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ON REAL FLAG MANIFOLDS WITH CUP-LENGTH EQUAL TO ITS DIMENSION

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Abstract. We prove that for any positive integers $n_1, n_2, \ldots, n_k$ there exists a real flag manifold $F(1, \ldots, 1, n_1, n_2, \ldots, n_k)$ with cup-length equal to its dimension. Additionally, we give a necessary condition that an arbitrary real flag manifold needs to satisfy in order to have cup-length equal to its dimension.

Keywords: cup-length; flag manifold; Lyusternik-Shnirel'man category

MSC 2010: 14M15, 55M30, 57N65

1. Introduction

The $\mathbb{Z}_2$*-cohomology cup-length* (or *cup-length*) of a path connected space X, denoted by $\operatorname{cup}(X)$, is the supremum of all positive integers m such that there exist classes $a_1, a_2, \ldots, a_m \in \widetilde{H}^*(X; \mathbb{Z}_2)$ with nonzero cup product, i.e., $a_1 a_2 \ldots a_m \neq 0$. It is well-known that $\operatorname{cup}(M)$ provides a lower bound for the Lyusternik-Shnirel'man category of M (recall that the Lyusternik-Shnirel'man category of M, denoted by $\operatorname{cat}(M)$, is the minimum number of open subsets of M covering M, each of which is contractible in M). In fact, one has

$$1 + \dim(M) \geqslant \operatorname{cat}(M) \geqslant 1 + \operatorname{cup}(M) \tag{1.1}$$

(in this paper dimension of a manifold M will be denoted by $\dim(M)$). A trivial upper bound for the cup-length of a manifold is its dimension. Furthermore, if $\operatorname{cup}(M) = \dim(M)$, then (1.1) implies $\operatorname{cat}(M) = 1 + \operatorname{cup}(M)$. In general, determining $\operatorname{cat}(M)$ poses a very difficult problem, so it is of interest to find manifolds M

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with cup-length equal to its dimension. In this paper, if $\mathrm{cup}(M) = \dim(M)$, then we say that the cup-length of M (or just $\mathrm{cup}(M)$) is *maximal*.

We consider this question for real flag manifolds (in this paper we only work with *real* flag manifolds, so we often use the term *flag manifold*). Let $n_1, n_2, \ldots, n_k \in \mathbb{N}$, $k \geqslant 2$. The *(real) flag manifold* $F(n_1, n_2, \ldots, n_k)$ is defined as the homogeneous space

$$O(n_1 + n_2 + \ldots + n_k)/O(n_1) \times O(n_2) \times \ldots \times O(n_k).$$

The numbers n_i, for $i \in [k]$, are the *steps* of this flag manifolds. Two special cases of flag manifolds are particularly important—flag manifolds with $k = 2$ are the *Grassmann manifolds*, and flag manifolds with $n_1 = \ldots = n_k = 1$ are the *complete flag manifolds*.

Although the cup-length of Grassmann manifold $F(m, n)$ is known only for $m \leqslant 4$ (see [3] and [10]), all Grassmann manifolds with maximal cup-length are known due to Berstein.

Theorem 1.1 ([1]). *The cup-length of Grasmmann manifold $F(m, n)$ is maximal if and only if $m = 1$, or $m = 2$ and $n = 2^t - 1$, for some $t \in \mathbb{N}$.*

The cup-length of all complete flag manifolds is maximal. In fact, the following stronger result holds (in this paper, we use the following notation: $a^{\cdots k} := \underbrace{a, \ldots, a}_{k}$).

