in \mathbb{S}^{n-1} \colon |t - x'| > 1 - |x|\}$ and $D_{|x|} := \{t \in \mathbb{S}^{n-1} \colon |t-x'| \leqslant 1 - |x|\}$. Since ω is almost increasing on $[0,2]$ we obtain

$$\int_{D_{|x|}} \frac{\omega(|t-x'|)}{|t-x|^n}\,\mathrm{d}\sigma(t) \leqslant C\omega(1-|x|) \int_{\mathbb{S}^{n-1}} \frac{\mathrm{d}\sigma(t)}{|t-x|^n} = C_1 \frac{\omega(1-|x|)}{1-|x|^2} \leqslant C_1 \frac{\omega(1-|x|)}{1-|x|}.$$

Further, since $|t-x'|/|t-x| \leqslant 2$ ($t \in \mathbb{S}^{n-1}, x \in \mathbb{B}^n$) and due to the Funk-Hecke type formula (2.5) we have

$$\begin{aligned} \int_{E_{|x|}} \frac{\omega(|t-x'|)}{|t-x|^n}\,\mathrm{d}\sigma(t) &\leqslant 2^n \int_{E_{|x|}} \frac{\omega(|t-x'|)}{|t-x'|^n}\,\mathrm{d}\sigma(t) \\ &= \frac{2^{n+1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_{\varphi_{|x|}}^{\pi} \frac{\omega(\sqrt{2-2\cos\phi_1})}{(\sqrt{2-2\cos\phi_1})^n} \sin^{n-2}\phi_1\,\mathrm{d}\phi_1, \end{aligned}$$

where $\varphi_{|x|} = 2\arcsin((1-|x|)/2)$ and $[\varphi_{|x|}, \pi] \subset [0,\pi]$. If $\varrho = \sqrt{2-2\cos\phi_1}$, we see that $\varrho \in (1-|x|, 2]$ and $\varrho\,\mathrm{d}\varrho = \sin\phi_1\,\mathrm{d}\phi_1$. Note also that $\varrho^4/4 = \varrho^2 - \sin^2\phi_1$, hence $\sin^2\varphi_1 \leqslant \varrho^2$. Then by (2.2)

$$\int_{E_{|x|}} \frac{\omega(|t-x'|)}{|t-x|^n}\,\mathrm{d}\sigma(t) \leqslant \frac{2^{n+1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_{1-|x|}^{2} \frac{\omega(\varrho)}{\varrho^2}\,\mathrm{d}\varrho \leqslant C\frac{\omega(1-|x|)}{1-|x|}.$$

Finally, by (3.4) and the above estimates we arrive at

$$|\nabla f(x)| \leqslant C\|f\|_{\#,A^\omega(\mathbb{B}^n)} \frac{\omega(1-|x|)}{1-|x|}, \tag{3.5}$$

where C does not depend on f. This proves the first statement.

Let us prove the second statement. Let $f \in B^\omega(\mathbb{B}^n)$ and ω satisfy (2.1). Without loss of generality we assume that $|x| \leqslant |y|$ $(x, y \in \mathbb{B}^n)$. Put

$$\overline{x} = (1-|x-y|)x', \quad \overline{y} = (1-|x-y|)y', \quad \text{where } x' = \frac{x}{|x|},\ y' = \frac{y}{|y|}.$$

It is clear that $\overline{x}, \overline{y} \in \mathbb{B}^n$. Let $h(s) = x - s(x-y)$, $0 \leqslant s \leqslant 1$, be the line segment between x and y. We have

$$\begin{aligned} |f(x)-f(y)| &\leqslant |x-y| \int_0^1 \left|\frac{\partial f(h(s))}{\partial s}\right| \mathrm{d}s \leqslant 2|x-y| \int_0^1 |\nabla f(sx+(1-s)y)|\,\mathrm{d}s \\ &\leqslant 2|x-y|\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_0^1 \frac{\omega(1-|xs+(1-s)y|)}{1-|xs+(1-s)y|}\,\mathrm{d}s. \end{aligned} \tag{3.6}$$

Now we split the rest of the proof into three cases.

