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Lin Yang; Sheng-Liang Yang; Tian-Xiao He

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GENERALIZED SCHRÖDER MATRICES ARISING FROM ENUMERATION OF LATTICE PATHS

Lin Yang, Sheng-Liang Yang, Lanzhou, Tian-Xiao He, Bloomington

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Abstract. We introduce a new family of generalized Schröder matrices from the Riordan arrays which are obtained by counting of the weighted lattice paths with steps $E=(1,0)$, $D=(1,1)$, $N=(0,1)$, and $D'=(1,2)$ and not going above the line $y=x$. We also consider the half of the generalized Delannoy matrix which is derived from the enumeration of these lattice paths with no restrictions. Correlations between these matrices are considered. By way of illustration, we give several examples of Riordan arrays of combinatorial interest. In addition, we find some new interesting identities.

Keywords: Riordan array; lattice path; Delannoy matrix; Schröder number; Schröder matrix

MSC 2010: 05A15, 05A19, 15A24, 11B83

1. Introduction

The Schröder numbers were introduced by Schröder in [23] while enumerating unrestricted bracketings of words, though at least the initial terms of this sequence were known to Hipparchus, as Stanley noted in his survey (see [28]) regarding these numbers. Rogers and Shapiro in [22] found bijections showing certain classes of lattice paths are enumerated by the Schröder numbers. While there are several such classes of lattice paths, we use the one given in the definition below.

Definition 1.1. Let $n\geqslant 1$. A lattice path from $(0,0)$ to (n,n) taking only East $E=(1,0)$, North $N=(0,1)$, and Diagonal $D=(1,1)$ steps while staying weakly below the main diagonal will be referred to as a *Schröder path of order* n. A small Schröder path is a Schröder path with no Diagonal $D=(1,1)$ steps on the line $y=x$.

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The set of all Schröder paths of order n will be denoted by $\mathcal{R}(n)$, and the set of all small Schröder paths of order n will be denoted by $\mathcal{S}(n)$. It is well known that $|\mathcal{R}(n)|$ is equal to the nth Schröder number

$$r_n = \frac{1}{n+1}\sum_{i=1}^{n}\binom{n+1}{i}\binom{n-1}{i-1}2^i,$$

and $|\mathcal{S}(n)|$ is equal to the nth small Schröder number $s_n = \frac{1}{2}r_n$ (see [7], [29]). Thus the first terms of the sequence $\{r_n\}_{n\geqslant 0}$ and $\{s_n\}_{n\geqslant 0}$ are $\{1, 2, 6, 22, 90, \ldots\}$ and $\{1, 1, 3, 11, 45, \ldots\}$.

The idea of translating a combinatorial theory into a theory of infinite matrices is nowadays a current trend in discrete mathematics. To confirm this statement, we cite Riordan arrays (see [10], [12], [16], [24]), [27], Aigner's admissible matrices (see [1]). Some matrices involving the Schröder numbers are studied by Rogers in [21] and Pergola and Sulanke in [19]. In [35], the authors observed that some Schröder matrices appear as the unsigned inverses of Delannoy matrices. Very recently, Ramirez and Sirvent in [20] introduced a new family of generalized Schröder matrices that are connected to inverse generalized Delannoy matrices.

In this paper, we consider the Schröder numbers, not as isolated sequences, but as belonging to Riordan arrays in which they form the first column as in (1.1) and (1.2). While there are other arrays that contain the Schröder numbers (for instance, see sequences A132372 and A104219 in [25]), these two seem very natural. In fact, if we recount the partial Schröder paths running from $(0,0)$ to $(n, n-k)$ with steps $E = (1,0)$, $N = (0,1)$ and $D = (1,1)$, we get the Schröder matrix (sequence A080247 in [25]):

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 6 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 22 & 16 & 6 & 1 & 0 & 0 & \ldots \\ 90 & 68 & 30 & 8 & 1 & 0 & \ldots \\ 394 & 304 & 146 & 48 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.1}$$

This Schröder matrix was studied by Rogers, see [21], by Rogers and Shapiro, see [22], by Pergola and Sulanke, see [19], and by others, see [10], [14], [17], [20], [35].

On the other hand, if we consider the partial small Schröder paths running from $(0,0)$ to $(n, n-k)$, we get the Schröder matrix of the second kind (sequence A186826 in [25]):

$$
\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 3 & 1 & 0 & 0 & 0 & \dots \\ 11 & 11 & 5 & 1 & 0 & 0 & \dots \\ 45 & 45 & 23 & 7 & 1 & 0 & \dots \\ 197 & 197 & 107 & 39 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.2}
$$

There are a huge number of papers on the enumeration of lattice paths for a specified set of steps (see [1], [8], [9], [11], [26], [31], [35]). In this paper, we add a fourth kind of steps $D' = (1,2)$, and consider the enumeration of these lattice paths consisting of steps $E = (1,0)$, $D = (1,1)$, $N = (0,1)$, and $D' = (1,2)$. So we give the following definitions formally.

Definition 1.2. A lattice path starting from the origin, ending on the first quadrant of $\mathbb{Z}^2$, and using steps $E = (1,0)$, $D = (1,1)$, $N = (0,1)$, and $D' = (1,2)$, with assigned weights 1, a, b, and c, respectively, will be referred to as a *weighted generalized Delannoy path*, or simply a *generalized Delannoy path*.

Definition 1.3. Let $n \geqslant 1$. A generalized Delannoy path from $(0,0)$ to (n,n) will be referred to as a *generalized central Delannoy path of order* n.

Definition 1.4. Let $n \geqslant 1$. A generalized Delannoy path from $(0,0)$ to (n,n) staying weakly below the main diagonal will be referred to as a *generalized Schröder path of order* n. A *generalized small Schröder path* is a generalized Schröder path with no Diagonal $D = (1,1)$ steps on the line $y = x$.

We illustrate all of the generalized Schröder paths from $(0,0)$ to $(2,2)$ in Figure 1, the first 6 are Schröder paths.