Lemma 1.2 ([5]). *For all $j, n \in \mathbb{N}$, the cup-length of $F(1^{\cdots j}, n)$ is maximal.*

Having in mind the previous two results, one may think that a similar (simple) classification of all flag manifolds with maximal cup-length can be found, but it seems that this question is much more difficult. There have been attempts in the literature to solve this problem, but only some partial results were obtained. In [5] a family of flag manifolds of the form $F(1^{\cdots j}, 2^{\cdots d}, n)$ with maximal cup-length was found. This family was extended in [6], where, additionally, a necessary and sufficient condition for $\mathrm{cup}(F(1^{\cdots j}, 2^{\cdots d}, n)) = \dim(F(1^{\cdots j}, 2^{\cdots d}, n))$ in cases $d = 1$ and $d = 2$ were obtained. Up to now, no infinite family of flag manifolds with maximal cup-length and at least two steps greater than 2 was known.

The main result of this paper is the following.

Theorem 1.3. *For any positive integers $n_1, n_2, \ldots, n_k$ there exists a positive integer j such that $\mathrm{cup}(F(1^{\cdots j}, n_1, \ldots, n_k))$ is maximal.*

We divide the proof of this result into two parts. In Section 3, we use the method of embedding the cohomology of a flag manifold in the cohomology of a complete flag manifold, developed by Korbaš and Lörinc in [5], and prove the result for $k = 2$

and $n_1 = n_2$. In Section 4, we complete the proof using the method of “fiberings” introduced by Horanská and Korbaš in [4].

In Section 5 we give a necessary condition that a flag manifold with maximal cup-length needs to satisfy. In particular, this implies the following result.

Theorem 1.4. *If the cup-length $F(n_1, \ldots, n_k)$ is maximal, then at least one of the numbers n_i, $i \in \{1, 2, \ldots, k\}$, is not greater than* 3.

2. Preliminaries and notation

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, for $k \in \mathbb{N}$ we denote $[k] := \{1, 2, \ldots, k\}$. Furthermore, for an m-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m$ we use the following notation:

$$|\alpha| := \sum_{j=1}^{m} \alpha_j \quad \text{and} \quad \|\alpha\| := \sum_{j=1}^{m} j\alpha_j.$$

Let $F := F(n_1, n_2, \ldots, n_k)$ be a flag manifold. Then $\dim(F) = \sum\limits_{1 \leqslant i < j \leqslant k} n_i n_j$. There are k canonical vector bundles over F, which we denote by γ_i, for $i \in [k]$ ($\dim \gamma_i = n_i$). By Borel's description from [2] (more precisely, its slight modification, see for example [7]), each class in $H^*(F; \mathbb{Z}_2)$ is a polynomial in Stiefel-Whitney of the vector bundles γ_i for $i \in [k-1]$. In this paper we denote by $w_{i,j}$ the jth Stiefel-Whitney of the vector bundles γ_i for $i \in [k-1]$ and $j \in [n_i]$.

For $i \in [k-1]$ and an n_i-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n_i}) \in \mathbb{N}_0^{n_i}$ we use the notation W_i^α for the monomial $w_{i,1}^{\alpha_1} w_{i,2}^{\alpha_2} \ldots w_{i,n_i}^{\alpha_{n_i}}$. Also, let

$$S_0 := n_k \quad \text{and} \quad S_i := n_k + n_1 + n_2 + \ldots + n_i, \quad i \in [k-1].$$

Furthermore, by an abuse of notation, we denote $W_i = \{w_{i,1}, \ldots, w_{i,n_i}\}$, $i \in [k-1]$. So,

$$\mathbb{Z}_2[W_1, \ldots, W_{k-1}] = \mathbb{Z}_2[w_{1,1}, \ldots, w_{1,n_1}, \ldots, w_{k-1,1}, \ldots, w_{k-1,n_{k-1}}].$$

Note that the cup-length of F is maximal if and only if $a_1 \ldots a_{\dim_F} \neq 0$ for some $a_i \in \widetilde{H}^*(F; \mathbb{Z}_2)$, $i \in [\dim(F)]$. Note that the latter implies $a_i \in H^1(F; \mathbb{Z}_2)$ for all $i \in [\dim(F)]$. Hence, the cup-length of F is maximal if and only if there exist $\alpha_i \in \mathbb{N}_0$, $i \in [k-1]$, such that $w_{1,1}^{\alpha_1} \ldots w_{k-1,1}^{\alpha_{k-1}} \neq 0$. Of course, a necessary condition for the last relation is that $\alpha_i \leqslant \mathrm{ht}(w_{i,1})$ for $i \in [k-1]$, where $\mathrm{ht}(a)$ denotes the height of the class a (the *height* of a class $a \in \widetilde{H}^*(X; \mathbb{Z}_2)$ is the largest positive integer n such that $a^n \neq 0$).

