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SOME APPROXIMATION RESULTS IN MUSIELAK-ORLICZ SPACES

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Abstract. We prove the continuity in norm of the translation operator in the Musielak-Orlicz L_M spaces. An application to the convergence in norm of approximate identities is given, whereby we prove density results of the smooth functions in L_M, in both the modular and norm topologies. These density results are then applied to obtain basic topological properties.

Keywords: approximate identity; Musielak-Orlicz space; density of smooth functions

MSC 2010: 46E30, 46B10

1. Introduction

Classical Lebesgue and Sobolev spaces with constant exponent arise in the modelling of most materials with sufficient accuracy. For certain materials with inhomogeneities, for instance electrorheological fluids, this is not adequate, but rather the exponent should be able to vary. This leads to studying those materials in Lebesgue and Sobolev spaces with variable exponent.

Historically, variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$, appeared in the literature for the first time in 1931 in a paper written by Orlicz, see [17]. The study of variable exponent Lebesgue spaces was then abandoned by Orlicz in favour of the theory of the function spaces $L_M(\Omega)$, built upon an N-function M, which now bears his name and which generalizes naturally the Lebesgue spaces with constant exponent. When we try to integrate both the functional structures of variable exponent Lebesgue spaces and Orlicz spaces, we are led to the so-called Musielak-Orlicz spaces. This later functional structure was extensively studied since the 1970's by the Polish school, notably by Musielak, Hudzik and Kamińska, see for instance [7], [8], [9], [15] and the references therein.

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Here we are interested in establishing some basic approximation results in Musielak-Orlicz spaces with respect to the modular and norm convergence, which then allows us to obtain some topological properties that constitute the basic tools needed in the existence theory for partial differential equations involving nonstandard growths described in terms of Musielak-Orlicz functions. Such results require to take into account the earlier ones studied deeply in the monographs (see [13], [15]) and those of the particular framework of variable Lebesgue and Sobolev spaces concerning completeness, density, reflexivity and separability obtained by Kováčik and Rákosník, see [12].

Throughout this paper, we denote by Ω an open subset of $\mathbb{R}^N$, $N \geqslant 1$. A real function $M\colon \Omega \times [0,\infty) \to [0,\infty]$ is called a φ-function, written $M \in \varphi$, if $M(x,\cdot)$ is a nondecreasing and convex function for all $x \in \Omega$ with $M(x,0) = 0$, $M(x,s) > 0$ for $s > 0$, $M(x,s) \to \infty$ as $s \to \infty$ and $M(\cdot,s)$ is a measurable function for every $s \geqslant 0$. A φ-function is called a Φ-function, denoted by $M \in \Phi$, if furthermore it satisfies

$$\lim_{s\to 0^+} \frac{M(x,s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{M(x,s)}{s} = \infty.$$

Define $\overline{M}\colon \Omega \times [0,\infty) \to [0,\infty]$ by

$$\overline{M}(x,s) = \sup_{t\geqslant 0}\{st - M(x,t)\} \quad \forall\, s \geqslant 0 \text{ and all } x \in \Omega.$$

It can be checked that $\overline{M} \in \varphi$. The Φ-function $\overline{M}$ is called the complementary function to M in the sense of Young. Given $M \in \varphi$, the Musielak-Orlicz space $L_M(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that $\int_\Omega M(x,|u(x)|/\lambda)\,\mathrm{d}x < \infty$ for some $\lambda > 0$. Equipped with the so-called Luxemburg norm

$$\|u\|_{L_M(\Omega)} = \inf\left\{\lambda > 0\colon \int_\Omega M(x,|u(x)|/\lambda)\,\mathrm{d}x \leqslant 1\right\},$$

where $L_M(\Omega)$ is a Banach space (see [15], Theorem 7.7). It is a particular case of the so-called modular function spaces, investigated by Nakano (see for instance [16]). We define $E_M(\Omega)$ as the subset of $L_M(\Omega)$ of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that $\int_\Omega M(x,|u(x)|/\lambda)\,\mathrm{d}x < \infty$ for all $\lambda > 0$.

A density result for smooth functions in Musielak-Orlicz-Sobolev spaces with respect to the modular topology was claimed for the first time in [2] in $\Omega = \mathbb{R}^N$ and then for a bounded star-shaped Lipschitz domain Ω in [3]. The authors assumed that the Φ-function M satisfies, among others, the log-Hölder continuity condition, that is to say there exists a constant $A > 0$ such that for all $s \geqslant 1$,

$$(1.1) \qquad \frac{M(x,s)}{M(y,s)} \leqslant s^{-A/\log|x-y|} \quad \forall\, x,y \in \Omega \quad \text{with } |x-y| \leqslant \frac{1}{2}.$$

Nonetheless, the proof involved an essential gap. The Jensen inequality was used for the infimum of convex functions, which obviously is not necessarily convex.

Unlike the classical Orlicz spaces, the spatial dependence of the φ-function M does not allow, in general, bounded functions to belong to Musielak-Orlicz spaces even if Ω has finite Lebesgue measure. Particularly, characteristic functions have no reason, in general, to lie in Musielak spaces. In the approach we use here, we only need M to be locally integrable that is for any constant number $c > 0$ and for every compact set $K \subset \Omega$

$$\int_K M(x, c)\,\mathrm{d}x < \infty. \tag{1.2}$$

Inequality (1.2) was introduced in [15], Definition 7.5 for measurable subsets of Ω with finite measure. Observe that (1.2) is not always satisfied as shown by the following example. Set $\Omega = (-1/2, 1/2)$ and set

$$M(x, s) = \begin{cases} s^{1/x}, & x \in (0, 1/2), \\ s^2, & x \in (-1/2, 0). \end{cases}$$

Note that M is a Φ-function. Consider the compact set $K = [0, 1/4]$, which is contained in Ω. Then for $c > 1$

$$\int_K M(x, c)\,\mathrm{d}x = \int_0^{1/4} c^{1/x}\,\mathrm{d}x = \infty.$$

From now on, $\mathcal{B}_c(\Omega)$ will stand for the set of bounded functions compactly supported in Ω and $\mathcal{C}_0^\infty(\Omega)$ will denote the set of infinitely differentiable functions compactly supported in Ω.

The condition (1.2) ensures that the set $\mathcal{B}_c(\Omega)$ is contained in $E_M(\Omega)$. Incidentally, the functions essentially bounded do not belong necessarily to $E_M(\Omega)$ even if (1.2) is satisfied. Here, we do not need to assume the condition (1.1).