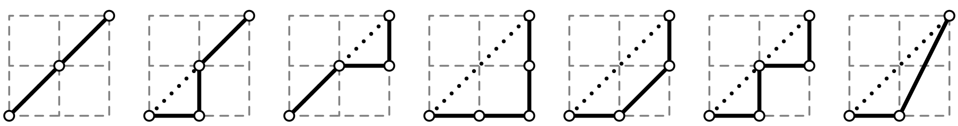

Figure 1. Generalized Schröder paths from $(0,0)$ to $(2,2)$.

The outline of this paper is as follows. In Section 2 we recall the definition of Riordan arrays and some basic properties. We show that $G = (G_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array, where $G_{n,k}$ is the total sum of weights of all weighted generalized Delannoy paths ending at $(k, n-k)$. In Section 3 we generalize the Schröder matrix by considering the enumeration of generalized Schröder paths ending at the point $(n, n-k)$. In Section 4 we generalize the Schröder matrix of the second kind by considering the enumeration of the generalized small Schröder paths ending at the point $(n, n-k)$. Then in Section 5 we consider the half of the generalized Delannoy matrix $G = (G_{n,k})_{n,k\geqslant 0}$. Moreover, the relationships between these Riordan arrays are studied. Meanwhile, some new interesting identities are obtained.

2. Riordan arrays

Since Shapiro et al. in [24] introduced the concept of Riordan arrays, many authors have applied them to several counting problems, see [4], [5], [10], [12], [14], [16], [27], [34]. An infinite lower triangular matrix $R = (r_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $g(t)f(t)^k$, where $g(t)$ and $f(t)$ are formal power series with $g(0) = 1$, $f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $g(t), f(t)$ is denoted by $R = (g(t), f(t))$. The set of all Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity $(1, t)$, called the Riordan group, see [24], [27]. The multiplication rule of Riordan arrays is given by

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))), \tag{2.1}$$

and the inverse of $(g(t), f(t))$ is

$$(g(t), f(t))^{-1} = \Big(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\Big), \tag{2.2}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$, i.e., $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

A Riordan array $R = (r_{n,k})_{n,k\in\mathbb{N}}$ can also be characterized by another matrix as follows, see [15], [16].

Lemma 2.1. *A lower triangular array $(r_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array if and only if there exists another array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$, with $\alpha_{0,0} \neq 0$, and s sequences $(\varrho_j^{[i]})_{j\in\mathbb{N}}$, $i = 1, 2, \ldots, s$, such that*

$$r_{n+1,k+1} = \sum_{i\geqslant 0}\sum_{j\geqslant 0} \alpha_{i,j} r_{n-i,k+j} + \sum_{i=1}^{s}\sum_{j\geqslant 0} \varrho_j^{[i]} r_{n+i,k+i+j+1}. \tag{2.3}$$

The array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ is called the A-matrix of the Riordan array $(r_{n,k})_{n,k\in\mathbb{N}} = (g(t), f(t))$. If $\Phi^{[i]}(t)$ denotes the generating functions of the ith row of the A-matrix and $\Psi^{[i]}(t)$ is the generating function for the sequence $(\varrho_j^{[i]})_{j\in\mathbb{N}}$, then the generating function $f(t)$ is determined by [15]

$$f(t) = \sum_{i\geqslant 0} t^{1+i}\Phi^{[i]}(f(t)) + \sum_{i=1}^{s} t^{1-i} f(t)^{i+1}\Psi^{[i]}(f(t)). \tag{2.4}$$

If column 0 of the Riordan array $(r_{n,k})_{n,k\in\mathbb{N}} = (g(t), f(t))$ is defined by

$$r_{n+1,0} = \sum_{i\geqslant 0}\sum_{j\geqslant 0}\beta_{i,j} r_{n-i,j} + \sum_{i=1}^{s}\sum_{j\geqslant 0}\eta_j^{[i]} r_{n+i,i+j}, \quad n \geqslant 0, \tag{2.5}$$

then the generating function $g(t)$ is given by the formula

$$g(t) = g_{0,0}\Big/\Big(1 - \sum_{i\geqslant 0} t^{i+1}R^{[i]}(f(t)) - \sum_{i=1}^{s} t^{1-i} f(t)^i S^{[i]}(f(t))\Big), \tag{2.6}$$

where $R^{[i]}(t) = \sum\limits_{j\geqslant 0}\beta_{i,j}t^j$, $i = 0, 1, \ldots$, and $S^{[i]}(t) = \sum\limits_{j\geqslant 0}\eta_j^{[i]}t^j$, $i = 1, \ldots, s$.

For an infinite lower triangular matrix $R = (r_{n,k})_{n,k\in\mathbb{N}}$, the half of R is defined to be the infinite lower triangular matrix $R^{(r)} = (r_{2n-k,n})_{n,k\in\mathbb{N}}$. It is known that if $R = (d(t), h(t))$ is a Riordan array, then its half is also a Riordan array, see [2], [33], [32], [35].

Lemma 2.2. *Let $R = (d(t), h(t)) = (r_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array. If $f(t)$ is the generating function defined by the functional equation $f(t)^2 = th(f(t))$, then the half of R is given by*

$$H = \Big(\frac{tf'(t)d(f(t))}{f(t)}, f(t)\Big). \tag{2.7}$$

Now we consider the generalized Delannoy paths in the Cartesian plane starting from $(0,0)$ that use the steps $E = (1,0)$, $D = (1,1)$, $N = (0,1)$, $D' = (1,2)$, where each step is labeled with weights 1, a, b, and c, respectively. Let α be a path. We define the weight $w(\alpha)$ to be the product of the weights of the steps. Let $G(n,k)$ be the set of all generalized Delannoy paths ending at the point $(k, n-k)$. The generalized Delannoy number $G_{n,k}$ is the sum of all $w(\alpha)$ with α in $G(n,k)$, as illustrated in Figure 2 (a), and the generalized Delannoy matrix is defined as $G = G(a,b,c) = (G_{n,k})_{n,k\in\mathbb{N}}$. In Figure 2 (b), we give a schematic illustration of the dependence of $G_{n+1,k+1}$ on the other elements in the array and get the recurrence

$$G_{n+1,k+1} = G_{n,k} + bG_{n,k+1} + aG_{n-1,k} + cG_{n-2,k}, \quad n,k \geqslant 0, \tag{2.8}$$

and initial conditions are $G_{n,0} = b^n$ and $G_{n,n} = 1$.

