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ANNIHILATORS OF SKEW DERIVATIONS WITH ENGEL CONDITIONS ON PRIME RINGS

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Abstract. Let R be a noncommutative prime ring of characteristic different from 2, with its two-sided Martindale quotient ring Q, C the extended centroid of R and $a \in R$. Suppose that δ is a nonzero σ-derivation of R such that $a[\delta(x^n), x^n]_k = 0$ for all $x \in R$, where σ is an automorphism of R, n and k are fixed positive integers. Then $a = 0$.

Keywords: prime ring; derivation; skew derivation; automorphism

MSC 2010: 16W20, 16W25

1. Introduction

Throughout this paper, unless specially stated, R always denotes an associative prime ring of characteristic different from 2, with extended centroid C and two-sided Martindale quotient ring Q. The definitions, the axiomatic formulations and the properties of these objects can be found in Beidar et al. [3]. For $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. Notice that an Engel condition is a polynomial $[x, y]_k = \sum\limits_{i=0}^{k} (-1)^i \binom{k}{i} y^i x y^{k-i}$ for all noncommutative indeterminates x, y. The ring R satisfies an Engel condition if there exists a positive integer k such that $[x, y]_k = 0$ for all $x, y \in R$. For a subset S of R, a mapping $f\colon S \to R$ is said to be commuting or centralizing on S if $[f(x), x] = 0$ or $[f(x), x] \in Z(R)$, respectively, for all $x \in S$. An additive mapping $d\colon R \to R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Also an additive mapping $g\colon R \to R$ is called a generalized derivation of R if $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation from R into itself. Basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R to itself. An important example is a map of the form $g(x) = ax + xb$ for some $a, b \in R$,

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and this generalized derivation is called an inner generalized derivation. Let R be an associative ring and σ an automorphism of R. By a skew derivation on R we mean an additive map $\delta\colon R \to R$ such that $\delta(xy) = \delta(x)y + \sigma(x)\delta(y)$ for all $x, y \in R$, and σ is called an associated automorphism of δ. For brevity, skew derivations are generally called σ-derivations. Let 1_R denote the identity automorphism of R. Clearly, the map $\sigma - 1_R$ is the simplest example of skew derivations and 1_R-derivations are just ordinary derivations. Another significant example is a map of the form $\delta(x) = ax + \sigma(x)b$ for some $a, b \in R$; such skew derivations are called inner skew derivations. The study of derivations on prime rings goes back to 1957 by Posner, see [23]. A variety of results have been motivated by this work [2], [5], [6]. A well known theorem of Posner (see [23]) states that if R is a prime ring and d a nonzero derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R must be commutative. Many authors have studied the relationship between the structure of a prime ring R and an additive map $f\colon R \to R$ which satisfies the Engel condition $[f(x), x]_k = 0$ for $k \geqslant 1$. In [19], Lanski generalized Posner's result to one-sided ideals as follows: Let R be a prime ring derivation d, I a left ideal of R and k, n two positive integers. Suppose $[d(r^k), r^k]_n = 0$ for all $r \in I$. Then either $d = 0$ or R is commutative. In [1], Albas et al. generalized this result to generalized derivations as follows: Let R be a noncommutative prime ring and I a nonzero left ideal of R. Let G be a generalized derivation of R such that $[G(r^k), r^k]_n = 0$ for all $r \in I$, where k, n are fixed positive integers. Then there exists $c \in U$: Utumi quotient ring, such that $G(x) = xc$ and $I(c - \alpha) = 0$ for suitable $\alpha \in C$. In particular, we have that $G(x) = \alpha x$ for all $x \in I$. Moreover, in [13], De Filippis proved: Let R be a prime ring of characteristic different from 2, d a nonzero derivation of R, L a non-central Lie ideal of R, $a \in R$. If $a[d(u), u] = 0$ for any $u \in L$ then $a = 0$. In [10], Chuang, Chou and Liu proved: Let R be a noncommutative prime ring and $a \in R$, let δ be a σ-derivation of R such that $a[\delta(x), x]_k = 0$ for all $x \in R$, where k is a fixed positive integer. Then $a = 0$ or $\delta = 0$ except when $R = M_2(GF(2))$. Also in [24] Shiue obtained: Let R be a prime ring, L a noncentral Lie ideal of R and $a \in R$. Suppose that d is a nonzero derivation of R is such that $a[d(u), u]_k = 0$ for all $u \in L$, where k is a fixed positive integer. Then $a = 0$ except when $\operatorname{char} R = 2$ and $\dim_C RC = 4$. Also, Shiue extended De Filippis's result to one-sided ideals as follows:

Theorem A ([25], Theorem 1). *Let R be a noncommutative prime ring with nonzero left ideal λ. Suppose that D is a nonzero derivation of R and $0 \neq a \in R$ is such that $a[D(u^k), u^k]_n = 0$ for all $u \in \lambda$, where k and n are fixed positive integers. Then $D = ad(b)$ for some $b \in Q$ such that $\lambda b = 0$ and $ab = 0$.*

Recently, in [7] Chou and Liu proved:

Theorem B ([7], Theorem 1.1). *Let R be a prime ring, L a noncentral Lie ideal of R and $a \in R$. Suppose that δ is a nonzero σ-derivation of R such that $a[\delta(x), x]_k = 0$ for all $x \in L$, where σ is an automorphism of R and k is a fixed positive integer. Then $a = 0$ except when* $\operatorname{char}(R) = 2$ *and $R \subseteq M_2(F)$, the 2×2 matrix ring over a field F.*

The main purpose of this article is to extend Theorem B to the case of power-commuting as follows:

Theorem 1.1. *Let R be a noncommutative prime ring of characteristic different from 2, with its two-sided Martindale quotient ring Q, C the extended centroid of R and $a \in R$. Suppose that δ is a nonzero σ-derivation of R such that $a[\delta(x^n), x^n]_k = 0$ for all $x \in R$, where σ is an automorphism of R, n and k are fixed positive integers. Then $a = 0$.*

Let σ be an automorphism of R. For $c \in R$, the map $\delta\colon\ x \in R \mapsto \sigma(x)c - cx$ defines a σ-derivation. A σ-derivation δ of R is called X-inner if its extension to Q is inner, that is, there exists $c \in Q$ such that $\delta(x) = \sigma(x)c - cx$ for all $x \in Q$. Otherwise, δ is called X-outer. Analogously, an automorphism σ of R is called X-inner if there exists a unit $q \in Q$ such that $\sigma(x) = qxq^{-1}$ for all $x \in Q$. Otherwise, σ is called X-outer. An automorphism σ of Q is called Frobenius (see [9]) if, in the case of $\operatorname{char}R = 0$, $\sigma(\lambda) = \lambda$ for all $\lambda \in C$ and if, in the case of $\operatorname{char}R = p \geqslant 2$, $\sigma(\lambda) = \lambda^{p^n}$ for all $\lambda \in C$, where n is a fixed integer, positive, zero or negative. We need some well-known facts and a remark which will be used in the sequel.