Let us note that if $M \in \varphi$ (or $\overline{M} \in \varphi$) satisfies $\lim\limits_{s\to\infty} \operatorname*{ess\,inf}\limits_{x\in\Omega} M(x, s)/s = \infty$ ($\lim\limits_{s\to\infty} \operatorname*{ess\,inf}\limits_{x\in\Omega} \overline{M}(x, s)/s = \infty$), then $\overline{M}$ (M) satisfies (1.2) not only for compact subsets $K \subset \Omega$ but for all measurable subsets of Ω having finite Lebesgue measure. Indeed, assume that $\lim\limits_{s\to\infty} \operatorname*{ess\,inf}\limits_{x\in\Omega} M(x, s)/s = \infty$ is fulfilled. Then for arbitrary $c > 0$, there exists $s_c > 0$ (not depending on x) such that for all $s > s_c$

$$\operatorname*{ess\,inf}_{x\in\Omega} \frac{M(x, s)}{s} > c + 1.$$

Thus, $\sup\limits_{s>s_c} (sc - M(x, s)) \leqslant 0$ and by the definition of $\overline{M}$ we obtain

$$\overline{M}(x, c) \leqslant \sup_{0\leqslant s\leqslant s_c} (sc - M(x, s)) \leqslant cs_c.$$

This inequality holds true for every $x \in \Omega$ and then we get (1.2) for subsets of finite Lebesgue measure.

In this paper, our main goal is to establish density results for smooth functions in Musielak spaces. To do so we first prove a result on the M-mean continuity of bounded functions compactly supported in Ω which we then apply to get the convergence in norm of approximate identities.

The paper is organized as follows. In Section 2 we give the main results. Section 3 is devoted to the proof of the main results. At the end we give two appendices that contain some basic properties of Musielak-Orlicz spaces that we prove using our main results.

2. Main results

For $h \in \mathbb{R}^N$, let $\tau_h u$ stand for the translation operator defined by

$$\tau_h u(x) = \begin{cases} u(x+h) & \text{if } x \in \Omega \text{ and } x+h \in \Omega, \\ 0 & \text{otherwise in } \mathbb{R}^N. \end{cases}$$

If the function u has a compact support, $\tau_h u$ is well-defined provided that $h < \operatorname{dist}(\operatorname{supp} u, \partial\Omega)$.

Theorem 2.1. *Let M be a Φ-function satisfying* (1.2). *Then any $u \in \mathcal{B}_c(\Omega)$ is M-mean continuous, that is to say for every $\varepsilon > 0$ there exists an $\eta = \eta(\varepsilon) > 0$ such that for $h \in \mathbb{R}^N$ with $|h| < \eta$ we have*

$$\|\tau_h u - u\|_{L_M(\Omega)} < \varepsilon.$$

Let J stand for the Friedrichs mollifier kernel defined on $\mathbb{R}^N$ by

$$J(x) = \begin{cases} k\mathrm{e}^{-1/(1-\|x\|^2)} & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| \geqslant 1, \end{cases}$$

where $k > 0$ is such that $\int_{\mathbb{R}^N} J(x)\,\mathrm{d}x = 1$. For $\varepsilon > 0$, we define $J_\varepsilon(x) = \varepsilon^{-N} J(x\varepsilon^{-1})$ and $u_\varepsilon = J_\varepsilon * u$ by

$$u_\varepsilon(x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)u(y)\,\mathrm{d}y = \int_{B(0,1)} u(x-\varepsilon y)J(y)\,\mathrm{d}y. \tag{2.1}$$

A direct consequence of Theorem 2.1 is the following approximation result.

Corollary 2.1. *Let M be a Φ-function satisfying* (1.2) *and let $u \in \mathcal{B}_c(\Omega)$. For any $\varepsilon > 0$ small enough, we have $u_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$. Furthermore,*

$$\|u_\varepsilon - u\|_{L_M(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Theorem 2.2. *Let M be a Φ-function satisfying* (1.2). *Then*

(1) $\mathcal{C}_0^\infty(\Omega)$ *is dense in $E_M(\Omega)$ with respect to the strong topology in $E_M(\Omega)$.*

(2) $\mathcal{C}_0^\infty(\Omega)$ *is dense in $L_M(\Omega)$ with respect to the modular topology in $L_M(\Omega)$.*

A special class of Φ-functions is introduced by the following:

Definition 2.1. We say that $M \in \Phi$ satisfies the Δ_2-condition, written $M \in \Delta_2$, if there exist a constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$M(x, 2t) \leqslant kM(x, t) + h(x) \tag{2.2}$$

for all $t \geqslant 0$ and for almost every $x \in \Omega$.

Remark 2.1. Let M be a Φ-function satisfying (1.2). In view of Lemma A.3, if $M \in \Delta_2$, then $\mathcal{C}_0^\infty(\Omega)$ is dense in $L_M(\Omega)$ with respect to the norm $\|\cdot\|_{L_M(\Omega)}$.

In general, if $u \in E_M(\Omega)$ we cannot expect that $\tau_h u$ belongs to $E_M(\Omega)$ as was proved first by Kamińska [8], Theorem 2.1 (see also [12], Example 2.9 and Theorem 2.10). In Theorem 2.1, we prove that the translation operator acts on the set of bounded functions compactly supported in Ω. In the case, where $M(x,t) = |t|^{p(x)}$, a similar result was proved in [5], page 261 by using the continuous imbedding between variable exponent Lebesgue spaces. Unfortunately, this result is not true in general in variable Lebesgue spaces, as shown in [8], Example 2^0 (see also [6], Proposition 3.6.1) unless the exponent is constant.

Remark 2.2. Note that the boundedness of the function u in Theorem 2.1 is necessary, else the result is false. Indeed, when we put ourselves in the particular case $M(x,t) = t^{p(x)}$, the authors in [12] gave the following example: $N = 1$, $\Omega = (-1, 1)$. For $1 \leqslant r < s < \infty$ they define the variable exponent

$$p(x) = \begin{cases} r & \text{if } x \in [0,1), \\ s & \text{if } x \in (-1,0) \end{cases}$$

and consider the function

$$f(x) = \begin{cases} x^{-1/s} & \text{if } x \in [0,1), \\ 0 & \text{if } x \in (-1,0). \end{cases}$$

They show that $\tau_h f \notin L^{p(\cdot)}(\Omega)$ although $f \in L^{p(\cdot)}(\Omega)$. Observe here, in this example, that the function f is compactly supported but not bounded on Ω.