The first case: $|y| + |x-y| \leqslant 1$. Since $\omega(t)/t$ is almost decreasing, using inequality (3.6) and Zygmund type condition (2.1) we obtain

$$\begin{aligned} &|f(x)-f(y)| \\ &\quad\leqslant C|x-y|\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_0^1 \frac{\omega(|x-y|(1-|y|)/(|x-y|-s))}{|x-y|(1-|y|)/(|x-y|-s)}\,\mathrm{d}s \\ &\quad\leqslant C_1\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_0^1 \frac{\omega(|x-y|(1-s))}{1-s}\,\mathrm{d}s = C_1\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_0^{|x-y|} \frac{\omega(u)}{u}\,\mathrm{d}u \\ &\quad\leqslant C_2\|f\|_{\#,B^\omega(\mathbb{B}^n)}\omega(|x-y|), \end{aligned}$$

where C, C_1, and C_2 do not depend on f.

For the second case, where $1-|y| < |x-y| \leqslant 1-|x|$, we have that

$$|f(x)-f(y)| \leqslant |f(x)-f(\overline{y})| + |f(\overline{y})-f(y)|.$$

Since $|x-\overline{y}| \leqslant |x-y|$, the first term on the right-hand side above can be estimated as in the first case, while the second term in view of (3.6) is estimated as

$$\begin{aligned}
|f(\overline{y})-f(y)| &\leqslant 2|\overline{y}-y|\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_0^1 \frac{\omega(1-|ys+(1-s)\overline{y}|)}{1-|ys+(1-s)\overline{y}|}\,\mathrm{d}s \\
&\leqslant C|\overline{y}-y|\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_0^1 \frac{\omega(1-(|y|s+(1-s)(1-|x-y|)))}{1-(|y|s+(1-s)(1-|x-y|))}\,\mathrm{d}s \\
&\leqslant C\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_{1-|x-y|}^{|y|} \frac{\omega(1-u)}{1-u}\,\mathrm{d}u = C\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_{1-|y|}^{|x-y|} \frac{\omega(x)}{x}\,\mathrm{d}x \\
&\leqslant C\|f\|_{\#,B^\omega(\mathbb{B}^n)} \int_0^{|x-y|} \frac{\omega(x)}{x}\,\mathrm{d}x \leqslant C_1\|f\|_{\#,B^\omega(\mathbb{B}^n)}\omega(|x-y|),
\end{aligned}$$

where we again used that $\omega(t)/t$ is almost decreasing and applied Zygmund type condition (2.1). Here C, C_1 do not depend on f.

Finally, consider the last (third) case $1-|x| < |x-y|$. In this case we have $|\overline{x}-\overline{y}| \leqslant |x-y|$ and

$$|f(x)-f(y)| \leqslant |f(x)-f(\overline{x})| + |f(\overline{x})-f(\overline{y})| + |f(y)-f(\overline{y})|.$$

The first and third terms on the right-hand side above can be estimated straightforwardly like in the second case, while the second term is estimated like in the first case. Thus the proof is completed. □

Corollary 3.1. *If ω satisfies the conditions* (2.1) *and* (2.2), *then the spaces $B^\omega(\mathbb{B}^n)$ and $A^\omega(\mathbb{B}^n)$ coincide up to the equivalence of norms.*

Theorem 3.2. *Let ω satisfy the conditions* (2.1) *and* (2.2). *Let f be harmonic in $\mathbb{B}^n$. Then $f \in A^\omega(\mathbb{B}^n)$ if and only if*

(1) *f is continuous in $\overline{\mathbb{B}^n}$,*

(2) *$|f(\tau)-f(\sigma)| \leqslant C\omega(|\tau-\sigma|)$, $\tau,\sigma \in \mathbb{S}^{n-1}$, where C is independent of τ, σ.*

Proof. Obviously, $f \in A^\omega(\mathbb{B}^n)$ implies the conditions (1) and (2). To prove the inverse implication we use the formula (3.4) and the arguments given after this formula to show that the conditions (1), (2) imply that $f \in B^\omega(\mathbb{B}^n)$, and then apply Corollary 3.1. □

We conclude this section with the particular case $n=2$. It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and use complex variables z, w as elements of $\mathbb{D}$.