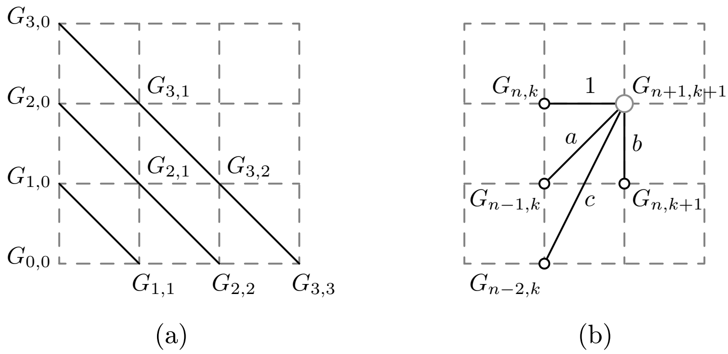

Figure 2. Generalized Delannoy matrix and recurrence relation of its entries.

The first few rows of $G(a,b,c)$ are

$$G(a,b,c)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ b & 1 & 0 & 0 & 0 & \cdots \\ b^2 & a+2b & 1 & 0 & 0 & \cdots \\ b^3 & 2ab+3b^2+c & 2a+3b & 1 & 0 & \cdots \\ b^4 & 3ab^2+4b^3+2bc & a^2+6ab+6b^2+2c & 3a+4b & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $k \geqslant 0$, let $g_k(t)=\sum\limits_{n=k}^{\infty} G_{n,k}t^n$. Then, from (2.8) we obtain

$$g_{k+1}(t)=tg_k(t)+btg_{k+1}(t)+at^2g_k(t)+ct^3g_k(t).$$

Thus,

$$g_{k+1}(t)=\frac{t+at^2+ct^3}{1-bt}g_k(t).$$

From this recurrence relation and the initial condition

$$g_0(t)=\sum_{n=k}^{\infty} G_{n,0}t^n=\frac{1}{1-bt},$$

we have

$$g_k(t)=\Big(\frac{t+at^2+ct^3}{1-bt}\Big)^k\frac{1}{1-bt}.$$

Therefore, we proved the following theorem (see also [20]).

Theorem 2.1. *The generalized Delannoy matrix can be represented by a Riordan array as*

$$G(a,b,c)=\Big(\frac{1}{1-bt},\frac{t+at^2+ct^3}{1-bt}\Big).$$

For example, in the case $a = b = c = 1$, the first few terms of the array $G(1,1,1)$ are

$$\left(\frac{1}{1-t}, \frac{t+t^2+t^3}{1-t}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 6 & 5 & 1 & 0 & 0 & 0 & \dots \\ 1 & 9 & 15 & 7 & 1 & 0 & 0 & \dots \\ 1 & 12 & 33 & 28 & 9 & 1 & 0 & \dots \\ 1 & 15 & 60 & 81 & 45 & 11 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $b = 1$, $a = c = 0$, $G(0,1,0) = (1/(1-t), t/(1-t))$ we have the famous Pascal matrix, see [2], [24].

For $a = b = 1$, $c = 0$, $G(1,1,0) = (1/(1-t), (t+t^2)/(1-t))$ we have the Pell matrix, see [18].

For $a = 0$, $b = 1$, $c = 1$, $G(0,1,1) = (1/(1-t), (t+t^3)/(1-t))$ we have the sequence A257365 in [25].

3. The generalized Schröder matrix

As mentioned above, a generalized Schröder path from $(0,0)$ to $(n, n-k)$ is a lattice path not rising above the line $y = x$ and using steps $E = (1,0)$, $D = (1,1)$, $N = (0,1)$, $D' = (1,2)$, with weights $1, a, b, c$, respectively.

Definition 3.1. Let $W(n,k)$ be the set of all generalized Schröder paths ending at $(n, n-k)$, and let $W_{n,k}$ be the sum of all $w(\alpha)$ with α in $W(n,k)$. We call the matrix $W(a,b,c) = (W_{n,k})_{n,k\in\mathbb{N}}$ the *generalized Schröder matrix*.

The last step of any path from $W(n,k)$ is one of the step set $\{E = (1,0), D = (1,1), N = (0,1), D' = (1,2)\}$, as shown in Figure 3.

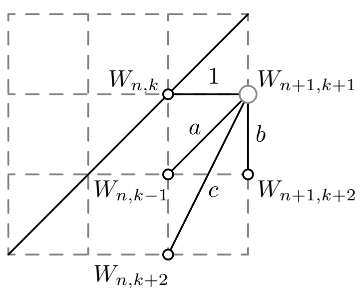

Figure 3. Recurrence relation of entries of the generalized Schröder matrix.

Therefore, the numbers $W_{n,k}$ satisfy the recurrence relations

$$W_{n+1,k+1} = W_{n,k} + aW_{n,k+1} + cW_{n,k+2} + bW_{n+1,k+2}, \tag{3.1}$$

$$W_{n+1,0} = aW_{n,0} + cW_{n,1} + bW_{n+1,1} \tag{3.2}$$

with $n, k \geqslant 0$ and boundary conditions $W_{n,n} = 1$ for $n \geqslant 0$. The first few rows of $W(a,b,c)$ are

$$W(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ a+b & 1 & 0 & 0 & \dots \\ a^2+3ab+2b^2+c & 2a+2b & 1 & 0 & \dots \\ a^3+6a^2b+10ab^2+5b^3+3ac+4bc & 3a^2+8ab+5b^2+2c & 3a+3b & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 3.1. *The generalized Schröder matrix $W(a,b,c) = (W_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array given by*

$$W(a,b,c) = \left(\frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2t(b+ct)}, \frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2(b+ct)}\right).$$