Theorem 2.2 is a unified generalization of the approximation results known in Lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$ and Orlicz spaces. The approach, now classical, is based upon reducing the study to continuous functions compactly supported in Ω and then using imbedding theorems and a sequence of mollifiers to conclude, see for instance [1], Corollary 2.30 and Theorem 8.21. This classical approach is based on the fact that the translation operator $u(\cdot + h)$ is continuous in norm when h tends to zero. This fails to hold in Musielak-Orlicz spaces as shown in Remark 2.2. Consequently, we cannot approximate in general the identities for a given function.

In the framework of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, Kováčik and Rákosník in [12], Theorem 2.11 proved first the density of the set $\mathcal{C}_0^\infty(\Omega)$ of infinitely differentiable functions compactly supported in Ω, provided only that the variable exponent is $p(\cdot) \in L^\infty(\Omega)$. Their idea consists in showing successively that the set of essentially bounded functions $L^\infty(\Omega) \cap L^{p(\cdot)}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and by means of Luzin's theorem the subset of continuous functions $\mathcal{C}(\Omega) \cap L^{p(\cdot)}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$, which finally leads to the density of the set $\mathcal{C}_0^\infty(\Omega)$ in $L^{p(\cdot)}(\Omega)$.

In Musielak-Orlicz spaces, the situation is more complicated. First, we note that although the assumption (1.2) is satisfied, the inclusion $L^\infty(\Omega) \subset L_{M(\cdot,\cdot)}(\Omega)$ does not hold true in general even if Ω is an open subset of $\mathbb{R}^N$ with finite Lebesgue measure. Secondly, the use of an idea similar to that of Cruz-Uribe and Fiorenza (see [5]) will require additional assumptions. Thirdly, in contrast to what is mentioned above, the translation operator is not acting, in general, between Musielak-Orlicz spaces (see [12], Example 2.9 and Theorem 2.10 and [6], Proposition 3.6.1). For these reasons and to the best of our knowledge, it is not possible to obtain approximation results using classical ideas. The approach we use consists in starting by proving the density of smooth functions $\mathcal{C}_0^\infty(\Omega)$ in $\mathcal{B}_c(\Omega)$ with respect to the norm in $L_M(\Omega)$ (see Corollary 2.1), and then the density of bounded functions compactly supported in norm in $E_M(\Omega)$ and in modular norm in $L_M(\Omega)$ (see Lemma B.1), which allows us to get the density of smooth functions $\mathcal{C}_0^\infty(\Omega)$ in $E_M(\Omega)$ and $L_M(\Omega)$ with respect to the norm and modular convergences, respectively. The idea we use in this paper is essentially based upon using the fact that for a function $u \in \mathcal{B}_c(\Omega)$, the translation operator $u(\cdot + h)$ is continuous with respect to the norm $\|\cdot\|_{L_M(\Omega)}$ as h tends to 0 (see Theorem 2.1). This constitutes the main reason why we introduce the space $\mathcal{B}_c(\Omega)$.

3. Proof of the main results

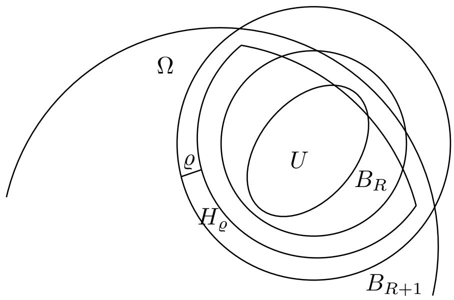
Proof of Theorem 2.1. For $u \in \mathcal{B}_c(\Omega)$, let $\operatorname{supp} u = U \subset B_R \cap \Omega$, where by B_R we denote a ball with radius $R > 0$. Let $h \in \mathbb{R}^N$ with $|h| < \min(1, \operatorname{dist}(U, \partial\Omega))$. We have $\operatorname{supp} \tau_h u \subset B := B_{R+1} \cap \Omega$. Let us define $\overline{B} = \overline{B}_{R+1} \cap \Omega$ where $\overline{B}_{R+1}$ stands for the closed ball with radius $R + 1$. Thanks to (1.2), for any constant number $C > 0$

and any compact subset $K \subset \Omega$ one has $M(x,C) \in L^1(K)$. Therefore for arbitrary $\varepsilon > 0$ there is $\nu > 0$ such that for every measurable subset $\Omega' \subset K$

$$\int_{\Omega'} M(x,C)\,\mathrm{d}x < \frac{\varepsilon}{2}, \quad \text{whenever } |\Omega'| < \nu. \tag{3.1}$$

For this ν there exists $\varrho \in (0,1)$ such that the Lebesgue measure is $|H_\varrho| < \frac{1}{4}\nu$, where

$$H_\varrho = \{x \in B\colon \operatorname{dist}(x, \partial B) \leqslant \varrho\}.$$



Define $U_\varrho = B \setminus H_\varrho$. Since u is measurable on U_ϱ, Luzin's theorem ensures that for $\nu > 0$ there exists a closed set $F_{1,\nu} \subset U_\varrho$ such that the restriction of u to $F_{1,\nu}$ is continuous and $|U_\varrho \setminus F_{1,\nu}| < \frac{1}{4}\nu$. We then have $|B \setminus F_{1,\nu}| < \frac{1}{2}\nu$. The function u is uniformly continuous on the compact set $F_{1,\nu}$. It follows that for $\varepsilon > 0$, there exists an $\eta \in (0, \varrho)$ such that for all x, $x + h \in F_{1,\nu}$ one has

$$|h| < \eta \Rightarrow |u(x+h) - u(x)| < \frac{\varepsilon}{2(\int_U M(x,1)\,\mathrm{d}x + 1)}. \tag{3.2}$$

Define two sets

$$F_{2,\nu} = \{x \in U, x + h \in F_{1,\nu}\} \quad \text{and} \quad F_\nu = F_{1,\nu} \cap F_{2,\nu}.$$

The set F_ν is a closed subset of Ω. In addition, we have $|B \setminus F_\nu| < \nu$. Indeed, since the Lebesgue measure is invariant by translation we get $|B \setminus F_{1,\nu}| = |B \setminus F_{2,\nu}|$. Therefore,

$$|B \setminus F_\nu| = |(B \setminus F_{1,\nu}) \cup (B \setminus F_{2,\nu})| \leqslant |B \setminus F_{1,\nu}| + |B \setminus F_{2,\nu}| < \nu. \tag{3.3}$$