By $A^\omega_\star(\mathbb{D})$ we denote the space of functions harmonic in $\mathbb{D}$ such that

$$|f(z)-f(w)| \leqslant C\omega(|1-z\overline{w}|), \quad z,w \in \mathbb{D},$$

where C does not depend on z, w. Since ω is a modulus of continuity, it follows that any $f \in A^{\omega}_{\star}(\mathbb{D})$ is continuous in $\overline{\mathbb{D}}$. The semi-norm and norm of a function $f \in A^{\omega}_{\star}(\mathbb{D})$ are given by

$$\|f\|_{\#, A^{\omega}_{\star}(\mathbb{D})} = \sup_{z,w\in\mathbb{D}} \frac{|f(z)-f(w)|}{\omega(|1-z\overline{w}|)}, \quad \|f\|_{A^{\omega}_{\star}(\mathbb{D})} = \|f\|_{\#, A^{\omega}_{\star}(\mathbb{D})} + \|f\|_{L^{\infty}(\mathbb{D})}.$$

Theorem 3.3. *Let ω satisfy the conditions* (2.1) *and* (2.2). *Let f be harmonic in $\mathbb{D}$. Then $A^{\omega}(\mathbb{D})$ and $A^{\omega}_{\star}(\mathbb{D})$ coincide up to the equivalence of norms.*

P r o o f. Indeed, let $f \in A^{\omega}(\mathbb{D})$. Since

$$\left|\frac{z-w}{1-z\overline{w}}\right| \leqslant 1$$

and ω is almost increasing we get that $\omega(|z-w|) \leqslant C\omega(|1-z\overline{w}|)$, hence $f \in A^{\omega}_{\star}(\mathbb{D})$. On the other hand, since f is continuous in $\overline{\mathbb{D}}$ and $f \in A^{\omega}_{\star}(\mathbb{D})$, one can see that $|f(\tau)-f(\sigma)| \leqslant C\omega(|1-\sigma\overline{\tau}|) = C\omega(|\tau-\sigma|)$, $\tau, \sigma \in \mathbb{T}$, hence, the desired statement follows by Theorem 3.2. □

4. Variable exponent generalized Hölder spaces $A^{\lambda(\cdot)}(\mathbb{B}^n)$ of harmonic functions in the unit ball $\mathbb{B}^n$

Let $\lambda\colon \mathbb{B}^n \to [0,1]$ be a continuous function satisfying the log-condition (2.3) in $\mathbb{B}^n$. By $A^{\lambda(\cdot)}(\mathbb{B}^n)$ we denote the space of functions f harmonic in $\mathbb{B}^n$ such that

$$|f(x)-f(y)| \leqslant C|x-y|^{\lambda(x)} \quad \forall\, x,y \in \mathbb{B}^n,$$

or, which is the same,

$$|f(x)-f(y)| \leqslant C|x-y|^{\lambda(y)} \quad \forall\, x,y \in \mathbb{B}^n,$$

where C does not depend on $x, y \in \mathbb{B}^n$. The semi-norm and norm of a function $f \in A^{\lambda(\cdot)}(\mathbb{B}^n)$ are given by

$$\|f\|_{\#, A^{\lambda(\cdot)}(\mathbb{B}^n)} = \sup_{x,y\in\mathbb{B}^n} \frac{|f(x)-f(y)|}{|x-y|^{\lambda(x)}}, \quad \|f\|_{A^{\lambda(\cdot)}(\mathbb{B}^n)} = \|f\|_{\#, A^{\lambda(\cdot)}(\mathbb{B}^n)} + \|f\|_{L^{\infty}(\mathbb{B}^n)}.$$