Remark 1.2. Let R be a prime ring, then the following statements hold:

(1) Every generalized derivation of R can be uniquely extended to Q, see [21], Theorem 3.

(2) Any automorphism of R can be uniquely extended to Q, see [8], Fact 2.

(3) Every generalized skew derivation of R can be uniquely extended to Q, see [4], Lemma 2.

Fact 1.3 ([9], Theorem 1). Let R be a prime ring and I a two-sided ideal of R. Then I, R and Q satisfy the same generalized polynomial identities with automorphisms.

Fact 1.4 ([11], Theorem 1). Let R be a prime ring with an X-outer σ-derivation δ. Then any generalized polynomial identity of R in the form $\Phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identitiy $\Phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Fact 1.5 ([11], Theorem 1). Let R be a prime ring with an X-outer automorphism σ. Suppose that δ is an X-outer σ-derivation of R. Then any generalized polynomial identity of R in the form $\Phi(x_i, \sigma(x_i), \delta(x_i)) = 0$ yields the generalized polynomial identity $\Phi(x_i, y_i, z_i) = 0$ of R, where x_i, y_i and z_i are distinct indeterminates.

Fact 1.6 ([9], Theorem 2)**.** Let R be a prime ring with an automorphism σ. Suppose that σ is not a Frobenius automorphism of R. Then any generalized polynomial identity of R in the form $\Phi(x_i, \sigma(x_i)) = 0$ yields the generalized polynomial identity $\Phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Fact 1.7 ([17], page 140)**.** Let R be a prime GPI-ring with an automorphism σ and extended centroid C. Suppose that $\sigma(\alpha) = \alpha$ for all $\alpha \in C$. Then σ is an X-inner automorphism.

2. Results

Let V_F be a right vector space over a field F. We denote by $\mathrm{End}(V)$ the ring of endomorphisms on V and by $\mathrm{End}(V_F)$ the ring of F-linear transformations on V_F. An additive map $T \in \mathrm{End}(V_F)$ is called a *semilinear transformation* if for some automorphism τ of F, $T(v\alpha) = T(v)\tau(\alpha)$ for all $v \in V$ and $\alpha \in F$, see [16], page 44.

The following lemma is proved in a way similar to the proof of Lemma 2.1 in [7] but to keep the integrity we prove this.

Lemma 2.1. *Let R be a dense subring of $\mathrm{End}(V_F)$ containing nonzero linear transformations of finite rank, where $\dim V_F \geqslant 3$, and let δ be a nonzero σ-derivation of R, where σ is an automorphism of R. If $a \in R$ and $a[\delta(x^n), x^n]_k = 0$ for all $x \in R$, where n and k are fixed positive integers, then $a = 0$.*

Proof. By [16], page 79, there exists an invertible semilinear transformation $T \in \mathrm{End}(V)$ such that $\sigma(x) = TxT^{-1}$ for all $x \in R$. That is, there is an automorphism τ of F such that $T(v\alpha) = (Tv)\tau(\alpha)$ for all $v \in V$ and $\alpha \in F$ and there exists $S \in \mathrm{End}(V)$ such that $\delta(x) = \sigma(x)S - Sx$ for all $x \in R$ by [12], Theorem 2.8. Hence we have $\delta(x) = TxT^{-1}S - Sx$ and by the hypothesis, we have

$$
\begin{aligned}
(2.1) \qquad 0 &= a[\delta(x^n), x^n]_k = a[Tx^nT^{-1}S - Sx^n, x^n]_k \\
&= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}(x^n)^i(Tx^nT^{-1}S - Sx^n)(x^n)^{k-i}
\end{aligned}
$$

for all $x \in R$. We claim that there exists $v_0 \in V$ such that v_0 and $T^{-1}Sv_0$ are F-independent. If not then v and $T^{-1}Sv$ are F-dependent for all $v \in V$. That is for every $v \in V$ there exists $\lambda_v \in F$ such that $T^{-1}Sv = v\lambda_v$. Moreover, by [10], Lemma 1, there exists $\lambda \in F$ such that $T^{-1}Sv = v\lambda$ for all $v \in V$. Then we conclude that $\delta(x)v = (TxT^{-1}S - Sx)v = T(xT^{-1}Sv) - Sxv = T((xv)\lambda) - Sxv = T(T^{-1}Sxv) - Sxv = 0$ for all $x \in R$ and $v \in V$. So this implies that $\delta = 0$, a contradiction. Now we obtain that v_0 and $T^{-1}Sv_0$ are F-independent for some $v_0 \in V$, as claimed. Observe that for $x \in R$ and for any $v_0 \in V$, by (2.1) we have

$$0 = a\sum_{i=0}^{k}(-1)^i\binom{k}{i}(x^n)^i(Tx^nT^{-1}S - Sx^n)(x^n)^{k-i}v_0. \tag{2.2}$$

Now, we divide the proof into several cases.

Case 1: $Sv_0 \notin v_0F + (T^{-1}Sv_0)F$. Then there exists $w \in V$ such that v_0, $T^{-1}Sv_0$ and w are F-independent and $Sv_0 = v_0\alpha + (T^{-1}Sv_0)\beta + w\gamma$, where $\alpha, \beta, \gamma \in F$ and $\gamma \neq 0$.