If $x \notin B$ then for $|h| < \eta$ we have $x + h \notin B_R \cap \Omega$. If not, we would get $x \in B$ which contradicts the fact that $x \notin B$. Hence, we obtain

$$\begin{aligned} \int_\Omega M(x, |\tau_h u(x) - u(x)|)\,\mathrm{d}x &= \int_B M(x, |\tau_h u(x) - u(x)|)\,\mathrm{d}x \\ &= \int_{B \cap F_\nu} M(x, |\tau_h u(x) - u(x)|)\,\mathrm{d}x + \int_{B \setminus F_\nu} M(x, |\tau_h u(x) - u(x)|)\,\mathrm{d}x. \end{aligned} \tag{3.4}$$

By (3.2) the first term on the right-hand side can be estimated as

$$\int_{B\cap F_\nu} M(x,|\tau_h u(x)-u(x)|)\,\mathrm{d}x \leqslant \int_{B\cap F_\nu} M\Big(x,\frac{\varepsilon}{2(\int_U M(x,1)\,\mathrm{d}x+1)}\Big)\,\mathrm{d}x < \frac{\varepsilon}{2}.$$

As regards the second term on the right-hand side of (3.4), we use the fact that $u\in\mathcal{B}_c(\Omega)$ is bounded by a constant number $c>0$ and then (3.1) (since $B\setminus F_\nu\subset K\colon = H_\varrho\cup\overline{U_\varrho}\cup U$) to obtain

$$\int_{B\setminus F\nu} M(x,|\tau_h u(x)-u(x)|)\,\mathrm{d}x \leqslant \int_{B\setminus F\nu} M(x,2c)\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}. \tag{3.5}$$

Putting together (3.4) and (3.5), we get

$$\forall\,\varepsilon>0,\quad \exists\,\eta>0\colon\ |h|<\eta\Rightarrow\int_\Omega M(x,|\tau_h u(x)-u(x)|)\,\mathrm{d}x<\varepsilon.$$

Let $\delta>0$ be arbitrary but fixed. As $u/\delta\in\mathcal{B}_c(\Omega)$, we get

$$\exists\,\eta>0\colon\ |h|<\eta\Rightarrow\int_\Omega M\Big(x,\frac{|\tau_h u(x)-u(x)|}{\delta}\Big)\,\mathrm{d}x\leqslant 1,$$

which gives

$$\|\tau_h u-u\|_{L_M(\Omega)}\leqslant\delta\quad\text{whenever } |h|<\eta.$$

$\square$

P r o o f of Corollary 2.1. Let $u\in\mathcal{B}_c(\Omega)$. The function u_ε defined in (2.1) belongs to $\mathcal{C}_0^\infty(\Omega)$ whenever $\varepsilon<\operatorname{dist}(\operatorname{supp}u,\partial\Omega)$ (see for example [1], Theorem 2.29). Let $\overline{M}$ stand for the complementary Φ-function of M and let $v\in L_{\overline{M}}(\Omega)$. By Fubini theorem and Hölder inequality (A.3) we can write

$$\begin{aligned}\int_\Omega|(u_\varepsilon(x)-u(x))v(x)|\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^N}\Big(\int_\Omega|u(x-\varepsilon y)-u(x)||v(x)|\,\mathrm{d}x\Big)J(y)\,\mathrm{d}y\\ &\leqslant 2\|v\|_{L_{\overline{M}}(\Omega)}\int_{|y|\leqslant 1}\|\tau_{-\varepsilon y}u-u\|_{L_M(\Omega)}J(y)\,\mathrm{d}y.\end{aligned}$$

Hence, by the definition of the Orlicz norm and the inequality (A.4) we obtain

$$\|u_\varepsilon-u\|_{L_M(\Omega)}\leqslant 2\int_{|y|\leqslant 1}\|\tau_{-\varepsilon y}u-u\|_{L_M(\Omega)}J(y)\,\mathrm{d}y.$$

We can now use Theorem 2.1. Given $\mu>0$, there exists $\eta>0$ such that for $\varepsilon<\eta$ we get

$$\|\tau_{-\varepsilon y}u(x)-u(x)\|_{L_M(\Omega)}\leqslant\mu$$

for every y with $|y| \leqslant 1$. Then we conclude that

$$\|u_\varepsilon - u\|_{L_M(\Omega)} \leqslant 2\mu \int_{|y|\leqslant 1} J(y)\,\mathrm{d}y = 2\mu,$$

which gives the result. $\square$

P r o o f of Theorem 2.2. (1) Combining Corollary 2.1 and Lemma B.1, we obtain the density of $\mathcal{C}_0^\infty(\Omega)$ in $E_M(\Omega)$ with respect to the strong topology.

(2) Let $u \in L_M(\Omega)$. According to Lemma B.1, there exist $w \in \mathcal{B}_c(\Omega)$ and $\lambda > 0$ such that for all $\eta \geqslant 0$

$$\int_\Omega M(x, |u(x) - w(x)|/\lambda)\,\mathrm{d}x \leqslant \eta.$$

Then by Corollary 2.1 there exists a function $v \in \mathcal{C}_0^\infty(\Omega)$ that converges strongly to w in $L_M(\Omega)$. But we know that the norm topology is stronger than the modular one, more precisely we have

$$\int_\Omega M(x, |w(x) - v(x)|)\,\mathrm{d}x \leqslant \eta.$$

Let us make the choice $\lambda_1 = \max\{1, \lambda\}$ and use the convexity of the Φ-function M; we can write

$$\int_\Omega M(x, |u(x) - v(x)|/2\lambda_1)\,\mathrm{d}x$$
$$\leqslant \frac{1}{2}\int_\Omega M(x, |u(x) - w(x)|/\lambda)\,\mathrm{d}x + \frac{1}{2}\int_\Omega M(x, |w(x) - v(x)|)\,\mathrm{d}x.$$