By $B^{\lambda(\cdot)}(\mathbb{B}^n)$ we denote the space of functions f harmonic in $\mathbb{B}^n$ such that

$$|\nabla f(x)| \leqslant C(1-|x|)^{\lambda(x)-1}, \quad x \in \mathbb{B}^n,$$

where C does not depend on x. The semi-norm and norm of a function $f \in B^{\lambda(\cdot)}(\mathbb{B}^n)$ are given by

$$\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} = \sup_{x\in\mathbb{B}^n} |\nabla f(x)|(1-|x|)^{1-\lambda(x)},$$
$$\|f\|_{B^{\lambda(\cdot)}(\mathbb{B}^n)} = \|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} + \|f\|_{L^\infty(\mathbb{B}^n)}.$$

Theorem 4.1. *Let λ satisfy the log-condition (2.3). The following statements hold.*

(1) *If $\sup\limits_{x\in\mathbb{B}^n} \lambda(x) < 1$, then $A^{\lambda(\cdot)}(\mathbb{B}^n) \hookrightarrow B^{\lambda(\cdot)}(\mathbb{B}^n)$ and*

$$\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} \leqslant C\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{B}^n)},$$

where C does not depend on f.

(2) *If $\inf\limits_{x\in\mathbb{B}^n} \lambda(x) > 0$, then $B^{\lambda(\cdot)}(\mathbb{B}^n) \hookrightarrow A^{\lambda(\cdot)}(\mathbb{B}^n)$ and*

$$\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{B}^n)} \leqslant C\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)},$$

where C does not depend on f.

Proof. The proof is similar to the proof of Theorem 3.1 with some changes. We provide the sketch of the proof. We have

$$\begin{aligned}(4.1)\qquad |\nabla f(x)| &\leqslant C\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{B}^n)} \int_{\mathbb{S}^{n-1}} \frac{|t-x'|^{\lambda(x')}}{|t-x|^n}\,\mathrm{d}\sigma(t)\\ &\leqslant C_1\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{B}^n)} \int_{\mathbb{S}^{n-1}} \frac{|t-x'|^{\lambda(x)}}{|t-x|^n}\,\mathrm{d}\sigma(t), \quad x\in\mathbb{B}^n,\end{aligned}$$

and splitting $\mathbb{S}^{n-1} = E_{|x|} \cup D_{|x|}$, we obtain

$$\begin{aligned}\int_{D_{|x|}} \frac{|t-x'|^{\lambda(x)}}{|t-x|^n}\,\mathrm{d}\sigma(t) &\leqslant C\frac{(1-|x|)^{\lambda(x)}}{1-|x|^2} \leqslant C(1-|x|)^{\lambda(x)-1},\\ \int_{E_{|x|}} \frac{|t-x'|^{\lambda(x)}}{|t-x|^n}\,\mathrm{d}\sigma(t) &\leqslant 2^n \int_{E_{|x|}} \frac{\mathrm{d}\sigma(t)}{|t-x'|^{n-\lambda(x)}} \leqslant \frac{2^{2n+1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_{1-|x|}^{2} \frac{\mathrm{d}\varrho}{\varrho^{2-\lambda(x)}}\\ &\leqslant \frac{(1-|x|)^{\lambda(x)-1}}{1-\lambda(x)} \leqslant C_1(1-|x|)^{\lambda(x)-1}.\end{aligned}$$

Here we have used that $\sup\limits_{x\in\mathbb{B}^n} \lambda(x) < 1$. The first statement is proved.

To prove the second statement, similarly to (3.6) we arrive at

$$\begin{aligned}(4.2)\qquad |f(x)-f(y)| &\leqslant 2|x-y| \int_0^1 |\nabla f(sx+(1-s)y)|\,\mathrm{d}s\\ &\leqslant 2|x-y|\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} \int_0^1 (1-|xs+(1-s)y|)^{\lambda(x)-1}\,\mathrm{d}s.\end{aligned}$$