Proof. From recurrence (3.1), the A-matrix is

$$A = \begin{pmatrix} 1 & a & c & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the entries $\alpha_{0,0}$, $\alpha_{0,1} = a$, $\alpha_{0,2} = c$, while the other entries are all equal to 0. Hence, by Lemma 2.1, the matrix $W(a,b,c) = (W_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array $(g(t), f(t))$. The generating functions of the rows of the A-matrix are $\Phi^{[0]}(t) = 1 + at + ct^2$, and $\Phi^{[i]}(t) = 0$ for all $i \geqslant 1$. The generating functions of the associated sequences are $\Psi^{[1]}(t) = b$, and $\Psi^{[i]}(t) = 0$ for all $i \geqslant 2$. Using (2.4), we get $f(t) = t(1 + af(t) + cf(t)^2) + bf(t)^2$, and consequently

$$f(t) = \frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2(b+ct)}.$$

From recurrence (3.2), we obtain that the generating function $g(t)$ satisfies the equation

$$g(t) = \frac{1}{1-t(a+cf(t))-bf(t)}.$$

Hence

$$g(t) = \frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2t(b+ct)}.$$

□

Theorem 3.2. *The inverse of the generalized Schröder matrix* $W(a,b,c) = (W_{n,k})_{n,k\in\mathbb{N}}$ *is given by*

$$W^{-1}(a,b,c) = \Big(\frac{1-bt}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Big).$$

P r o o f. A direct computation shows that the composition inverse of

$$\frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2(b+ct)}$$

is $(t-bt^2)/(1+at+ct^2)$. Hence using Theorem 3.1 and (1.2) we get

$$W^{-1} = \Big(\frac{1-bt}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Big).$$

□

Theorem 3.3. *The general term of the generalized Schröder matrix* $W(a,b,c)$ *is given by*

$$\begin{aligned} W_{n,k} &= \frac{k+1}{n+1}\sum_{m=0}^{2n+2}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{2n-m-k}{n-m-k}\binom{n+1}{i}\binom{n+1-i}{m-2i}a^{m-2i}b^{n-m-k}c^i \\ &= \frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\sum_{p=0}^{i}\binom{i}{i-p}\binom{n-k-p-1}{i-1}(a+b)^{i-p}b^{n-k-i-p}c^p. \end{aligned}$$

P r o o f. Let $W = (g(t), f(t))$ and $h(t) = (t-bt^2)/(1+at+ct^2)$. Then $W^{-1} = (h(t)/t, h(t))$.

Thus, using the definition of the Riordan array and the Lagrange inversion formula (see [6], [34]), we obtain

$$\begin{aligned} W_{n,k}(a,b,c) &= [t^n]g(t)f(t)^k = [t^n]\frac{\bar{h}(t)}{t}\bar{h}(t)^k = [t^{n+1}]\bar{h}(t)^{k+1} \\ &= \frac{k+1}{n+1}[t^{n-k}]\Big(\frac{t}{h(t)}\Big)^{n+1} = [t^{n-k}]\frac{k+1}{n+1}(1+at+ct^2)^{n+1}(1-bt)^{-(n+1)} \\ &= [t^{n-k}]\frac{k+1}{n+1}\sum_{m=0}^{2n+2}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{n+1}{i}\binom{n+1-i}{m-2i}a^{m-2i}c^it^m\cdot\sum_{j=0}^{\infty}\binom{n+j}{j}b^jt^j \\ &= [t^{n-k}]\frac{k+1}{n+1}\sum_{j=0}^{\infty}\binom{n+j}{j}\sum_{m=0}^{2n+2}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{n+1}{i}\binom{n+1-i}{m-2i}b^ja^{m-2i}c^it^{m+j} \\ &= \frac{k+1}{n+1}\sum_{m=0}^{2n+2}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{2n-m-k}{n-m-k}\binom{n+1}{i}\binom{n+1-i}{m-2i}b^{n-m-k}a^{m-2i}c^i. \end{aligned}$$

On the other hand,

$$W_{n,k}(a,b,c) = \frac{k+1}{n+1}[t^{n-k}]\Big(\frac{t}{h(t)}\Big)^{n+1} = [t^{n-k}]\frac{k+1}{n+1}\Big(\frac{1+at+ct^2}{1-bt}\Big)^{n+1}$$
$$= [t^{n-k}]\frac{k+1}{n+1}\Big(1+\frac{(a+b)t+ct^2}{1-bt}\Big)^{n+1}$$
$$= [t^{n-k}]\frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\Big(\frac{(a+b)t+ct^2}{1-bt}\Big)^{i}$$
$$= [t^{n-k}]\frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\sum_{m=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\binom{i+m-1}{m}(a+b)^j b^m c^{i-j} t^{m+2i-j}$$
$$= \frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\sum_{p=0}^{i}\binom{i}{i-p}\binom{n-k-p-1}{i-1}(a+b)^{i-p}b^{n-k-i-p}c^p.$$

This completes the proof. □

The generalized Schröder matrix $W(a,b,c)$ has some interesting interpolating properties. Indeed, we have:

(i) $W(a,b,0)$ is the $(a+b,b)$ Catalan number triangle (see [10]) with a general element

$$W_{n,k}(a,b,0) = \frac{k+1}{n+1}\sum_{m=0}^{2n+2} b^{n-m-k}\binom{2n-m-k}{n-m-k}\binom{n+1}{m}a^m$$
$$= \frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\binom{n-k-1}{i-1}(a+b)^i b^{n-k-i}.$$

(ii) $W(a,0,c)$ is the weighted Motzkin matrix (see [32]) with a general element

$$W_{n,k}(a,0,c) = \frac{k+1}{n+1}\sum_{i=0}^{\lfloor (n-k)/2\rfloor}\binom{n+1}{i}\binom{n+1-i}{n-k-2i}a^{n-k-2i}c^i.$$

(iii) $W(0,b,c)$ is the matrix with a general element

$$W_{n,k}(0,b,c) = \frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{2n-2i-k}{n-2i-k}\binom{n+1}{i}b^{n-2i-k}c^i.$$

Since $W_{n,0}(a,b,c)$ is the total sum of weights of generalized Schröder paths ending at (n,n), we call it the generalized Schröder number and denote it by $W_n(a,b,c)$.