This yields the result. $\square$

Appendix A. Background

Here, we recall some known facts about Musielak-Orlicz spaces. More details can be found in the papers by Musielak, Kamińska and Hudzik. Observe first that equivalently a Φ-function M can be represented as (see [15], Theorem 13.2)

$$M(x,t) = \int_0^t a(x,s)\,\mathrm{d}s \quad \text{for } t \geqslant 0,$$

where $a(x,\cdot)$ is a right continuous and increasing function, $a(x,s)>0$ for $s>0$, $a(x,0)=0$, $a(x,s)\to\infty$ as $s\to\infty$ for every $x\in\Omega$. The complementary of a Φ-function M (see [15], Definition 13.4) can be also expressed as

$$\overline{M}(x,t)=\int_0^t a^*(x,s)\,\mathrm{d}s \quad \text{for } t\geqslant 0,$$

where $a^*(x,s)=\sup\{v, a(x,v)\leqslant s\}$. Moreover, we have the Young inequality

$$uv\leqslant M(x,u)+\overline{M}(x,v) \quad \forall\, u,v\geqslant 0,\ \forall\, x\in\Omega, \tag{A.1}$$

which reduces to an equality when $v=a(x,u)$ or $u=a^*(x,v)$. It's easy to check that

$$\|u\|_{L_M(\Omega)}\leqslant 1 \Leftrightarrow \int_\Omega M(x,|u(x)|)\,\mathrm{d}x\leqslant 1. \tag{A.2}$$

We also have the following Hölder inequality (see [15], Theorem 13.13)

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x\leqslant \|u\|_M\|v\|_{L_{\overline{M}}(\Omega)} \tag{A.3}$$

for all $u\in L_M(\Omega)$ and $v\in L_{\overline{M}}(\Omega)$, where $\|u\|_M=\sup\limits_{\|v\|_{L_{\overline{M}}}\leqslant 1}\int_\Omega |u(x)v(x)|\,\mathrm{d}x$ is the Orlicz norm.

The equivalence between Orlicz and Luxemburg norms is well-known in the Orlicz spaces setting, see [14], Theorem 3.8.5, while in the Musielak-Orlicz framework this result was proved by Musielak in [15], Theorem 13.11 using a local integrability condition upon measurable sets with finite measure.

Lemma A.1. *Let M be a Φ-function satisfying* (1.2). *Then, for all $u\in L_M(\Omega)$*

$$\|u\|_{L_M(\Omega)}\leqslant \|u\|_M\leqslant 2\|u\|_{L_M(\Omega)}. \tag{A.4}$$

Proof. The inequality on the right-hand side is an easy consequence of the Young inequality. We only need to prove the left-hand side inequality. To this end, it is sufficient to prove that

$$\int_\Omega M(x,|u(x)|/\|u\|_M)\,\mathrm{d}x\leqslant 1.$$

This can be done by using (1.2) and following exactly the lines of [14], Lemma 3.7.2. $\square$

We denote by M^{-1} the inverse of the Φ-function M with respect to its second argument defined as follows:

$$M^{-1}(x,t) = \inf\{s \geqslant 0,\, M(x,s) \geqslant t\}.$$

Thus, we have

$$M^{-1}(x, M(x,s)) = M(x, M^{-1}(x,s)) = s. \tag{A.5}$$

A sequence $\{u_n\}$ is said to converge to u in $L_M(\Omega)$ in the modular sense, if there exists $\lambda > 0$ such that

$$\int_\Omega M\Big(x, \frac{|u_n(x) - u(x)|}{\lambda}\Big)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

We say that $\{u_n\}$ converges to u in norm in $L_M(\Omega)$, if $\|u_n - u\|_{L_M(\Omega)} \to 0$ as $n \to \infty$.

The following lemma was proved in [10], Proposition 3.1, [15], Theorem 8.14 using the local integrability (see [15], Definition 7.5). For the convenience of the reader, we give here a simple proof using Vitali's theorem.

Lemma A.2. *Let $M \in \Phi$. If $M \in \Delta_2$, then the norm convergence and the modular convergence are equivalent.*

Proof. We only need to prove that the modular convergence implies the norm convergence, the converse is an easy task. Let $\{u_n\}$ be a sequence of functions belonging to $L_M(\Omega)$ such that $\int_\Omega M(x, u_n(x)/\lambda)\,\mathrm{d}x \to 0$ as $n \to \infty$ for some $\lambda > 0$. Thus, $M(x, u_n(x)/\lambda) \to 0$ strongly in $L^1(\Omega)$. Hence, for a subsequence still indexed by n, we can assume that $u_n \to u$ a.e. in Ω. Let p be a fixed integer, by the Δ_2-condition we can write

$$M(x, 2^p u_n(x)/\lambda) \leqslant k^p M(x, u_n(x)/\lambda) + (k^{p-1} + \ldots + k + 1)h(x).$$

Therefore, by Vitali's theorem we get $\lim\limits_{n\to\infty} \int_\Omega M(x, 2^p u_n)\,\mathrm{d}x = 0$. For an arbitrary $\lambda > 0$ there exists m such that $\lambda \leqslant 2^m$. Then we can write

$$\int_\Omega M(x, \lambda u_n)\,\mathrm{d}x \leqslant \frac{\lambda}{2^m} \int_\Omega M(x, 2^m u_n)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty,$$

which gives $\|u_n\|_{L_M(\Omega)} \to 0$ as $n \to \infty$. $\square$

Note that $E_M(\Omega)$ is a closed subset of $L_M(\Omega)$. Indeed, let $\{u_n\} \subset E_M(\Omega)$ be such that $u_n \to u \in L_M(\Omega)$. For any $\lambda > 0$, we have $\int_\Omega M(x, 2\lambda|u_n - u|)\,\mathrm{d}x \to 0$. This implies $M(x, 2\lambda|u_n - u|) \to 0$ in $L^1(\Omega)$. So there is $h \in L^1(\Omega)$ such that $M(x, 2\lambda|u_n - u|) \leqslant h$. Hence,

$$\lambda|u_n(x) - u(x)| \leqslant \tfrac{1}{2} M^{-1}(x, h(x)),$$

which yields

$$\lambda|u(x)| \leqslant \lambda|u_n(x)| + \tfrac{1}{2} M^{-1}(x, h(x)).$$

By the convexity of M, we get

$$M(x, \lambda|u(x)|) \leqslant \tfrac{1}{2} M(x, 2\lambda|u_n(x)|) + \tfrac{1}{2} h(x).$$

Thus

$$\int_\Omega M(x, \lambda|u(x)|)\,\mathrm{d}x \leqslant \frac{1}{2} \int_\Omega M(x, 2\lambda|u_n(x)|)\,\mathrm{d}x + \frac{1}{2} \int_\Omega h(x)\,\mathrm{d}x < \infty.$$

So $u \in E_M(\Omega)$.