Here again we have to deal with three cases. In the first case $|y| + |x-y| \leqslant 1$, by using (4.2) we obtain

$$\begin{aligned}|f(x) - f(y)| &\leqslant 2|x-y|\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} \int_0^1 (|x-y|(1-s))^{\lambda(x)-1}\,\mathrm{d}s \\ &= 2\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} \frac{|x-y|^{\lambda(x)}}{\lambda(x)} \leqslant \frac{2}{\lambda_0}\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)}|x-y|^{\lambda(x)},\end{aligned}$$

where $\lambda_0 = \inf\limits_{x\in\mathbb{B}^n} \lambda(x) > 0$. Now, in the second case, $1-|y| < |x-y| \leqslant 1-|x|$, we have

$$|f(x) - f(y)| \leqslant |f(x) - f(\overline{y})| + |f(\overline{y}) - f(y)|.$$

Since $|x-\overline{y}| \leqslant |x-y|$, the first term on the right-hand side above is covered by the first case, while the second term is estimated by the use of (4.2),

$$\begin{aligned}|f(\overline{y}) - f(y)| &\leqslant 2|\overline{y}-y|\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} \int_0^1 (1-|ys+(1-s)\overline{y}|)^{\lambda(x)-1}\,\mathrm{d}s \\ &\leqslant 2\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} \int_{1-|x-y|}^{|y|} (1-u)^{\lambda(x)-1}\,\mathrm{d}u \\ &\leqslant 2\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)} \int_0^{|x-y|} t^{\lambda(x)-1}\,\mathrm{d}t \leqslant \frac{2}{\lambda_0}\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{B}^n)}|x-y|^{\lambda(x)}.\end{aligned}$$

Finally, the last (third) case, where $1-|x| < |x-y|$, follows by the same arguments as those in Theorem 3.1. □

Corollary 4.1. *Let* λ *satisfy the log-condition* (2.3) *and*

$$0 < \inf_{x\in\mathbb{B}^n} \lambda(x) \leqslant \sup_{x\in\mathbb{B}^n} \lambda(x) < 1, \tag{4.3}$$

then the spaces $B^{\lambda(\cdot)}(\mathbb{B}^n)$ *and* $A^{\lambda(\cdot)}(\mathbb{B}^n)$ *coincide up to the equivalence of norms.*

Theorem 4.2. *Let* λ *satisfy the log-condition* (2.3) *and the condition* (4.3). *Let* f *be harmonic in* $\mathbb{B}^n$. *Then* $f \in A^{\lambda(\cdot)}(\mathbb{B}^n)$ *if and only if*

(1) f *is continuous in* $\overline{\mathbb{B}^n}$,

(2) $|f(\tau) - f(\sigma)| \leqslant C|\tau-\sigma|^{\lambda(\tau)}$, $\tau, \sigma \in \mathbb{S}^{n-1}$, *where* C *is independent of* τ, σ.

P r o o f. It is clear that if $f \in A^{\lambda(\cdot)}(\mathbb{B}^n)$, then the conditions (1) and (2) are satisfied. Now, the condition (2) yields the inequality (4.1), hence $f \in B^{\lambda(\cdot)}(\mathbb{B}^n)$. By Corollary 4.1 we obtain that $f \in A^{\lambda(\cdot)}(\mathbb{B}^n)$. □

Finally, consider the particular case $n = 2$. It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and use complex variables z, w as elements of $\mathbb{D}$.

By $A_*^{\lambda(\cdot)}(\mathbb{D})$ we denote the space of functions f harmonic in $\mathbb{D}$ such that

$$|f(z) - f(w)| \leqslant C|1 - z\overline{w}|^{\lambda(z)} \quad \forall\, z, w \in \mathbb{D},$$

or, which is the same,

$$|f(z) - f(w)| \leqslant C|1 - z\overline{w}|^{\lambda(w)} \quad \forall\, z, w \in \mathbb{D},$$

where C does not depend on z, w. Obviously, any $f \in A_*^{\omega}(\mathbb{D})$ is continuous in $\overline{\mathbb{D}}$. The semi-norm and norm of a function $f \in A_*^{\lambda(\cdot)}(\mathbb{D})$ are given by

$$\|f\|_{\#, A_*^{\lambda(\cdot)}(\mathbb{D})} = \sup_{z,w\in\mathbb{D}} \frac{|f(z) - f(w)|}{|1 - z\overline{w}|^{\lambda(z)}}, \quad \|f\|_{A_*^{\lambda(\cdot)}(\mathbb{D})} = \|f\|_{\#, A_*^{\lambda(\cdot)}(\mathbb{D})} + \|f\|_{L^\infty(\mathbb{D})}.$$