From Theorem 3.3 we have

$$W_n(a,b,c) = \frac{1}{n+1}\sum_{m=0}^{n}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{2n-m}{n-m}\binom{n+1}{i}\binom{n+1-i}{m-2i}a^{m-2i}b^{n-m}c^i$$
$$= \frac{1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\sum_{p=0}^{i}\binom{i}{i-p}\binom{n-p-1}{i-1}(a+b)^{i-p}b^{n-i-p}c^p.$$

and the generating function is

$$W(t) = \sum_{n=0}^{\infty} W_n(a,b,c)t^n = \frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2t(b+ct)}. \tag{3.3}$$

Theorem 3.4. *Let $W_n = W_n(a,b,c)$. Then*

$$\begin{aligned} b(n+1)W_n &= ((4b^2+2ab-c)n-2b^2-ab-c)W_{n-1} \\ &\quad + ((8bc+2ac-a^2b)n+2a^2b-10bc-ac)W_{n-2} \\ &\quad + ((4c^2-a^2c)n+2a^2c-8c^2)W_{n-3}, \quad n \geqslant 2, \end{aligned}$$

and $W_{-1} = 0$, $W_0 = 1$, $W_1 = a+b$.

Proof. Let

$$W(t) = \sum_{n=0}^{\infty} W_n t^n = \frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2t(b+ct)}$$

and let $P(t) = 2t(b+ct)W(t)$. Then $P(t) = 2bt\sum_{n=0}^{\infty} W_n t^n + 2ct^2\sum_{n=0}^{\infty} W_n t^n$. Consequently,

$$P(t) = \sum_{n=1}^{\infty} 2bW_{n-1}t^n + \sum_{n=2}^{\infty} 2cW_{n-2}t^n. \tag{3.4}$$

$$P'(t) = \sum_{n=0}^{\infty} 2b(n+1)W_n t^n + \sum_{n=1}^{\infty} 2c(n+1)W_{n-1}t^n. \tag{3.5}$$

From the closed form of the generating function

$$P(t) = 1-at-\sqrt{(1-at)^2-4t(b+ct)},$$

we obtain

$$P'(t) = -a + \frac{2b+a-(a^2-4c)t}{\sqrt{(1-at)^2-4t(b+ct)}}.$$

By some computation we find that $P(t)$ satisfies the differential equation

$$(3.6)\quad (1-2(2b+a)t+(a^2-4c)t^2)P'(t)+(2b+a-(a^2-4c)t)P(t) = 2b+2(ab+2c)t.$$

Taking W_n to be zero for n less than zero, substituting equations (3.4) and (3.5) into (3.6), and equating the coefficients of t^n in the resulting equation we get the desired result. □

For $a = b = c = 1$, the generalized Schröder numbers $W_n = W_n(1, 1, 1)$ form the sequence A064641 of [25], and they satisfy the recurrence relation

$$(n+1)W_n = (5n-4)W_{n-1} + (9n-9)W_{n-2} + (3n-6)W_{n-3}, \quad n \geqslant 2,$$

with initial conditions $W_{-1} = 0$, $W_0 = 1$, $W_1 = 2$.

The generalized Schröder numbers $W_n(a, b, c)$ also have the following interesting special cases:

(i) For $a = 0$,

$$W_n(0, b, c) = \frac{1}{n+1}\sum_{i=0}^{n+1}\binom{2n-2i}{n-2i}\binom{n+1}{i}b^{n-2i}c^i,$$

and they satisfy

$$b(n+1)W_n = ((4b^2-c)n-2b^2-c)W_{n-1}+(8n-10)bcW_{n-2}+(4n-8)c^2W_{n-3}, \quad n \geqslant 2,$$

with initial conditions $W_{-1} = 0$, $W_0 = 1$, $W_1 = b$.

(ii) For $b = 0$, $W_n = W_n(a, 0, c)$ are the weighted Motzkin numbers

$$W_n(a, 0, c) = \frac{1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\binom{n+1-i}{n-2i}a^{n-2i}c^i,$$

and they satisfy [13]

$$(n+2)W_n = a(2n+1)W_{n-1} + (4c-a^2)(n-1)W_{n-2}, \quad n \geqslant 2,$$

with initial conditions $W_0 = 1$, $W_1 = a$.

(iii) For $c = 0$, $W_n(a,b,0)$ are the weighted Schröder numbers:

$$W_n(a,b,0) = \frac{1}{n+1}\sum_{m=0}^{n}\binom{2n-m}{n-m}\binom{n+1}{m}a^m b^{n-m}$$
$$= \frac{1}{n+1}\sum_{i=1}^{n}\binom{n+1}{i}\binom{n-1}{i-1}(a+b)^i b^{n-i},$$

and they satisfy the recurrence relation in [30]

$$(n+1)W_n = ((2a+4b)n - a - 2b)W_{n-1} - a^2(n-2)W_{n-2}, \quad n \geqslant 2$$

with initial conditions $W_0 = 1$, $W_1 = a + b$. Particularly,

$$W_n(q,1,0) = \sum_{j=0}^{n}\frac{1}{n}\binom{n}{j}\binom{2n-j}{n-1}q^j = \sum_{i=1}^{n}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}(q+1)^i$$

are the q-analog of the Schröder numbers defined in [3]. The first few rows of $W(q,1,0)$ are

$$W(q,1,0)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q+1 & 1 & 0 & 0 & 0 & \cdots \\ q^2+3q+2 & 2q+2 & 1 & 0 & 0 & \cdots \\ q^3+6q^2+10q+5 & 3q^2+8q+5 & 3q+3 & 1 & 0 & \cdots \\ q^4+10q^3+30q^2+35q+14 & 4q^3+20q^2+30q+14 & 6q^2+15q+9 & 4q+4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4. Generalized Schröder matrix of the second kind

In this section, we consider the generalized small Schröder paths, i.e., the generalized Schröder paths having no diagonal steps on the line $y = x$. Let $V(n,k)$ be the set of all partial generalized Schröder paths ending at the point $(n, n-k)$ and having no diagonal steps on the line $y = x$. Let $V_{n,k}$ be the sum of all $w(\alpha)$ with α in $V(n,k)$. We call the matrix $V(a,b,c) = (V_{n,k})_{n,k\in\mathbb{N}}$ the generalized Schröder matrix of the second kind. Then the entries of column 0 of this matrix are the generalized small Schröder numbers. By considering the steps incident at $(n+1, n-k)$ we have

$$V_{n+1,k+1} = V_{n,k} + aV_{n,k+1} + cV_{n,k+2} + bV_{n+1,k+2}, \tag{4.1}$$
$$V_{n+1,0} = cV_{n,1} + bV_{n+1,1} \tag{4.2}$$

with $n, k \geqslant 0$ and boundary conditions $V_{n,n} = 1$ for $n \geqslant 0$.