Although the cup-length of a general flag manifold is far from being understood, the heights of the first Stiefel-Whitney classes are known by the following result of Korbaš and Lörinc (see [5]).

Proposition 2.1. *Let t be the unique integer such that $2^t < S_{k-1} \leqslant 2^{t+1}$, and let $m_i = \min\{n_i, S_{k-1} - n_i\}$. Then*

$$\operatorname{ht}(w_{i,1}) = \begin{cases} S_{k-1} - 1 & \textit{if } m_i = 1, \\ 2^{t+1} - 2 & \textit{if } m_i = 2, \textit{ or } m_i = 3 \textit{ and } S_{k-1} = 2^t + 1, \\ 2^{t+1} - 1 & \textit{otherwise.} \end{cases}$$

We also denote $\operatorname{ht}(n_i) := \operatorname{ht}(w_{i,1})$ for $i \in [k-1]$.

3. Evaluation of cup-length and complete flags

In this section we prove our main result in the case $k = 2$ and $n_1 = n_2$. The method that we use in the proof is the one introduced by Korbaš and Lörinc in [5]. First, we explain this method.

Let $m \geqslant 2$ and observe the complete flag manifold $F(1^{\cdots m})$. Denote by $e_i := w_1(\gamma_i)$ the first Stiefel-Whitney class of the canonical line bundle γ_i over $F(1^{\cdots m})$, $i \in [m]$. The following lemma is well-known (see [5], [10]).

Lemma 3.1. *A monomial $e_1^{a_1} \dots e_m^{a_m} \in H^{\binom{m}{2}}(F(1^{\cdots m}); \mathbb{Z}_2) \cong \mathbb{Z}_2$ is nonzero if and only if $(a_1, a_2, \dots, a_m)$ is a permutation of the m-tuple $(m-1, m-2, \dots, 1, 0)$.*

Let $n_1, n_2, \dots, n_k$ ($k \geqslant 2$) be positive integers, $\nu_i = n_1 + n_2 + \dots + n_i$, $i \in [k]$ (it is understood that $\nu_0 = 0$), and $m = \nu_k$. For the flag manifold $F(n_1, n_2, \dots, n_k)$ we have the map $p\colon F(1^{\cdots m}) \to F(n_1, n_2, \dots, n_k)$, given by

$$p(V_1, \dots, V_{n_1}, \dots, V_{\nu_{k-1}+1}, \dots, V_m) = (V_1 \oplus \dots \oplus V_{n_1}, \dots, V_{\nu_{k-1}+1} \oplus \dots \oplus V_m).$$

Our proof is based on the following result from [5], page 154.

Lemma 3.2. *If $F = F(n_1, n_2, \dots, n_k)$, $u \in H^{\dim F}(F; \mathbb{Z}_2)$ and*

$$v = e_1^{n_1-1} \dots e_{n_1-1} e_{n_1+1}^{n_2-1} \dots e_{n_1+n_2-1} \dots e_{\nu_{k-1}+1}^{n_k-1} \dots e_{\nu_k-1} \in H^*(F(1^{\cdots m}); \mathbb{Z}_2),$$