Choose $u \in V$ such that

$$u = 0 \quad \text{if } \dim V_F = 3,$$

and

$$u \notin (v_0)F + (T^{-1}Sv_0)F + wF \quad \text{if} \quad \dim V_F \geqslant 4.$$

By the density of R, there exists $x \in R$ such that

$$xv_0 = 0, \quad xT^{-1}Sv_0 = T^{-1}Sv_0, \quad xw = w, \quad xu = 0. \tag{2.3}$$

So by (2.2) we may obtain

$$0 = (-1)^k a((T^{-1}Sv_0)\beta + w\gamma). \tag{2.4}$$

Note that $Sv_0 = v_0(\alpha - \gamma) + (T^{-1}Sv_0)\beta + (w + v_0)\gamma$. Replacing w by $w + v_0$ in (2.3) and (2.4), we have

$$0 = (-1)^k a((T^{-1}Sv_0)\beta + w\gamma + v_0\gamma). \tag{2.5}$$

Since $\gamma \neq 0$ it follows from (2.4) and (2.5) that

$$av_0 = 0. \tag{2.6}$$

On the other hand, $Sv_0 = (v_0)\alpha + (T^{-1}Sv_0)\beta + (w+u)\gamma - u\gamma$. Similarly, replacing w by $w + u$ in (2.3) and (2.4) we get

$$0 = (-1)^k a(T^{-1}Sv_0\beta + w\gamma + u\gamma). \tag{2.7}$$

By (2.4) and (2.7), we conclude that

$$au = 0 \quad \text{for every } u \notin v_0F + (T^{-1}Sv_0)F + wF. \tag{2.8}$$

Choose $u_0 \in V$ such that $u_0 \notin v_0F + (T^{-1}Sv_0)F + wF$ if $\dim V_F \geqslant 4$. Then $u_0 + T^{-1}Sv_0 \notin v_0F + (T^{-1}Sv_0)F + wF$ and $u_0 + w \notin v_0F + (T^{-1}Sv_0)F + wF$. Hence (2.8) yields that $au_0 = a(u_0 + T^{-1}Sv_0) = a(u_0 + w) = 0$. This implies $aT^{-1}Sv_0 = aw = 0$. Recall that $av_0 = 0$ by (2.6). Consequently, $a = 0$, as desired. So we may assume that $\dim V_F = 3$. In this case, $\{v_0, T^{-1}Sv_0, w\}$ is a basis of V over F.

Suppose first that $\beta = 0$. In this situation, $Sv_0 = v_0\alpha + w\gamma$, $\gamma \neq 0$, and using (2.4) we conclude that $aw = 0$.

Subcase 1.1: $Tv_0 = v_0\alpha^* + (T^{-1}Sv_0)\beta^* + w\gamma^*$, where $\alpha^*, \beta^*, \gamma^* \in F$ with $\beta^* \neq 0$. Let $S' = S+T$. Then $S'v_0 = v_0(\alpha+\alpha^*)+(T^{-1}Sv_0)\beta^*+w(\gamma+\gamma^*) = v_0(\alpha + \alpha^* - \beta^*)+(T^{-1}S'v_0)\beta^* + w(\gamma + \gamma^*)$, $T^{-1}S'v_0 = v_0 + T^{-1}Sv_0$ and $\delta(x) = TxT^{-1}S - Sx = TxT^{-1}S - Sx + Tx - Tx = TxT^{-1}S' - S'x$.

Clearly, $\{v_0, T^{-1}S'v_0, w\}$ is a basis of V over F. Replacing S by S' in (2.3) and (2.4), we obtain

$$0 = a(-1)^k(T^{-1}S'v_0\beta^* + w(\gamma + \gamma^*)). \tag{2.9}$$

Recall that $av_0 = 0$ by (2.6) and $aw = 0$. From (2.9) we conclude that $aT^{-1}S'v_0 = 0$. Consequently, $a = 0$, as desired.

Subcase 1.2: $Tv_0 = v_0\alpha^* + w\gamma^*$, where $\alpha^*, \gamma^* \in F$. Recall that $\{v_0, T^{-1}Sv_0, w\}$ is a basis of V over F and $Sv_0 = v_0\alpha + w\gamma$, where $\alpha, \gamma \in F$ and $\gamma \neq 0$. By the density of R, there exists $x \in R$ such that

$$xv_0 = 0, \quad xT^{-1}Sv_0 = T^{-1}Sv_0, \quad xw = T^{-1}Sv_0. \tag{2.10}$$

Then $x^nT^{-1}Sv_0 = T^{-1}Sv_0$. In view of (2.2) we obtain that

$$0 = a(-1)^k(x^n)^kTx^nT^{-1}Sv_0 = a(-1)^k(x^n)^kSv_0.$$

So we have

$$0 = a(-1)^kT^{-1}Sv_0\gamma. \tag{2.11}$$

So the last relation implies that $aT^{-1}Sv_0 = 0$ since $\gamma \neq 0$. Recall that $av_0 = 0$ by (2.6) and $aw = 0$. Consequently, we obtain that $a = 0$, as desired.

Suppose next that $\beta \neq 0$. In this case $Sv_0 = v_0\alpha + (T^{-1}Sv_0)\beta + w\gamma$, $\beta \neq 0$, $\gamma \neq 0$. Let $Tv_0 = v_0\alpha^* + (T^{-1}Sv_0)\beta^* + w\gamma^*$, where $\alpha^*, \beta^*, \gamma^* \in F$. From (2.4) and (2.6), we conclude that

$$a(-1)^k((T^{-1}Sv_0)\beta + w\gamma) = 0 \quad \text{and} \quad av_0 = 0. \tag{2.12}$$

Subcase 1.3: $(T^{-1}Sv_0)\beta + w\gamma$ and $(T^{-1}Sv_0)\beta^* + w\gamma^*$ are F-independent. In this case β^* and γ^* are not both zero. Given $d \in D$, let $r_d\colon V \to V$ be the map defined by $r_d(v) = vd$ for $v \in V$. First we assume that $\gamma^* \neq 0$. Recall that $Sv_0 = v_0\alpha + (T^{-1}Sv_0)\beta + w\gamma$ and $Tv_0 = v_0\alpha^* + T^{-1}Sv_0\beta^* + w\gamma^*$, where $\beta \neq 0$, $\gamma \neq 0$, $\gamma^* \neq 0$. Thus we have

$$v_0\alpha = Sv_0 - (T^{-1}Sv_0)\beta - w\gamma, \tag{2.13}$$

$$v_0\alpha^* = Tv_0 - (T^{-1}Sv_0)\beta^* - w\gamma^*. \tag{2.14}$$

Now right multiplying (2.14) with $(\gamma^*)^{-1}\gamma$, we have $v_0\alpha^*(\gamma^*)^{-1}\gamma = Tv_0(\gamma^*)^{-1}\gamma - (T^{-1}Sv_0)\beta^*(\gamma^*)^{-1}\gamma - w\gamma$ and if we write $(\gamma^*)^{-1}\gamma = d$, we get

$$v_0\alpha^* d = Tv_0 d - (T^{-1}Sv_0)\beta^* d - w\gamma. \tag{2.15}$$

Using (2.13) and (2.15), we have $v_0(\alpha - \alpha^* d) = Sv_0 - (Tv_0)d - (T^{-1}Sv_0)(\beta - \beta^* d)$, thus

$$Sv_0 - (Tv_0)d = v_0(\alpha - \alpha^* d) + T^{-1}Sv_0\beta', \quad \text{where } \beta' = \beta - \beta^* d. \tag{2.16}$$