Theorem 4.3. *Let λ satisfy the log-condition* (2.3) *and*

$$0 < \inf_{z\in\mathbb{D}} \lambda(z) \leqslant \sup_{z\in\mathbb{D}} \lambda(z) < 1.$$

Let f be harmonic in $\mathbb{D}$. Then $A^{\lambda(\cdot)}(\mathbb{D})$ and $A_*^{\lambda(\cdot)}(\mathbb{D})$ coincide up to the equivalence of norms.

P r o o f. The proof is similar to the proof of Theorem 3.3. □

5. Hölder type spaces of harmonic functions on the half space $\mathbb{R}^n_+$

5.1. Generalized Hölder spaces of harmonic functions on the half space $\mathbb{R}^n_+$ with prescribed modulus of continuity. Let $\omega\colon\ [0, 2] \to \mathbb{R}_+$ be a modulus of continuity. Put

$$H(x) = |x - S|^{n-2} = 2^{n-2}|\Phi(x) - S|^{2-n}, \quad x \in \mathbb{R}^n_+,$$

where Φ is the conformal transformation defined in Section 2 and $S = (0, \dots, 0, -1)$.

Here we consider the spaces $A^\omega(\mathbb{R}^n_+)$ and $B^\omega(\mathbb{R}^n_+)$ of complex-valued harmonic functions in $\mathbb{R}^n_+$ with the weight H.

By $A^\omega(\mathbb{R}^n_+)$ we denote the space of functions harmonic in $\mathbb{R}^n_+$ such that

$$|H(x)f(x) - H(y)f(y)| \leqslant C\omega(\alpha(x, y))$$

for all $x, y \in \mathbb{R}^n_+$, where C does not depend on x, y. The semi-norm and norm of a function $f \in A^\omega(\mathbb{R}^n_+)$ are given by

$$\|f\|_{\#,A^\omega(\mathbb{R}^n_+)} = \sup_{x,y\in\mathbb{R}^n_+} \frac{|H(x)f(x) - H(y)f(y)|}{\omega(\alpha(x,y))}$$

and

$$\|f\|_{A^\omega(\mathbb{R}^n_+)} = \|f\|_{\#,A^\omega(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)},$$

respectively. Since ω is a modulus of continuity, it follows that any $f \in A^\omega(\mathbb{R}^n_+)$ is continuous in $\overline{\mathbb{R}^n_+}$. This implies that

$$|H(u)f(u) - H(v)f(v)| \leqslant C\omega(\alpha(u,v))$$

for all $u, v \in \mathbb{R}^{n-1} \times \{0\}$, where C does not depend on u, v.

By $B^\omega(\mathbb{R}^n_+)$ we denote the space of functions harmonic in $\mathbb{R}^n_+$ such that

$$\bigl|\nabla\bigl(H(x)f(x)\bigr)\bigr| \leqslant \frac{C}{x_n}\omega\Bigl(\frac{x_n}{|x-S|^2}\Bigr), \quad x \in \mathbb{R}^n_+,$$

where C does not depend on x and x_n is the nth component of x. The semi-norm and norm of a function $f \in B^\omega(\mathbb{R}^n_+)$ are given by

$$\|f\|_{\#,B^\omega(\mathbb{R}^n_+)} = \sup_{x\in\mathbb{R}^n_+}\Bigl(|\nabla(H(x)f(x))|\frac{x_n}{\omega(x_n/|x-S|^2)}\Bigr),$$
$$\|f\|_{B^\omega(\mathbb{R}^n_+)} = \|f\|_{\#,B^\omega(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}.$$

Theorem 5.1. *The following statements are true.*

(1) *Let ω satisfy the condition* (2.2), *then $A^\omega(\mathbb{R}^n_+) \hookrightarrow B^\omega(\mathbb{R}^n_+)$ and $\|f\|_{\#,B^\omega(\mathbb{R}^n_+)} \leqslant C\|f\|_{\#,A^\omega(\mathbb{R}^n_+)}$, where C does not depend on f.*