Lemma A.3. *Let $M \in \varphi$, then the following assertions are equivalent*

(i) $E_M(\Omega) = L_M(\Omega)$.

(ii) $M \in \Delta_2$.

Proof. (i)$\Rightarrow$(ii) For any $u \in L_M(\Omega)$ we have $2u \in L_M(\Omega)$. Then

$$L_M(\Omega) \subset L_{\widetilde{M}}(\Omega), \quad \text{where } \widetilde{M}(x, u) = M(x, 2u).$$

Therefore, by [15], Theorem 8.5 (b) there exist a constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\widetilde{M}(x, u) = M(x, 2u) \leqslant k M(x, u) + h(x).$$

(ii)$\Rightarrow$(i) For $u \in L_M(\Omega)$ there exists $\lambda > 0$ such that $\int_\Omega M(x, |u(x)|/\lambda)\,\mathrm{d}x < \infty$. We shall prove that for every $\mu > 0$ we have $\int_\Omega M(x, |u(x)|/\mu)\,\mathrm{d}x < \infty$. Indeed, there exists an integer m such that $\lambda/\mu \leqslant 2^m$. Thus, we can write

$$\begin{aligned}\int_\Omega M(x, |u(x)|/\mu)\,\mathrm{d}x &\leqslant \frac{\lambda}{2^m \mu} \int_\Omega M\Big(x, \frac{2^m |u(x)|}{\lambda}\Big)\,\mathrm{d}x \\ &\leqslant \frac{\lambda}{2^m \mu} \Big(k^m \int_\Omega M\Big(x, \frac{|u(x)|}{\lambda}\Big)\,\mathrm{d}x + (k^{m-1} + \ldots + k + 1) \int_\Omega h(x)\,\mathrm{d}x\Big) < \infty.\end{aligned}$$

$\square$

In the following lemma we prove approximation results in $E_M(\Omega)$ and $L_M(\Omega)$ with respect to the strong and modular topologies respectively by bounded functions compactly supported in Ω.

Lemma B.1. *Let M be a Φ-function satisfying* (1.2). *Then*
(1) *$\mathcal{B}_c(\Omega)$ is dense in $E_M(\Omega)$ with respect to the strong topology in $L_M(\Omega)$.*
(2) *$\mathcal{B}_c(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology in $L_M(\Omega)$.*

Proof. (i) If $u \in E_M(\Omega)$, then for all $\lambda > 0$ one has $M(x, |u|/\lambda) \in L^1(\Omega)$. Denote by T_j, $j > 0$, the truncation function at levels $\pm j$ defined on $\mathbb{R}$ by $T_j(s) = \max\{-j, \min\{j, s\}\}$. We define the sequence $\{u_j\}$ by

$$u_j = T_j(u)\chi_{K_j}, \tag{B.1}$$

where χ_{K_j} stands for the characteristic function of the set

$$K_j = \Big\{x \in \Omega\colon\ |x| \leqslant j, \operatorname{dist}(x, \Omega^c) \geqslant \frac{1}{j}\Big\}.$$

Hence, the function u_j belongs to $\mathcal{B}_c(\Omega)$ and converges almost everywhere to u in Ω. Thus $M(x, |u_j(x) - u(x)|/\lambda) \to 0$ a.e. in Ω and

$$M(x, |u_j(x) - u(x)|/2\lambda) \leqslant M(x, |u(x)|/\lambda) \in L^1(\Omega). \tag{B.2}$$

So that by the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega M(x|u_j(x) - u(x)|/2\lambda)\,\mathrm{d}x \leqslant 1 \quad \text{for } j \text{ large enough,}$$

which yields $\lim\limits_{j\to\infty} \|u_j - u\|_{L_M(\Omega)} \leqslant \lambda$. Being $\lambda > 0$ arbitrary, we get

$$\lim_{j\to\infty} \|u_j - u\|_{L_M(\Omega)} = 0.$$

(ii) Now if $u \in L_M(\Omega)$ then for some $\lambda > 0$ one has $M(x, |u|/\lambda) \in L^1(\Omega)$. Let $\{u_j\}$ be the sequence defined in (B.1). The inequality (B.2) holds for some $\lambda > 0$ and since $u_j \in \mathcal{B}_c(\Omega)$ and converges a.e. to u in Ω, we get $M(x, |u_j(x) - u(x)|/2\lambda) \to 0$ a.e. in Ω. Thus, Lebesgue's dominated convergence theorem yields

$$\int_\Omega M(x, |u_j(x) - u(x)|/2\lambda) \to 0 \quad \text{as } j \to \infty.$$

$\square$

In the framework of classical Lebesgue or Orlicz spaces, the separability of the closure of bounded functions compactly supported in $\overline{\Omega}$ is well-known, see for instance [1], Theorems 2.21 and 8.21, while for bounded variable exponent spaces one can see in [12], Corollary 2.12. Here, using the density results obtained in Theorem 2.2 we prove the separability of $E_M(\Omega)$. Our proof is totally different from that given by Musielak in [15], Theorem 7.10, since the author used the density of simple functions assuming that the Φ-function M satisfies the local integrability condition on a measure space (Ω, Σ, μ), where μ is a positive complete measure, that is to say $\int_D M(x,s)\,\mathrm{d}\mu < \infty$ for every $s > 0$ and $D \in \Sigma$ with $\mu(D) < \infty$ (see [15], Definition 7.5).

Theorem B.1. *For any $M \in \Phi$ satisfying* (1.2), *the space $E_M(\Omega)$ is separable.*

Remark B.1. In view of Lemma A.3, if the Φ-function $M \in \Delta_2$ then $L_M(\Omega)$ is a separable space.