On the other hand, we assume that $\beta^* \neq 0$. Now right multiplying (2.14) with $(\beta^*)^{-1}\beta$, and writing $d' = (\beta^*)^{-1}\beta$, we have

$$v_0\alpha^* d' = Tv_0 d' - T^{-1}Sv_0\beta - w\gamma^* d'. \tag{2.17}$$

Using (2.13) and (2.17), we have

$$Sv_0 - (Tv_0)d' = v_0(\alpha - \alpha^* d') + w\gamma', \quad \text{where } \gamma' = \gamma - \gamma^* d'. \tag{2.18}$$

Let $S - r_dT = S'$. Then by (2.16) and (2.18) we have $S'v_0 = v_0(\alpha - \alpha^* d) + T^{-1}Sv_0\beta'$ or $S'v_0 = v_0(\alpha - \alpha^* d) + w\gamma'$. Note that $\sigma(x)T = Tx$ and $r_dx = xr_d$ for all $x \in R$. Thus $\delta(x) = \sigma(x)S - Sx = \sigma(x)S - Sx + \sigma(x)r_dT - \sigma(x)r_dT = \sigma(x)(S - r_dT) + \sigma(x)Td - Sx = \sigma(x)(S - r_dT) + Txd - Sx = \sigma(x)(S - r_dT) - (S - r_dT)x = \sigma(x)S' - S'x$. Clearly $v_0, T^{-1}S'v_0, w$ are F-independent. Replacing S by S' in (2.2), (2.3) and (2.4) we obtain that $xv_0 = 0$, $xT^{-1}S'v_0 = T^{-1}S'v_0$, $xw = w$ and $(-1)^k a(x^n)^k S'v_0 = 0$.

This implies that

$$(2.19)\qquad \text{either}\quad aT^{-1}Sv_0\beta' = 0,\ \text{where } \gamma^* \neq 0 \quad \text{or} \quad aw\gamma' = 0,\ \text{where } \beta^* \neq 0.$$

In view of (2.12) and (2.19), we get $av_0 = aT^{-1}Sv_0 = aw = 0$. Consequently, $a = 0$, as desired.

Subcase 1.4: $(T^{-1}Sv_0)\beta^* + w\gamma^* = ((T^{-1}Sv_0)\beta + w\gamma)l$ for some $l \in F$. Recall that $Sv_0 = v_0\alpha + (T^{-1}Sv_0)\beta + w\gamma$ and $Tv_0 = v_0\alpha^* + (T^{-1}Sv_0)\beta^* + w\gamma^*$, $\beta \neq 0$, $\gamma \neq 0$. So $Sv_0 = v_0\alpha + w'$ and $Tv_0 = v_0\alpha^* + (T^{-1}Sv_0)\beta^* + w\gamma^* = v_0\alpha^* + w'l$, where $w' = (T^{-1}Sv_0)\beta + w\gamma$. Clearly $\{v_0, T^{-1}Sv_0, w'\}$ is a basis of V over F. Replacing w by w' in (2.3) and (2.4), we obtain $aw' = 0$. On the other hand, replacing w by w' in (2.10) and (2.11) and using $aw' = 0$, we obtain $aT^{-1}Sv_0 = 0$. Using these facts and (2.12) we get $av_0 = aT^{-1}Sv_0 = aw' = 0$. Consequently, $a = 0$, as desired.

Case 2: $Sv_0 \in v_0F + (T^{-1}Sv_0)F$. First we may assume that $Tv_0 \notin v_0F + (T^{-1}Sv_0)F$. Let $S + T = S'$, then $S'v_0 \notin v_0F + (T^{-1}S'v_0)F$. If not, we have $Tv_0 \in v_0F + T^{-1}Sv_0F$, a contradiction. Thus $S'v_0 \notin v_0F + T^{-1}S'v_0F$. Recall that for all $x \in R$, $\delta(x) = TxT^{-1}S' - S'x$. Replacing S by S', by Case 1 we are done. Hence we may assume that $Tv_0 \in v_0F + (T^{-1}Sv_0)F$. So there exist $\alpha, \alpha^*, \beta, \beta^* \in F$ such that

$$(2.20)\qquad Sv_0 = v_0\alpha + T^{-1}Sv_0\beta \quad \text{and} \quad Tv_0 = v_0\alpha^* + T^{-1}Sv_0\beta^*.$$

Let $S' = S + T$, then $S'v_0 = Sv_0 + Tv_0 = v_0(\alpha + \alpha^*) + T^{-1}Sv_0(\beta + \beta^*)$ and for all $x \in R$, $\delta(x) = TxT^{-1}S' - S'x$. Clearly β and β^* are not both zero since Sv_0 and Tv_0 are F-independent. Replace S by S' if $\beta = 0$. So we may assume that $\beta \neq 0$. By (2.3), there exists $x \in R$ such that $xv_0 = 0$, $xT^{-1}Sv_0 = T^{-1}Sv_0$, $xw = w$ and using (2.2), we get $0 = a(-1)^k(x^n)^kTx^nT^{-1}Sv_0 = a(-1)^k(x^n)^kSv_0 = a(-1)^k(x^n)^k(v_0\alpha + (T^{-1}Sv_0)\beta)$. This implies that $aT^{-1}Sv_0 = 0$. We claim that

$$(2.21)\qquad \text{if} \quad Tv_0 \notin v_0F \quad \text{then} \quad a = 0.$$

Let $w \in V$ and $w \notin v_0F + (T^{-1}Sv_0)F$. Then $\{v_0, T^{-1}Sv_0, w\}$ are F-independent. So we can take $Tw = v_0\alpha^{**} + (T^{-1}Sv_0)\beta^{**} + w\gamma^{**} + u\eta$, where $\alpha^{**}, \beta^{**}, \gamma^{**}, \eta \in F$ and $u \in V$ are such that $u = 0$ if $\dim V_F = 3$ and $u \notin v_0F + (T^{-1}Sv_0)F + wF$ if $\dim V_F \geqslant 4$.