The first few rows of $V(a,b,c)$ are

$$V(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ b & 1 & 0 & 0 & \cdots \\ ab+2b^2+c & a+2b & 1 & 0 & \cdots \\ a^2b+5ab^2+5b^3+ac+4bc & a^2+5ab+5b^2+2c & 2a+3b & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 4.1. *The generalized Schröder matrix of the second kind* $V(a,b,c) = (V_{n,k})_{n,k\in\mathbb{N}}$ *can be represented by Riordan arrays as*

$$V(a,b,c) = \Big(\frac{1+at-\sqrt{(1-at)^2-4t(b+ct)}}{2t(a+b+ct)}, \frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2(b+ct)}\Big),$$

$$V^{-1}(a,b,c) = \Big(\frac{(1+at)(1-bt)}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Big).$$

Proof. The proof is very similar to that of Theorems 3.1 and 3.2, so we omit the details. □

Theorem 4.2. *The generalized Schröder matrix* $W(a,b,c)$ *and the generalized Schröder matrix of the second kind* $V(a,b,c)$ *are related by*

$$W(a,b,c) = V(a,b,c)\cdot(1+at,t),$$

where $(1+at,t)$ *is the Riordan array whose elements on the main diagonal are* 1*, subdiagonal are* a*, and all other elements are* 0*. Consequently,* $W_{n,k} = V_{n,k} + aV_{n,k+1}$.

Proof. From Theorems 3.1 and 4.1,

$$\begin{aligned} W^{-1}(a,b,c) &= \Big(\frac{1-bt}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Big) \\ &= \Big(\frac{1}{1+at}, t\Big)\Big(\frac{(1+at)(1-bt)}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Big) \\ &= \Big(\frac{1}{1+at}, t\Big)\cdot V^{-1}(a,b,c). \end{aligned}$$

Thus, $W(a,b,c) = V(a,b,c)\cdot(1+at,t)$. □

From this theorem, we deduce $V(a,b,c) = W(a,b,c)\cdot(1/(1+at),t)$, which implies the following results.

Corollary 4.1. *The general terms of the generalized Schröder matrix of the second kind $V(a,b,c)$ is given by*

$$V_{n,k} = \sum_{j=k}^{n} (-1)^{j-k} W_{n,j} a^{j-k},$$

where $W_{n,j}$ are terms of the generalized Schröder matrix.

Recall that $V_{n,0}$ are the generalized small Schröder numbers which are the sum of weights of the generalized Schröder paths ending at the point (n,n) and having no diagonal steps on the line $y = x$. We will denote the $V_{n,0}$ simply by V_n. Then

$$V(t) = \sum_{n=0}^{\infty} V_n t^n = \frac{1 + at - \sqrt{(1-at)^2 - 4t(b+ct)}}{2t(a+b+ct)}.$$

By doing minor computations, we obtain

$$W(t) = V(t) + \frac{a}{b+ct}(V(t) - 1),$$

where

$$W(t) = \frac{1 - at - \sqrt{(1-at)^2 - 4t(b+ct)}}{2t(b+ct)}$$

is the generating function of the generalized Schröder numbers. This implies that

$$W_n = V_n + \frac{a}{b} \sum_{i=1}^{n} (-1)^{n-i} \left(\frac{c}{b}\right)^{n-i} V_i, \quad b \neq 0,\ n \geqslant 1.$$

Particularly, in the case $b = 1$ and $c = 0$, we get

$$W_n = (1+a)V_n, \quad n \geqslant 1. \tag{4.3}$$

For $a = 1$, $b = 1$ and $c = 0$, $W(1,1,0)$ and $V(1,1,0)$ are the Schröder matrix (1.1) and the Schröder matrix of the second kind (1.2). Equation (4.3) reduce to the well known relation linking the Schröder numbers, large and small, see [7].

For $a = 3$, $b = 1$ and $c = 0$, the first 5 rows of the matrices $W(3,1,0)$ and $V(3,1,0)$ are

$$W(3,1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 20 & 8 & 1 & 0 & 0 \\ 116 & 56 & 12 & 1 & 0 \\ 740 & 392 & 108 & 16 & 1 \end{pmatrix}, \quad V(3,1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 \\ 29 & 29 & 9 & 1 & 0 \\ 185 & 185 & 69 & 3 & 1 \end{pmatrix}.$$

Column 0 of $W(3,1,0)$ appears as the sequence A082298 in [25], and column 0 of $V(3,1,0)$ as A059231.

5. The half of the generalized Delannoy matrix

Let $W^*(n,k)$ be the set of all generalized Delannoy paths ending at $(n, n-k)$ with no restriction. Let $W^*_{n,k}$ be the sum of all $w(\alpha)$ with α in $W^*(n,k)$. Then the array $W^*(a,b,c) = (W^*_{n,k})_{n,k\in\mathbb{N}}$ is the half of the generalized Delannoy matrix $G = (G_{n,k})_{n,k\in\mathbb{N}}$, i.e, $W^*_{n,k} = G_{2n-k,n}$. The first few rows of $W^*(a,b,c)$ are

$$W^*(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ a+2b & 1 & 0 & 0 & \cdots \\ a^2+6ab+6b^2+2c & 2a+3b & 1 & 0 & \cdots \\ a^3+12a^2b+30ab^2+20b^3+6ac+12bc & 3a^2+12ab+10b^2+3c & 3a+4b & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 5.1. *The matrix $W^*(a,b,c)$ is a Riordan array and can be represented as*

$$W^*(a,b,c) = \Bigl(\frac{1}{\sqrt{(1-at)^2-4t(b+ct)}}, \frac{1-at-\sqrt{(1-at)^2-4t(b+ct)}}{2(b+ct)}\Bigr), \tag{5.1}$$

$$W^*_{n,k}(a,b,c) = \sum_{m=0}^{n}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{2n-k-m}{n}\binom{n}{m-i}\binom{m-i}{i}a^{m-2i}c^ib^{n-k-m}. \tag{5.2}$$