P r o o f of Theorem B.1. Let $u \in E_M(\Omega)$. By virtue of Theorem 2.2, we can assume that $u \in \mathcal{C}_c(\Omega)$ (the set of continuous functions with compact support in Ω). Hence, it is sufficient to show that there exists a countable set dense in $\mathcal{C}_c(\Omega)$ with respect to the strong topology in $E_M(\Omega)$. Let Ω_n be the sequence of compact subsets of $\mathbb{R}^N$ defined by

$$\Omega_n = \Big\{x \in \Omega\colon\ |x| \leqslant n \text{ and } \operatorname{dist}(x, \partial\Omega) \geqslant \frac{1}{n}\Big\}.$$

Recall that $\Omega = \bigcup\limits_{i=1}^{\infty} \Omega_n$. Let $\mathcal{P}$ be the set of all polynomials on $\mathbb{R}^N$ with rational coefficients and

$$\mathcal{P}_n = \{v\chi_{\Omega_n}\colon\ v \in \mathcal{P}\},$$

where χ_{Ω_n} is the characteristic function of Ω_n. If $\operatorname{dist}(\operatorname{supp} u, \partial\Omega) > 1/n$ then u belongs to $\mathcal{C}(\Omega_n)$ and by using the density of $\mathcal{P}_n$ in $\mathcal{C}(\Omega_n)$, see [1], Corollary 1.32, there exists a sequence $u_j \in \mathcal{P}_n$ that converges uniformly to u, that is to say

$$\forall\,\varepsilon > 0,\ \exists\, j_0 \in \mathbb{N},\ \forall\, j \geqslant j_0,\ \sup_{x \in \Omega_n} |u_j(x) - u(x)| \leqslant \frac{\varepsilon}{\int_{\Omega_n} M(x,1)\,\mathrm{d}x + 1}.$$

So for every $\varepsilon > 0$ there exists $j_0 \in \mathrm{N}$ such that for any $j \geqslant j_0$ one has

$$\int_{\Omega_n} M\Big(x, \frac{|u(x) - u_j(x)|}{\varepsilon}\Big)\,\mathrm{d}x \leqslant 1.$$

Therefore, $\mathcal{P}_n$ is dense in $\mathcal{C}(\Omega_n)$ for the strong topology in $L_M(\Omega)$. Consequently, the countable set $\bigcup\limits_{n=1}^{\infty} \mathcal{P}_n$ is dense in $E_M(\Omega)$ and $E_M(\Omega)$ is separable. $\square$

In the Orlicz spaces, the density of simple functions, denoted $\mathcal{S}$, in $E_M(\Omega)$ is an important step in the proof of the duality result (see for instance [1], Theorem 8.19). In the Musielak-Orlicz spaces such approximation result needs to assume a local integrability condition (see [15], Theorem 7.6) which allows to get the inclusion $\mathcal{S} \subset E_M(\Omega)$.

Here, we use the density of the set S_c of simple functions compactly supported in Ω (see Lemma B.2 below) and prove a duality result in the following theorem.

Theorem B.2. *Let M be a Φ-function satisfying* (1.2) *and let $\overline{M}$ stand for its complementary function. Then, the dual space $(E_M(\Omega))'$ of $E_M(\Omega)$ is isomorphic to $L_{\overline{M}}(\Omega)$; denote $(E_M(\Omega))' \simeq L_{\overline{M}}(\Omega)$.*

We point out that this duality result is well known (see [4]). The general case of Köthe duality of Musielak-Orlicz spaces was given for the first time by Kamińska-Kubiak in [11]. The result we obtain here can be deduced from [11]. For the convenience of the reader we give the proof after proving the following four lemmas.

We denote by $\mathcal{S}$ the family of finite linear combinations of characteristic functions of measurable sets B_i with finite Lebesgue measure, expressed as

$$\sum_{i=1}^{p} \alpha_i \chi_{B_i}(x) \quad \text{with } \alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R} \text{ and } |B_i| < \infty.$$

Let $\mathcal{S}_c$ stand for the set of the functions belonging to $\mathcal{S}$ with the additional property that $\bigcup_{i=1}^{p} B_i \subset K$ for some compact subset K of Ω. In the next lemma we prove the density of $\mathcal{S}_c$ in $E_M(\Omega)$.

Lemma 3.1. *Let M be a Φ-function satisfying* (1.2). *Then the set $\mathcal{S}_c$ is dense in $E_M(\Omega)$ with respect to the strong topology in $E_M(\Omega)$.*

Proof. Let $u \in \mathcal{B}_c(\Omega)$. Since u is a measurable function, by classical result (see [1]) we know that there exists a sequence $\{u_n\} \subset \mathcal{S}$ converging pointwise to u in Ω and satisfying $|u_n(x)| \leqslant |u(x)|$ for all $n \in \mathbb{N}$ and $x \in \Omega$. Since $u \in \mathcal{B}_c(\Omega)$, we can assume that $u_n \in \mathcal{S}_c$. Hence,

$$M(x, |u_n(x) - u(x)|/\lambda) \leqslant M(x, 2|u(x)|/\lambda) \in L^1(\Omega).$$

As for all $\lambda > 0$

$$M(x, |u_n(x) - u(x)|/\lambda) \to 0 \quad \text{a.e. in } \Omega,$$

by the Lebesgue dominated convergence theorem we obtain

$$\int_\Omega M(x, |u_n(x) - u(x)|/\lambda)\,\mathrm{d}x \leqslant 1 \quad \text{for } n \text{ large enough},$$

which yields $\|u_n - u\|_{L_M(\Omega)} \leqslant \lambda$ for n large enough. Being $\lambda > 0$ arbitrary, we get

$$\|u_n - u\|_{L_M(\Omega)} \to 0 \text{ as } n \to \infty.$$

$\square$

Lemma B.3. *Let M be a Φ-function satisfying* (1.2). *For every nonempty subset $E \subset K$, where K is a compact subset of Ω, there exist two constant numbers $c_1, c_2 \geqslant 0$ such that*

$$\|\chi_E\|_{L_M(\Omega)} \leqslant \frac{1}{M^{-1}(c_1, c_2/|E|)}. \tag{B.3}$$

P r o o f. Let $x_0 \in \Omega$ be fixed. By (1.2), the measurable function

$$x \mapsto M\Big(x, M^{-1}\Big(x_0, \frac{1}{2|E|}\Big)\chi_E\Big)$$

belongs to $L^1(\Omega)$. Hence, there is an $\eta > 0$ such that for any measurable subset Ω' of Ω one has

$$|\Omega'| < \eta \Rightarrow \int_{\Omega'} M\Big(x, M^{-1}\Big(x_0, \frac{1}{2|E|}\Big)\chi_E\Big)\,\mathrm{d}x < \frac{1}{2}.$$

As $M(\cdot, s)$ is measurable on E, Luzin's theorem implies that for $\eta > 0$ there exists a closed set $F_\eta \subset E$ such that the restriction of $M(\cdot, s)$ to F_η is continuous and $|E \setminus F_\eta| < \eta$. Let k be the point, where the supremum of $M(\cdot, s)$ is reached in the set F_η. Then

$$\begin{aligned}&\int_E M\Big(x, M^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)\,\mathrm{d}x \\ &\quad= \int_{F_\eta} M\Big(x, M^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)\,\mathrm{d}x + \int_{E\setminus F_\eta} M\Big(x, M^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)\,\mathrm{d}x.\end{aligned}$$