Case 2.1: Now we assume that $\beta^{**} = 0$. Then $Tw = v_0\alpha^{**} + w\gamma^{**} + u\eta$. If $\gamma^{**} = 0$, then $\eta \neq 0$ since $\{Tv_0, Tw, Sv_0\}$ are F-independent. Suppose first that $\gamma^{**} \neq 0$. Consider $x \in R$ such that $xv_0 = 0$, $xT^{-1}Sv_0 = w$, $xw = w$ and $xu = 0$. Then we have $0 = (-1)^ka(x^n)^k(v_0\alpha^{**} + w\gamma^{**} + u\eta)$ and using $xv_0 = 0$, $xu = 0$ and $\gamma^{**} \neq 0$ in the last relation, we get $aw = 0$. On the other hand, if $\gamma^{**} = 0$ then

$\eta \neq 0$. Let $x \in R$ such that $xv_0 = 0$, $xT^{-1}Sv_0 = w$, $xw = w$, $xu = w$. In this case we have $0 = (-1)^k a(x^n)^k T x^n T^{-1} S v_0 = (-1)^k a(x^n)^k Tw = (-1)^k a(x^n)^k (v_0\alpha^{**} + u\eta)$ and using $xv_0 = 0$ and $\eta \neq 0$, this implies $aw = 0$.

Case 2.2: $\beta^{**} \neq 0$. Let $d \in F$ be such that $\beta^{**} + \beta\tau(d) = 0$ and let $w' = w + (T^{-1}Sv_0)d$. Then $\{v_0, T^{-1}Sv_0, w'\}$ are F-independent and $Tw' = v_0(\alpha^{**} + \alpha\tau(d)) + w\gamma^{**} + u\eta$. In Case 2.1, when $Tw = v_0\alpha^{**} + w\gamma^{**} + u\eta$, we have concluded that $aw = 0$. Now we have $Tw' = v_0(\alpha^{**} + \alpha\tau(d)) + w\gamma^{**} + u\eta$ so by the same process as in Case 2.1, we get $aw' = 0$. Since $aT^{-1}Sv_0 = 0$, we obtain $aw = 0$. We see that if either $\beta^{**} = 0$ or $\beta^{**} \neq 0$, then we conclude that $aw = 0$ for all $w \notin v_0F + (T^{-1}Sv_0)F$. Particularly $a(v_0 + w) = 0$ and $a(T^{-1}Sv_0 + w) = 0$ for all $w \notin v_0F + (T^{-1}Sv_0)F$. This implies $av_0 = aT^{-1}Sv_0 = aw = 0$ for all $w \notin v_0F + T^{-1}Sv_0F$. Consequently, $a = 0$, as desired.

Assume on the contrary that $a \neq 0$. By Case 1 and (2.21) we conclude that for every $v \in V$, v and $T^{-1}Sv$ are F-dependent or $Tv \in vF$. So we assume that for every $v \in V$, we have

$$Sv \in (Tv)F \quad \text{or} \quad Tv \in vF. \tag{2.22}$$

In particular, the relation (2.20) reduces to $Tv_0 = v_0\alpha^*$.

Let $w \in V$ and $w \notin v_0F + T^{-1}Sv_0F$. Note that $\{Tv_0, Sv_0, Tw\}$ are F-independent. Suppose $Tw \notin wF$. Then $T(w\lambda) \notin (w\lambda)F$ for all $0 \neq \lambda \in F$. By (2.22), we obtain that $S(w\lambda) \in (T(w\lambda))\gamma$ for some $\gamma \in F$. If $S(w\lambda + v_0) = T(w\lambda + v_0)\eta$ for some $\eta \in F$ then we conclude that $Tw(\tau(\lambda)(\gamma - \eta)) - (Sv_0) - (Tv_0)\eta = 0$ implying $\{Sv_0, Tw, Tv_0\}$ are F-dependent, a contradiction. Hence by (2.22), we have $T(w\lambda + v_0) \in (w\lambda + v_0)F$. That is, for all $0 \neq \lambda \in F$, $T(w\lambda + v_0) = (w\lambda + v_0)\mu_\lambda$, where $\mu_\lambda \in F$ depends on λ. Using $Tv_0 = v_0\alpha^*$, we obtain

$$Tw\tau(\lambda) = w\lambda\mu_\lambda + v_0(\mu_\lambda - \alpha^*). \tag{2.23}$$

Clearly, from $T(w + v_0) = (w + v_0)\mu_1$, it follows that $Tw = w\mu_1 + v_0(\mu_1 - \alpha^*)$. Due to this and (2.23) we obtain $w(\mu_1\tau(\lambda) - \lambda\mu_\lambda) + v_0((\mu_1 - \alpha^*)\tau(\lambda) - \mu_\lambda + \alpha^*) = 0$. This implies

$$\mu_1\tau(\lambda) - \lambda\mu_\lambda = 0 \tag{2.24}$$

and

$$(\mu_1 - \alpha^*)\tau(\lambda) - \mu_\lambda + \alpha^* = 0 \tag{2.25}$$

for all $0 \neq \lambda \in F$.

Left multiplying (2.25) with λ, we have $\lambda(\mu_1 - \alpha^*)\tau(\lambda) - \lambda\mu_\lambda + \lambda\alpha^* = 0$ and using (2.24), we have

$$\lambda(\mu_1 - \alpha^*)\tau(\lambda) - \mu_1\tau(\lambda) + \lambda\alpha^* = 0 \quad \forall\, \lambda \in F. \tag{2.26}$$

Replacing λ in (2.26) by $\lambda + \beta$, we get

$$\beta(\mu_1 - \alpha^*)\tau(\lambda) + \lambda(\mu_1 - \alpha^*)\tau(\beta) = 0 \quad \forall\, \lambda, \beta \in F. \tag{2.27}$$

Assume that $\tau(\lambda) \neq \lambda$ for some $0 \neq \lambda \in F$. Replacing λ by λ^2 in (2.27), we obtain

$$\beta(\mu_1 - \alpha^*)\tau(\lambda)\tau(\lambda) + \lambda^2(\mu_1 - \alpha^*)\tau(\beta) = 0. \tag{2.28}$$