(2) *Let ω satisfy the condition* (2.1), *then $B^\omega(\mathbb{R}^n_+) \hookrightarrow A^\omega(\mathbb{R}^n_+)$ and $\|f\|_{\#,A^\omega(\mathbb{R}^n_+)} \leqslant C\|f\|_{\#,B^\omega(\mathbb{R}^n_+)}$, where C does not depend on f.*

P r o o f. The proof is based on Theorem 3.1. A function f belongs to $A^\omega(\mathbb{R}^n_+)$ if and only if $\mathcal{K}f$ belongs to $A^\omega(\mathbb{B}^n)$ with the equivalence of the corresponding semi-norms. To verify the above said we take into account the properties (1) and (3) of the function Φ, the following equation, which can be checked directly,

$$|(x-S)|y-S|^2 - (y-S)|x-S|^2| = |x-S||y-S||x-y| \tag{5.1}$$

valid for any $x, y \in \mathbb{B}^n$ or $x, y \in \mathbb{R}^n_+$, and the doubling property of ω.

A similar relation can be proved for the functions in spaces $B^\omega(\mathbb{R}^n_+)$ and $B^\omega(\mathbb{B}^n)$. Indeed, suppose that $f = \mathcal{K}g$ and $g \in B^\omega(\mathbb{B}^n)$. Let $x \in \mathbb{R}^n_+, y = \Phi(x) \in \mathbb{B}^n$, and $\nabla_x\Phi$ stand for the matrix whose ith row is $\nabla_x\Phi_i$. Each component of $\nabla_x\Phi(x)$ is estimated by $C/|x-S|^2$ with some absolute $C > 0$. We obtain

$$\begin{aligned}|\nabla_x(H(x)f(x))| &= 2^{(n-2)/2}|\nabla_x g(\Phi(x))| \leqslant 2^{(n-2)/2}|\nabla_y g(y)|_{y=\Phi(x)}\nabla_x\Phi(x)| \\ &\leqslant C\frac{\omega(1-|\Phi(x)|)}{1-|\Phi(x)|}\frac{1}{|x-S|^2} \leqslant C_1\frac{\omega(1-|\Phi(x)|^2)}{1-|\Phi(x)|^2}\frac{1}{|x-S|^2} \\ &= C_1\frac{1}{4x_n}\omega\Big(\frac{4x_n}{|x-S|^2}\Big) \leqslant C_2\frac{1}{x_n}\omega\Big(\frac{x_n}{|x-S|^2}\Big).\end{aligned}$$

Here we used the facts that ω is almost increasing on $[0,2]$ and $1+|\Phi(x)| \leqslant 2$ for any $x \in \mathbb{R}^n_+$.

Suppose that $f = \mathcal{K}g$ and $g \in B^\omega(\mathbb{R}^n_+)$. We have for $x \in \mathbb{B}^n$, $y = \Phi(x) \in \mathbb{R}^n_+$:

$$\begin{aligned}|\nabla_x f(x)| &= |\nabla_x\mathcal{K}g(x)| = 2^{(n-2)/2}|\nabla_x|x-S|^{2-n}g(\Phi(x))| \\ &= 2^{(n-2)/2}|\nabla_x H(\Phi(x))g(\Phi(x))| = |\nabla_y H(y)g(y)|_{y=\Phi(x)}\nabla_x\Phi(x)| \\ &\leqslant \frac{C}{|x-S|^2}\frac{1}{2(x_n+1)/|x-S|^2-1}\omega\Big(\frac{2(x_n+1)/|x-S|^2-1}{|\Phi(x)-S|^2}\Big) \\ &= C\frac{\omega((1-|x|^2)/4)}{1-|x|^2} \leqslant C_1\frac{\omega(1-|x|)}{1-|x|}.\end{aligned}$$

□

Corollary 5.1. *If ω satisfies the conditions* (2.1) *and* (2.2), *then the spaces $B^\omega(\mathbb{R}^n_+)$ and $A^\omega(\mathbb{R}^n_+)$ coincide up to the equivalence of norms.*

5.2. Variable exponent generalized Hölder spaces $A^{\lambda(\cdot)}(\mathbb{R}^n_+)$ of harmonic functions on the half space $\mathbb{R}^n_+$. Let $\lambda\colon \mathbb{R}^n_+ \to [0,1]$ satisfy the global log-condition (2.4).