For the first term on the right-hand side of the last equality, we use (A.5) obtaining

$$\int_{F_\eta} M\Big(x, M^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)\,\mathrm{d}x \leqslant \int_{F_\eta} M\Big(k, M^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)\,\mathrm{d}x \leqslant \frac{1}{2},$$

while for the second, since $|E \setminus F_\eta| < \eta$ we have

$$\int_{E\setminus F_\eta} M\Big(x, M^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)\,\mathrm{d}x \leqslant \frac{1}{2}.$$

Thus, we get

$$\int_\Omega M\Big(x, M^{-1}\Big(k, \frac{1}{2|E|}\Big)\chi_E\Big)\,\mathrm{d}x \leqslant 1.$$

$\square$

Remark B.2. Formula (B.3) remains valid either for every nonempty bounded subset E of Ω or for every nonempty subset E of Ω if Ω is bounded and open.

Lemma B.4. *Let M and $\overline{M}$ be two complementary Φ-functions. Let $v \in L_{\overline{M}}(\Omega)$ be a fixed function. Define*

$$L_v(u) = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \forall\, u \in L_M(\Omega). \tag{B.4}$$

Then L_v defines a linear continuous functional on $L_M(\Omega)$. Furthermore,

$$\|v\|_{L_{\overline{M}}(\Omega)} \leqslant \|L_v\| \leqslant 2\|v\|_{L_{\overline{M}}(\Omega)},$$

where $\|L_v\| = \sup\{|L_v(u)|, \|u\|_{L_M(\Omega)} \leqslant 1\}$.

Proof. We omit the proof, since it is similar to the one given in [1], Lemma 8.17 in the framework of Orlicz spaces.

The above result holds also when L_v is restricted to $E_M(\Omega)$. In general, continuous linear functionals defined on $L_M(\Omega)$ can be expressed in a form different from that defined in (B.4) (see [14], Theorem 3.13.5). Hence, we cannot have the Riesz representation theorem as in classical Lebesgue spaces. In the next lemma we give an "almost complete" analogue of Riesz representation theorem. A similar result in the Orlicz framework can be found in [14], Theorem 3.13.6. $\square$

Lemma B.5. *Let M be a Φ-function satisfying* (1.2) *and let $L \in [E_M(\Omega)]'$. Then there exists a unique function $v \in L_{\overline{M}}(\Omega)$ such that*

$$L(u) = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \forall\, u \in E_M(\Omega). \tag{B.5}$$

Proof. We begin first by assuming that u belongs to $\mathcal{S}_c$ i.e. u is of the form $\sum\limits_{i=1}^{p} \alpha_i \chi_{B_i}(x)$, where B_i are measurable sets of finite Lebesgue measure such that $\bigcup\limits_{i=1}^{p} B_i \subset K$ for some compact subset $K \subset \Omega$ and $\alpha_i \in \mathbb{R}$ for $i = 1, 2, \ldots, p$. Let μ be the complex measure defined for a measurable set of finite Lebesgue measure $A \subset K \subset \Omega$ for some compact K as

$$\mu(A) = L(\chi_A).$$

By (B.3) there exist two constants $c_1, c_2 \geqslant 0$ such that

$$|\mu(A)| \leqslant \|L\| \|\chi_A\|_{L_M(\Omega)} \leqslant \frac{\|L\|}{M^{-1}(c_1, c_2/|A|)} \to 0 \quad \text{as } |A| \to 0.$$

Thus, the measure μ is absolutely continuous with respect to the Lebesgue measure and it follows by Radon-Nikodym's theorem that there exists a nonnegative measurable function $v \in L^1(\Omega)$, unique up to sets of Lebesgue measure zero, such that

$$\mu(A) = \int_A v(x)\,\mathrm{d}x.$$

Hence, we can write

$$\begin{aligned}
\text{(B.6)} \qquad L(u) &= \sum_{i=1}^{p} \alpha_i L(\chi_{B_i}) = \sum_{i=1}^{p} \alpha_i \mu(B_i) = \sum_{i=1}^{p} \alpha_i \int_{B_i} v(x)\,\mathrm{d}x \\
&= \sum_{i=1}^{p} \alpha_i \int_\Omega v(x)\chi_{B_i}\,\mathrm{d}x = \int_\Omega u(x)v(x)\,\mathrm{d}x.
\end{aligned}$$

Now for arbitrary $u \in E_M(\Omega)$, by Lemma B.2 we can found a sequence of functions $u_j \in \mathcal{S}_c$ such that $u_j \to u$ a.e. in Ω and strongly in $E_M(\Omega)$. Therefore, by Fatou's lemma we obtain

$$\begin{aligned}
\left|\int_\Omega u(x)v(x)\,\mathrm{d}x\right| &\leqslant \liminf_{j\to\infty} \int_\Omega |u_j(x)v(x)|\,\mathrm{d}x = \liminf_{j\to\infty} L(|u_j|\operatorname{sgn} v) \\
&\leqslant \|L\| \liminf_{j\to\infty} \|u_j\|_{L_M(\Omega)} \leqslant \|L\|\|u\|_{L_M(\Omega)}.
\end{aligned}$$

This implies that $v \in L_{\overline{M}}(\Omega)$. Let $L_v(u) = \int_\Omega u(x)v(x)\,\mathrm{d}x$, the linear functional defined by (B.4). By (B.5), L_v and L coincide on the set $\mathcal{S}_c$ and by Lemma B.2, they coincide everywhere in $E_M(\Omega)$. □

P r o o f of Theorem B.2. It is immediate that from Lemma B.5 we get the isomorphism

$$L_{\overline{M}}(\Omega) \simeq [E_M(\Omega)]'.$$

□

Theorem B.3. *Let $M, \overline{M} \in \Phi$ be a pair of complementary Φ-functions satisfying both* (1.2) *and the Δ_2-condition. Then, the Musielak-Orlicz space $L_M(\Omega)$ is reflexive.*

P r o o f. Since M and $\overline{M}$ both satisfy the Δ_2-condition, by Lemma A.3 and Theorem B.2 we get

$$L_{\overline{M}}(\Omega) \simeq [L_M(\Omega)]' \quad \text{and} \quad L_M(\Omega) \simeq [L_{\overline{M}}(\Omega)]'$$

and then the conclusion follows. □

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