then $p^*(u) \cdot v \in H^{\binom{m}{2}}(F(1^{\cdots m}); \mathbb{Z}_2)$ and

$$u \neq 0 \quad \textit{if and only if} \quad p^*(u) \cdot v \neq 0.$$

In [5], page 155 the authors also gave a description of the map p^* from the previous lemma. If $w_{i,j}$ is the jth Stiefel-Whitney class of the canonical bundle γ_i over $F(n_1, n_2, \ldots, n_k)$, $i \in [k]$, $j \in [n_i]$, then $p^*(w_{i,j})$ is the jth elementary symmetric polynomial in the variables $e_{\nu_{i-1}+1}, e_{\nu_{i-1}+2}, \ldots, e_{\nu_i}$. In our application the most important will be the case $j = 1$, when one has

$$p^*(w_{i,1}) = e_{\nu_{i-1}+1} + e_{\nu_{i-1}+2} + \ldots + e_{\nu_i}. \tag{3.1}$$

For $a_1, a_2, \ldots, a_k \in \mathbb{N}_0$, we denote the multinomial coefficient by

$$\binom{a_1 + a_2 + \ldots + a_k}{a_1, a_2, \ldots, a_k} := \frac{(a_1 + a_2 + \ldots + a_k)!}{a_1!\, a_2! \ldots a_k!}.$$

The following lemma is probably well-known, but we prove it for the sake of completeness.

Lemma 3.3. *Let $a_1, \ldots, a_k$ be nonnegative integers and $a_i = (\alpha_{1,i}, \ldots, \alpha_{s_i,i})_2$, for $i \in [k]$, their representations in base 2. Then*

$$\binom{a_1 + a_2 + \ldots + a_k}{a_1, a_2, \ldots, a_k} \text{ is odd}$$

if and only if for all $i, j \in [k]$, $i \neq j$, and $l \in [\max\{s_i, s_j\}]$ at least one of the numbers $\alpha_{l,i}$ and $\alpha_{l,j}$ is equal to zero.

P r o o f. ($\Rightarrow$) By symmetry, it is enough to prove the claim for $i = k-1$ and $j = k$. Since

$$\binom{a_1 + a_2 + \ldots + a_k}{a_1} \cdots \binom{a_{k-1} + a_k}{a_{k-1}} = \binom{a_1 + a_2 + \ldots + a_k}{a_1, a_2, \ldots, a_k} \equiv 1 \pmod 2,$$

the number $\binom{a_{k-1}+a_k}{a_k}$ is odd. Now, the result follows from [6], Lemma 2.3.

($\Leftarrow$) Note that the given condition implies that for every $i \in [k]$ and every $l \in \mathbb{N}_0$ at most one of the numbers a_i and $a_{i+1} + \ldots + a_k$ has digit 1 in position l. By [6], Lemma 2.3, this implies that $\binom{a_i+a_{i+1}+\ldots+a_k}{a_i}$ is odd (for all $i \in [k-1]$), which completes our proof. $\square$

Lemma 3.4. *For every $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $\operatorname{cup}(F(1^{\cdot\cdot j}, n, n))$ is maximal.*

P r o o f. If $n = 1$, then by Lemma 1.2 we can take $j = 1$. So, we may assume that $n \geqslant 2$. Let s be the unique integer such that $2^{s-1} < n \leqslant 2^s$. We will prove that $j = 2^{s+n-1}$ has the desired property. Let $a_1, a_2, \ldots, a_j$ be the sequence obtained from the sequence $n, n+1, \ldots, 2n+j-1$ by removing numbers $2^{s+i}+i$ for $i \in [n-1] \cup \{0\}$ (note that $2n+j-1 > 2^{s+n-1}+n-1 > 2^s \geqslant n$). Finally, let $m = 2^s + 2^{s+1} + \ldots + 2^{s+n-1} = 2^{s+n} - 2^s$.