Left multiplying (2.27) with λ, we have

$$\lambda\beta(\mu_1 - \alpha^*)\tau(\lambda) + \lambda^2(\mu_1 - \alpha^*)\tau(\beta) = 0. \tag{2.29}$$

Using (2.28) and (2.29), we get

$$\lambda\beta(\mu_1 - \alpha^*)\tau(\lambda) - \beta(\mu_1 - \alpha^*)\tau(\lambda)\tau(\lambda) = 0. \tag{2.30}$$

Similarly; replacing β in (2.27) by β^2 and using a process similar to the above, we have

$$\beta\lambda(\mu_1 - \alpha^*) - \lambda(\mu_1 - \alpha^*)\tau(\beta) = 0. \tag{2.31}$$

Since $\tau(\lambda) \neq 0$, by (2.27) and (2.30) we obtain

$$\beta(\mu_1 - \alpha^*) + (\mu_1 - \alpha^*)\tau(\beta) = 0. \tag{2.32}$$

And using (2.27) and (2.31) together, we get

$$(\mu_1 - \alpha^*)\tau(\lambda) + \lambda(\mu_1 - \alpha^*) = 0. \tag{2.33}$$

By the relations (2.27), (2.32) and (2.33), we have $\tau(\lambda) = \lambda$ or $\mu_1 = \alpha^*$ for all $0 \neq \lambda \in F$. By assumption, we get $\mu_1 = \alpha^*$ and moreover, by (2.25), we have $\alpha^* = \mu_\lambda = \mu_1$ for all $0 \neq \lambda \in F$. Thus $Tw = w\alpha^*$, a contradiction.

So we conclude that

$$Tw \in wF \quad \text{for every } w \in V \text{ with } w \notin v_0F + T^{-1}Sv_0F. \tag{2.34}$$

Choose $w \in V$ such that $w \notin v_0F + T^{-1}Sv_0F$. Clearly $w + v_0$, $w + T^{-1}Sv_0 \notin v_0F + T^{-1}Sv_0F$. By (2.34), $Tw = w\mu$, $T(w + v_0) = (w + v_0)\xi$, $T(w + T^{-1}Sv_0) = (w + T^{-1}Sv_0)\varepsilon$ for some $\mu, \xi, \varepsilon \in F$. By the F-independence of $v_0, T^{-1}Sv_0, w$ and by (2.20), we get $\varepsilon = \mu = \xi = \alpha^*$. This implies $Tv = v\alpha^*$ for all $v \in V$. So $\sigma(x) = TxT^{-1} = x$ for all $x \in R$. In this case by Theorem A, $\delta = 0$, a contradiction. $\square$

Lemma 2.2. *Let R be a dense subring of $\mathrm{End}(V_F)$, containing nonzero linear transformations of finite rank, where $\dim V_F = 2$, and let δ be a nonzero σ-derivation of R, where σ is an automorphism of R. If $a \in R$ and $a[\delta(x^n), x^n]_k = 0$ for all $x \in R$, where n and k are fixed positive integers, then $a = 0$.*

Proof. In view of the proof of Lemma 2.1, there exist $c \in \mathrm{End}(V)$ and an invertible semilinear transformation $q \in \mathrm{End}(V)$ such that $\sigma(x) = qxq^{-1}$ and $\delta(x) = cx - \sigma(x)c = cx - qxq^{-1}c$ for all $x \in R$. So we have $a[cx^n - qx^nq^{-1}c, x^n]_k = 0$ for all $x \in R$. Since $\dim V_F = 2$ we have $a[cx^n - qx^nq^{-1}c, x^n]_k = 0$ for all $x \in M_2(F)$.

By [18], Theorem 4.23 there exists $e = e^2 \in M_2(F)$ such that $Ra = Re$.

If $e = 0$, then $a = 0$, as desired.

If $e = 1$ then we have $Ra = R$ and for all $x \in R$

$$[\delta(x^n), x^n]_k = 0. \tag{2.35}$$

By [20], Theorem 1, we get $\delta = 0$, a contradiction.

Let $e \neq 0, 1$. Then by [18], Proposition 21.20, we have $Ra \cong Re$, $e = e^2 \in M_2(F)$. So we have for all $x \in M_2(F)$ and $e = e^2 \in M_2(F)$

$$e[cx^n - qx^nq^{-1}c, x^n]_k = 0. \tag{2.36}$$

Denote $p = q^{-1}c = \sum_{i,j} e_{ij}p_{ij}$, $q = \sum_{i,j} e_{ij}q_{ij}$, where $q_{ij}, p_{ij} \in F$ and e_{ij} is the usual matrix unit, with 1 in (i,i)-entry and zero elsewhere. Now, let us make some calculations:

For $e = x = e_{11}$ in (2.36) and right multiplying this relation by e_{22}, we have

$$q_{11}p_{12} = 0. \tag{2.37}$$

For $e = x = e_{22}$ in (2.36) and right multiplying this relation by e_{11}, we get

$$q_{22}p_{21} = 0. \tag{2.38}$$

For $e = x = e_{11} + e_{21}$ in (2.36), right multiplying this relation by e_{22} and using (2.37), we have

$$q_{12}p_{12} = 0. \tag{2.39}$$

For $e = x = e_{12} + e_{22}$ in (2.36), right multiplying this relation by e_{11} and using (2.38) we obtain

$$q_{21}p_{21} = 0. \tag{2.40}$$

If $p_{12} \neq 0$, then by the relations (2.37) and (2.39), we have $q_{11} = 0 = q_{12}$, so $q = \begin{pmatrix} 0 & 0 \\ q_{21} & q_{22} \end{pmatrix}$, a contradiction to the invertibility of q.