By $A^{\lambda(\cdot)}(\mathbb{R}^n_+)$ we denote the space of functions f harmonic in $\mathbb{R}^n_+$ such that

$$|H(x)f(x) - H(y)f(y)| \leqslant C\alpha(x,y)^{\lambda(x)} \quad \forall\, x,y \in \mathbb{R}^n_+,$$

or, which is the same,

$$|H(x)f(x) - H(y)f(y)| \leqslant C\alpha(x,y)^{\lambda(y)} \quad \forall\, x,y \in \mathbb{R}^n_+,$$

where C does not depend on x, y. The semi-norm and norm of a function $f \in A^{\lambda(\cdot)}(\mathbb{R}^n_+)$ are given by

$$\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{R}^n_+)} = \sup_{x,y\in\mathbb{R}^n_+}\frac{|H(x)f(x)-H(y)f(y)|}{\alpha(x,y)^{\lambda(x)}}$$

and

$$\|f\|_{A^{\lambda(\cdot)}(\mathbb{R}^n_+)} = \|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}.$$

By $B^{\lambda(\cdot)}(\mathbb{R}^n_+)$ we denote the space of functions f harmonic in $\mathbb{B}^n$ such that

$$|\nabla\bigl(H(x)f(x)\bigr)| \leqslant \frac{C}{x_n}\Bigl(\frac{x_n}{|x-S|^2}\Bigr)^{\lambda(x)}, \quad x \in \mathbb{R}^n_+,$$

where C does not depend on x. The semi-norm and norm of a function $f \in B^{\lambda(\cdot)}(\mathbb{R}^n_+)$ are given by

$$\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{R}^n_+)} = \sup_{x\in\mathbb{R}^n_+} |\nabla(H(x)f(x))| x_n \Bigl(\frac{|x-S|^2}{x_n}\Bigr)^{\lambda(x)}$$

and

$$\|f\|_{B^{\lambda(\cdot)}(\mathbb{R}^n_+)} = \|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}.$$

Theorem 5.2. *Let λ satisfy the log-condition* (2.4). *The following statements hold.*

(1) *If $\sup\limits_{x\in\mathbb{R}^n_+} \lambda(x) < 1$, then $A^{\lambda(\cdot)}(\mathbb{R}^n_+) \hookrightarrow B^{\lambda(\cdot)}(\mathbb{R}^n_+)$ and*

$$\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{R}^n_+)} \leqslant C\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{R}^n_+)},$$

where C does not depend on f.

(2) *If $\inf\limits_{x\in\mathbb{R}^n_+} \lambda(x) > 0$, then $B^{\lambda(\cdot)}(\mathbb{R}^n_+) \hookrightarrow A^{\lambda(\cdot)}(\mathbb{R}^n_+)$ and*

$$\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{R}^n_+)} \leqslant C\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{R}^n_+)},$$

where C does not depend on f.

P r o o f. The proof is similar to the proof of Theorem 5.1 and is based on Theorem 4.1. We note that λ satisfies the log-condition (2.4) on $\mathbb{R}^n_+$ if and only if $\lambda \circ \Phi$ satisfies the log-condition (2.3) on $\mathbb{B}^n$. □

Corollary 5.2. *Let λ satisfy the log-condition* (2.4) *and*

$$0 < \inf_{x\in\mathbb{R}^n_+} \lambda(x) \leqslant \sup_{x\in\mathbb{R}^n_+} \lambda(x) < 1.$$

Then the spaces $B^{\lambda(\cdot)}(\mathbb{R}^n_+)$ and $A^{\lambda(\cdot)}(\mathbb{R}^n_+)$ coincide up to the equivalence of norms.

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