Proof. Since the matrix $W^*(a,b,c)$ is the half of

$$G(a,b,c) = \Bigl(\frac{1}{1-bt}, \frac{t+at^2+ct^3}{1-bt}\Bigr),$$

by Lemma 1.2, we obtain that $W^*(a,b,c)$ is a Riordan array and its expression is (5.1). From Theorem 2.2, we get the formula (5.2). □

Corollary 5.1. *For the inverse of the Riordan arrays $W^*(a,b,c)$, we have*

$$W^{*-1}(a,b,c) = \Bigl(\frac{1-2bt-(c+ab)t^2}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Bigr).$$

Note that $W^*_{n,0}(a,b,c)$ are the generalized central Delannoy numbers, and we denote them as $W^*_n(a,b,c)$. By the previous theorem, we have

$$\sum_{n=0}^{\infty} W^*_n(a,b,c)t^n = \frac{1}{\sqrt{(1-at)^2-4t(b+ct)}}, \tag{5.3}$$

$$W^*_n(a,b,c) = \sum_{m=0}^{n}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{2n-m}{n}\binom{n}{m-i}\binom{m-i}{i}a^{m-2i}c^ib^{n-m}. \tag{5.4}$$

Some interesting special cases are:

(i) $W_n^*(a,b,0)$ *are the weighted central Delannoy numbers:*

$$W_n^*(a,b,0)=\sum_{m=0}^{n}\binom{2n-m}{n}\binom{n}{m}a^m b^{n-m}. \tag{5.5}$$

(ii) $W_n^*(a,0,c)$ *are the weighted central trinomial coefficients:*

$$W_n^*(a,0,c)=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{n-i}\binom{n-i}{i}a^{n-2i}c^i. \tag{5.6}$$

(iii) $W_n^*(0,b,c)$ *are the weighted version of the sequence* A006139 *in* [25]*:*

$$W_n^*(0,b,c)=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{2n-2i}{n}\binom{n}{i}c^i b^{n-2i}. \tag{5.7}$$

Theorem 5.2. *Let* $W_n^*=W_n^*(a,b,c)$*. Then*

$$(n+1)W_{n+1}^*=((2a+4b)n+a+2b)W_n^*-(a^2-4c)nW_{n-1}^*,\quad n\geqslant 2,$$

and $W_0^*=1$, $W_1^*=a+2b$*.*

Proof. Let

$$Q(t)=\frac{1}{\sqrt{(1-at)^2-4t(b+ct)}}=\sum_{n=0}^{\infty}W_n^*t^n. \tag{5.8}$$

Then we have

$$Q'(t)=\frac{2b+4ct-a(at-1)}{(\sqrt{(1-at)^2-4t(b+ct)})^3}. \tag{5.9}$$

By some computation we find that $Q(t)$ satisfies the differential equation

$$((a^2-4c)t^2-(2a+4b)t+1)Q'(t)+((a^2-4c)t-a-2b)Q(t)=0. \tag{5.10}$$

Taking W_n^* to be zero for n less than zero, substituting equations (5.8) and (5.9) into (5.10), and equating the coefficients of t^n in the result equation we get the desired result. □

Now we will explore the relationship between the number $W^*_{n,k}$ of unrestricted paths and the number $W_{n,k}$ of restricted paths. It can be shown that these numbers are connected by the generalized Pell numbers. Indeed, we have the following result.

Theorem 5.3. *For the Riordan arrays $W^*(a,b,c)$ and $W(a,b,c)$ we have*

$$W^*(a,b,c) = W(a,b,c)\cdot\Big(\frac{1-bt}{1-2bt-(c+ab)t^2}, t\Big).$$

Proof. From Corollary 5.1 and Theorems 3.3,

$$\begin{aligned} W^{*-1}(a,b,c) &= \Big(\frac{1-2bt-(c+ab)t^2}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Big) \\ &= \Big(\frac{1-2bt-(c+ab)t^2}{1-bt}, t\Big)\Big(\frac{1-bt}{1+at+ct^2}, \frac{t-bt^2}{1+at+ct^2}\Big) \\ &= \Big(\frac{1-2bt-(c+ab)t^2}{1-bt}, t\Big)\cdot W^{-1}(a,b,c). \end{aligned}$$

Thus, $W^*(a,b,c) = W(a,b,c)\cdot((1-bt)/(1-2bt-(c+ab)t^2), t)$. □

Now we define the generalized Pell sequence $(P_n)_{n\geqslant 0}$ by recurrence relation $P_n = 2bP_{n-1} + (ab+c)P_{n-2}$ with initial values $P_0 = 1, P_1 = b$. Then the generating function is

$$\sum_{n=0}^{\infty} P_n t^n = \frac{1-bt}{1-2bt-(c+ab)t^2}.$$

Hence Theorem 5.2 implies the following identities:

$$W^*_{n,k} = \sum_{j=k}^{n} W_{n,j} P_{j-k}. \tag{5.11}$$

Particularly,

$$W^*_n = \sum_{j=0}^{n} W_{n,j} P_j. \tag{5.12}$$

For $a = b = 1, c = 0$, we have

$$W^*(1,1,0) = \left(\frac{1}{\sqrt{1-6t+t^2}}, \frac{1-t-\sqrt{1-6t+t^2}}{2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 13 & 5 & 1 & 0 & 0 \\ 63 & 25 & 7 & 1 & 0 \\ 321 & 129 & 41 & 9 & 1 \end{pmatrix}.$$

The central Delannoy numbers (A001850) appear in Column 0 of this matrix. From Theorem 5.3, we have

$$W^*(1,1,0) = W(1,1,1) \cdot \Big(\frac{1-t}{1-2t-t^2}, t\Big),$$

where $(1-t)/(1-2t-t^2)$ is the generating function of the Pell numbers, see [18].