Similarly if $p_{21} \neq 0$, then by the relations (2.38) and (2.40), we have $q_{22} = 0 = q_{21}$, so $q = \begin{pmatrix} q_{11} & q_{12} \\ 0 & 0 \end{pmatrix}$, a contradiction. So we have both $p_{12} = 0 = p_{21}$. In this case p must be a diagonal matrix in $M_2(F)$. Let us define $\psi(x) = (1 + e_{12})x(1 - e_{12}) = x - xe_{12} + e_{12}x - e_{12}xe_{12}$. Since p is a diagonal matrix and the identity in the hypothesis is invariant under the action of automorphism ψ, $\psi(p)$ is also diagonal. As $\psi(p) = p - pe_{12} + e_{12}p - e_{12}pe_{12}$ and $p = \sum_s e_{ss}p_{ss}$ we have $\psi(p) - p = -\sum_s e_{ss}p_{ss}e_{12} + e_{12}\sum_s e_{ss}p_{ss} - e_{12}\sum_s e_{ss}p_{ss}e_{12} = -p_{11}e_{12} + p_{22}e_{12}$. We know that the left hand side of the above relation is diagonal, so we have $p_{22} = p_{11}$. In this case $p = \lambda I_2$, where I_2 is an identity matrix in $M_2(F)$, which implies $\delta = 0$, a contradiction. □

Theorem 2.3. *Let R be a prime ring, $n, k \geqslant 1$ fixed integers, $c, q \in Q$ such that q is invertible. Suppose that $a \in R$ and $a \neq 0$. If $a[cx^n - qx^nq^{-1}c, x^n]_k = 0$ for all $x \in R$ then $q^{-1}c \in C$ or $q, c \in C$.*

P r o o f. By the hypothesis, we denote for all $x \in R$,

$$\phi(x) = a[cx^n - qx^nq^{-1}c, x^n]_k = 0. \tag{2.41}$$

By assumption we know that R satisfies (2.41). That is, $\phi(x)$ is a generalized polynomial identity for R. By Fact 1.3, R and Q satisfy the same generalized polynomial identity with the automorphism Q also satisfying (2.41). If $q^{-1}c \in C$ then there is nothing to be proved. If $q \in C$, then by (2.41) we get $a[[c, x^n], x^n]_k = 0$. And by Theorem A, we have $c \in C$, as desired. So we may assume that both $q^{-1}c \notin C$ and $q \notin C$. In this case (2.41) is a nontrivial generalized polynomial identity for Q. By [22], Q is a primitive ring having a nonzero socle with C as the associated division ring and by [16], page 75, Q is isomorphic to a dense ring of linear transformations on some vector space V over C. Since R is a noncommutative ring we may assume that $\dim_C V \geqslant 2$. By Lemma 2.1 and Lemma 2.2, in case of either $\dim_C V \geqslant 3$ or $\dim_C V = 2$ we have $a = 0$, a contradiction. □

Now we are ready for the proof of Theorem 1.1.

P r o o f of Theorem 1.1. Assume that $a \neq 0$. We will show that this assumption will lead to a number of contradictions. Assume first that δ is X-inner, that is there exists c, $0 \neq c \in Q$, such that $\delta(x) = cx - \sigma(x)c$ for all $x \in R$. Hence we have $a[cx^n - \sigma(x^n)c, x^n]_k = 0$ for all $x \in R$ and also for all $x \in Q$ by Fact 1.3. By Theorem 2.3, we may assume σ is X-outer.

Case 1: σ is not Frobenius. Since $a[cx^n - \sigma(x^n)c, x^n]_k = 0$ for all $x \in Q$, by Fact 1.6 we have $a[cx^n - y^nc, x^n]_k = 0$ for all $x \in Q$. Let $x = y$, then $a[d(x^n), x^n]_k = 0$ for all $x \in Q$, where $d(x) = [c, x]$ is a derivation. And by Theorem A, we obtain that either $a = 0$ or $c \in C$. By assumption we conclude $c \in C$ and $a[y^n, x^n]_k = 0$ for all $x \in Q$. Then by the proof of [25], Proposition 3, we obtain that R is commutative, a contradiction.

Case 2: σ is Frobenius. We may assume $\mathrm{char} R = p > 0$. Otherwise, if $\mathrm{char} R = 0$ then the Frobenius automorphism σ fixes C and hence must be X-inner by Fact 1.7, a contradiction. So for all $\lambda \in C$, $\sigma(\lambda) = \lambda^{p^n}$ for some nonzero fixed integer n. Also we may assume that $n \neq 0$. Let F be the algebraic closure of C if C is infinite and set $F = C$ if C is finite. Clearly, the map $Q \ni q \mapsto q \otimes 1 \in Q \otimes_C F$ gives a ring embedding. So we may assume Q is a subring of $Q \otimes_C F$. By [15], Theorem 3.5, $Q \otimes_C F$ is a prime ring with F as its extended centroid. Since taking pth powers or pth roots is an automorphism of C, it is also an automorphism of F. So σ can be extended to an automorphism of $Q \otimes_C F$ and remains Frobenius. Moreover, by the same proof as in [20], page 144. The relation $\phi(x) = a[cx^n - \sigma(x^n)c, x^n]_k$ is a nontrivial generalized polynomial identity with automorphisms of $Q \otimes_C F$. By Chuang's theorem (see [8]), $Q \otimes_C F$ is a primitive ring having nonzero socle with F as its associated division ring. By [16], page 75, $Q \otimes_C F$ is isomorphic to a dense subring of $\mathrm{End}(V_F)$ for some vector space V over F and $Q \otimes_C F$ contains nonzero linear transformations of finite rank. By Lemmas 2.1 and 2.2, we get $a = 0$, a contradiction.

Assuming now that δ is X-outer, we have

$$0 = a\left[\sum_{i=0}^{n-1} \sigma(x^i)\delta(x)x^{n-i-1}, x^n\right]_k \tag{2.42}$$

for all $x \in R$. So by Fact 1.4, we get

$$0 = a\left[\sum_{i=0}^{n-1} \sigma(x^i)yx^{n-i-1}, x^n\right]_k$$

for all $x \in R$ and $y \in R$. If σ is X-outer then by Fact 1.5, we have

$$0 = a\left[\sum_{i=0}^{n-1} z^i yx^{n-i-1}, x^n\right]_k$$

and for $z = 0$ we obtain $a[yx^{n-1}, x^n]_k = 0$ and replacing y by yx, we get $a[yx^n, x^n]_k = 0$. Now [14], Theorem 1.2 forces $a = 0$ or R is commutative. But both cases lead to a contradiction.

Thus we may assume that σ is an X-inner automorphism. In this case there exists an invertible element $q \in Q$ such that $\sigma(x) = qxq^{-1}$ for all $x \in Q$. By (2.42), R satisfies

$$a\left[\sum_{i=0}^{n-1} (qxq^{-1})^i yx^{n-i-1}, x^n\right]_k$$

and $\sigma \neq 1$. Clearly, $\sigma = 1$ gives a contradiction since if $\sigma = 1$, then δ is an ordinary derivation and by Theorem A we get $a = 0$, a contradiction. Then this identity is a nontrivial generalized polynomial identity for R. By [16], page 75 and [22], Q is a primitive ring having a nonzero socle with C as its associated division ring and Q is isomorphic to a dense ring of linear transformations on some vector space V over C.