Example 5.1. For $a = b = c = 1$,

$$W^*(1,1,1) = W(1,1,1)\Big(\frac{1-t}{1-2t-2t^2}, t\Big), \quad W(1,1,1) = V(1,1,1)(1+t,t).$$

The first few rows of the matrices $W^*(1,1,1)$, $W(1,1,1)$ and $V(1,1,1)$ are

$$W^*(1,1,1) = \Big(\frac{1}{\sqrt{1-6t-3t^2}}, \frac{1-t-\sqrt{1-6t-3t^2}}{2(1+t)}\Big) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 15 & 5 & 1 & 0 & 0 \\ 81 & 28 & 7 & 1 & 0 \\ 459 & 161 & 45 & 9 & 1 \end{pmatrix},$$

$$W(1,1,1) = \Big(\frac{1-t-\sqrt{1-6t-3t^2}}{2t(1+t)}, \frac{1-t-\sqrt{1-6t-3t^2}}{2(1+t)}\Big) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 7 & 4 & 1 & 0 & 0 \\ 29 & 18 & 6 & 1 & 0 \\ 133 & 86 & 33 & 8 & 1 \end{pmatrix},$$

$$V(1,1,1) = \Big(\frac{1+t-\sqrt{1-6t-3t^2}}{2t(2+t)}, \frac{1-t-\sqrt{1-6t-3t^2}}{2(1+t)}\Big) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & 3 & 1 & 0 & 0 \\ 16 & 13 & 5 & 1 & 0 \\ 73 & 60 & 26 & 7 & 1 \end{pmatrix}.$$

The first columns of the matrices $W(1,1,1)$ and $W^*(1,1,1)$ correspond to the sequences A064641 and A122868 in [25], respectively.

Example 5.2. If $b=0$, $a=c=1$, then

$$W^*(1,0,1)=W(1,0,1)\Big(\frac{1}{1-t^2},t\Big),\ W(1,0,1)=V(1,0,1)(1+t,t).$$

The first few rows of the matrices $W^*(1,0,1)$, $W(1,0,1)$, and $V(1,0,1)$ are

$$\begin{aligned} W^*(1,0,1) &= \left(\frac{1}{\sqrt{1-2t-3t^2}},\quad \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right)\\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 3 & 2 & 1 & 0 & 0\\ 7 & 6 & 3 & 1 & 0\\ 19 & 16 & 10 & 4 & 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} W(1,0,1) &= \left(\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}, \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right)\\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 2 & 2 & 1 & 0 & 0\\ 4 & 5 & 3 & 1 & 0\\ 9 & 12 & 9 & 4 & 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} V(1,0,1) &= \left(\frac{1+t-\sqrt{1-2t-3t^2}}{2(t+t^2)}, \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right)\\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0\\ 1 & 3 & 2 & 1 & 0 & 0\\ 3 & 6 & 6 & 3 & 1 & 0\\ 6 & 15 & 15 & 10 & 4 & 1 \end{pmatrix}. \end{aligned}$$

The first columns of the matrices $W^*(1,0,1), W(1,0,1), V(1,0,1)$ correspond to the central trinomial coefficients (A002426), Motzkin numbers (A001006), and the Riordan numbers (A005043) respectively.

Example 5.3. If $b=0$, $a=3$, $c=1$, then

$$W^*(3,0,1)=W(3,0,1)\Big(\frac{1}{1-t^2},t\Big),\quad W(3,0,1)=V(3,0,1)(1+3t,t).$$

The first few rows of the matrices $W^*(3,0,1)$, $W(3,0,1)$, and $V(3,0,1)$ are

$$W^*(3,0,1) = \left(\frac{1}{\sqrt{1-6t+5t^2}}, \frac{1-3t-\sqrt{1-6t+5t^2}}{2t}\right)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 11 & 6 & 1 & 0 & 0 \\ 45 & 30 & 9 & 1 & 0 \\ 195 & 144 & 58 & 12 & 1 \end{pmatrix},$$

$$W(3,0,1) = \left(\frac{1-3t-\sqrt{1-6t+5t^2}}{2t^2}, \frac{1-3t-\sqrt{1-6t+5t^2}}{2t}\right)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 1 & 0 & 0 \\ 36 & 29 & 9 & 1 & 0 \\ 137 & 132 & 57 & 12 & 1 \end{pmatrix},$$

$$V(3,0,1) = \left(\frac{1+3t-\sqrt{1-6t+5t^2}}{2(3t+t^2)}, \frac{1-3t-\sqrt{1-6t+5t^2}}{2t}\right)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 3 & 11 & 6 & 1 & 0 \\ 11 & 42 & 30 & 9 & 1 \end{pmatrix}.$$

The first columns of the matrices $W^*(3,0,1), W(3,0,1), V(3,0,1)$ correspond to the sequence A026375, A002212 (restricted hexagonal numbers), and A117641 (3-Riordan numbers), respectively.

Example 5.4. If $a=0$, $b=c=1$, then we have $W(0,1,1) = V(0,1,1)$ and

$$W^*(0,1,1) = W(0,1,1)\left(\frac{1-t}{1-2t-t^2}, t\right).$$

The first few rows of the matrices $W^*(0,1,1)$ and $W(0,1,1)$ are

$$W^*(0,1,1) = \left(\frac{1}{\sqrt{1-4t-4t^2}}, \frac{1-\sqrt{1-4t-4t^2}}{2(1+t)}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 8 & 3 & 1 & 0 & 0 \\ 32 & 13 & 4 & 1 & 0 \\ 136 & 55 & 19 & 5 & 1 \end{pmatrix},$$

$$W(0,1,1) = \left(\frac{1-\sqrt{1-4t-4t^2}}{2t(1+t)}, \frac{1-\sqrt{1-4t-4t^2}}{2(1+t)}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 9 & 7 & 3 & 1 & 0 \\ 31 & 24 & 12 & 4 & 1 \end{pmatrix}.$$

The entries of the first column of the matrix $W(0,1,1)$ occur as the sequence A052709 in [25], and that of the matrix $W^*(0,1,1)$ occur as the sequence A006139 in [25].

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Authors' addresses: Lin Yang, Sheng-Liang Yang (corresponding author), Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, 730050, Gansu, P. R. China, e-mail: `yanglimath@163.com`, `slyang@lut.cn`; Tian-Xiao He, Department of Mathematics, Illinois Wesleyan University, Bloomington, IL, 61702-2900, USA, e-mail: `the@iwu.edu`.