First we consider $\dim_C V \geqslant 3$. Since $q \notin C$, there exists $v \in V$ such that $\{q^{-1}v, v\}$ are linearly C-independent. Since $\dim_C V \geqslant 3$ there exists $w \in V$ such that $\{q^{-1}v, v, w\}$ are linearly C-independent. By the density of Q, there exist $x, y \in Q$ such that $xw = 0$, $xv = v$, $yw = v$, $xq^{-1}v = q^{-1}v$. So by (2.42) we get

$$\begin{aligned}
0 &= a\Big[\sum_{i=0}^{n-1}(qxq^{-1})^i yx^{n-i-1}, x^n\Big]_k \\
&= a\sum_{j=0}^{k}(-1)^j(x^n)^j\Big(\sum_{i=0}^{n-1}(qxq^{-1})^i yx^{n-i-1}\Big)(x^n)^{k-j}w \\
&= a(-1)^k(x^n)^k\sum_{i=0}^{n-1}(qxq^{-1})^i yx^{n-i-1}w = a(-1)^k(x^n)^k(qxq^{-1})^{n-1}yw \\
&= a(-1)^k(x^n)^k qx^{n-1}q^{-1}v = a(-1)^k(x^n)^k v = a(-1)^k v.
\end{aligned}$$

So we have

$$av = 0. \tag{2.43}$$

Since $v + w$ is also C-independent of w and $q^{-1}v$, using $v + w$ instead of v, we also have $a(w + v) = 0$, implying that

$$aw = 0. \tag{2.44}$$

And by the density of Q there exist $x, y \in Q$ such that $xw = 0$, $yw = qv$, $xv = q^{-1}v$, $xq^{-1}v = q^{-1}v$, we conclude that

$$0 = a\Big[\sum_{i=0}^{n-1}(qxq^{-1})^i yx^{n-i-1}, x^n\Big]_k w = a(-1)^k q^{-1}v.$$

Then we have

$$aq^{-1}v = 0.$$

By using (2.43), (2.44) and the last equation, we have $aV = 0$, which implies that $a = 0$, a contradiction.

Now we may assume that $\dim_C V = 2$. Then $Q \cong M_2(C)$ is the ring of all 2×2 matrices over C.

Denote $q = \sum_{r,s} q_{rs}e_{rs}$, $a = \sum_{r,s} a_{rs}e_{rs}$, $q^{-1} = \sum_{r,s} d_{rs}e_{rs}$ for $q_{rs}, a_{rs}, d_{rs} \in C$. It is clear that if $\begin{pmatrix} q_{22} & -q_{12} \\ -q_{21} & q_{11} \end{pmatrix} \in M_2(C)$ is invertible, hence its inverse is the form

$$q^{-1} = \frac{1}{\det(q)} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.$$

By the hypothesis we obtain

$$0 = a\sum_{j=0}^{k}(-1)^j \binom{k}{j}(x^n)^j \left(\sum_{i=0}^{n-1}(qxq^{-1})^i yx^{n-i-1}\right)(x^n)^{k-j}. \tag{2.45}$$

For $x = e_{11}$, $y = e_{22}$ in (2.45) and left multiplying this relation by e_{11} we get

$$a_{11}q_{22}q_{12} = 0. \tag{2.46}$$

For $x = e_{11}$, $y = e_{22}$ in (2.45) and left multiplying this relation by e_{12} we arrive at

$$a_{21}q_{22}q_{12} = 0. \tag{2.47}$$

For $x = e_{11}$, $y = e_{12}$ in (2.45) and left multiplying this relation by e_{11} we have

$$a_{11}q_{11}q_{22} = 0. \tag{2.48}$$

For $x = e_{11}$, $y = e_{12}$ in (2.45) and left multiplying this relation by e_{22} we obtain

$$a_{21}q_{11}q_{22} = 0. \tag{2.49}$$

For $x = e_{22}$, $y = e_{21}$ in (2.45) and left multiplying this relation by e_{22} we conclude that

$$a_{22}q_{22}q_{11} = 0. \tag{2.50}$$

For $x = e_{22}$, $y = e_{21}$ in (2.45) and left multiplying this relation by e_{11} we get

$$a_{12}q_{22}q_{11} = 0. \tag{2.51}$$

For $x = e_{22}$, $y = e_{11}$ in (2.45) and left multiplying this relation by e_{11} we arrive that

$$a_{12}q_{11}q_{21} = 0. \tag{2.52}$$

For $x = e_{22}$, $y = e_{11}$ in (2.45) and left multiplying this relation by e_{22} we obtain

$$a_{22}q_{11}q_{21} = 0. \tag{2.53}$$

Now we define the following automorphisms of Q:

$$\begin{aligned}
\varphi(x) &= (1 - e_{12})x(1 + e_{12}) = x + xe_{12} - e_{12}x - e_{12}xe_{12},\\
\psi(x) &= (1 + e_{12})x(1 - e_{12}) = x - xe_{12} + e_{12}x - e_{12}xe_{12},\\
\chi(x) &= (1 - e_{21})x(1 + e_{21}) = x + xe_{21} - e_{21}x - e_{21}xe_{21},\\
\beta(x) &= (1 + e_{21})x(1 - e_{21}) = x - xe_{21} + e_{21}x - e_{21}xe_{21}.
\end{aligned}$$

Of course the identity $\xi(a[\delta(x^n), x^n]_k)$ is satisfied by Q, where $\xi \in \{\varphi, \psi, \chi, \beta\}$. Hence we have for all $x \in Q$

$$\xi(a)\biggl[\sum_{i=0}^{n-1}(\xi(q)x\xi(q)^{-1})^i yx^{n-i-1}, x^n\biggr]_k = 0.$$

Therefore the matrices $\xi(a)$ and $\xi(q)$ must satisfy the above conditions (2.46)–(2.53). We may assume that $q_{11} = 0$. Since q is invertible, q_{12} and q_{21} must be nonzero elements. It is easy to see that $a = 0$ by using some basic computations. Similarly, if one of the elements q_{12}, q_{21}, and q_{22} is equal to zero then we have $a = 0$. Hence we assume that $q_{ij} \neq 0$ for $i, j \in \{1, 2\}$. So by (2.46)–(2.53), we have $a = 0$, a contradiction. □

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