Note that

$$m + \sum_{i=1}^{j} a_i = n^2 + 2nj + \binom{j}{2} = \dim(F(1^{\cdots j}, n, n)),$$

so it is enough to prove that $\prod\limits_{i=1}^{j} w_{i,1}^{a_i} \cdot w_{j+1,1}^{m}$ is nonzero (in $H^*(F(1^{\cdots j}, n, n); \mathbb{Z}_2)$). By Lemma 3.2 and (3.1), this is equivalent with (in $H^*(F(1^{\cdots j+2n}); \mathbb{Z}_2)$)

$$\begin{aligned}(3.2)\quad 0 &\neq (e_{j+1} + \ldots + e_{j+n})^m e_{j+1}^{n-1} \ldots e_{j+n-1} e_{j+n+1}^{n-1} \ldots e_{j+2n-1} \prod_{i=1}^{j} e_i^{a_i} \\ &= \sum_{t_1+\ldots+t_n=m} \binom{m}{t_1, \ldots, t_n} e_{j+1}^{t_1+n-1} \ldots e_{j+n-1}^{t_{n-1}+1} e_{j+n}^{t_n} e_{j+n+1}^{n-1} \ldots e_{j+2n-1} \prod_{i=1}^{j} e_i^{a_i}.\end{aligned}$$

By Lemma 3.1, a summand in the last expression is nonzero if and only if the multinomial coefficient $\binom{m}{t_1,\ldots,t_n}$ is odd and $(t_1+n-1, \ldots, t_{n-1}+1, t_n)$ is a permutation of the n-tuple $(2^{s+n-1}+n-1, \ldots, 2^{s+1}+1, 2^s)$.

Let $(t_1, \ldots, t_n)$ be an n-tuple satisfying these conditions. Since $t_i + n - i \geqslant 2^s > n-1$, we have that $t_i > 0$ for all $i \in [n]$, i.e. t_i has at least one nonzero digit in the binary expansion. On the other hand, m has exactly n digits in the binary expansion, so by Lemma 3.3, $\binom{m}{t_1,\ldots,t_n}$ is odd if and only if

$$\{t_1, \ldots, t_{n-1}, t_n\} = \{2^{s+n-1}, \ldots, 2^{s+1}, 2^s\}.$$

Additionally, $\{t_1+n-1, \ldots, t_{n-1}+1, t_n\} = \{2^{s+n-1}+n-1, \ldots, 2^{s+1}+1, 2^s\}$, so there is an index $i \in [n]$ such that $2^{s+n-1}+n-1 = t_i+n-i$. Since $2^{s+n-1} \geqslant t_i$ and $n-1 \geqslant n-i$, we have that $i = 1$ and $t_1 = 2^{s+n-1}$. Continuing in the same way we conclude that $t_2 = 2^{s+n-2}, \ldots, t_n = 2^s$.

Hence, $(t_1, \ldots, t_n) = (2^{s+n-1}, \ldots, 2^{s+1}, 2^s)$ is the only n-tuple for which the corresponding summand in (3.2) is nonzero. This completes our proof. $\square$

Note that j constructed in the previous lemma satisfies $j = 2^{s+n-1} \leqslant (n-1)2^n$.

4. Fiberings and cup-length

To complete the proof of our main result we use the method of "fiberings" introduced by Horanská and Korbaš in [4]. This method proved very useful in cup-length calculation (see [4], [5], [9]). It is based on the following result.

Theorem 4.1 ([4]). *Let $p\colon E \to B$ be a smooth fiber bundle with connected base B and connected fiber F. Suppose that the fiber inclusion induces an epimorphism in $\mathbb{Z}_2$-cohomology. Then* $\mathrm{cup}(E) \geqslant \mathrm{cup}(F) + \mathrm{cup}(B)$.

Let us observe the following fiber bundle (see [5]):

$$\begin{tikzcd} F(n_{l+1},\dots,n_k) \arrow[r, hook] & F(n_1,\dots,n_k) \arrow[d] \\ & F(n_1,\dots,n_l,n_{l+1}+\dots+n_k). \end{tikzcd}$$

Since the inclusion $i\colon F(n_{l+1},\dots,n_k) \to F(n_1,\dots,n_k)$ induces an epimorphism in $\mathbb{Z}_2$-cohomology (see [5]), we can apply Theorem 4.1 on this fiber bundle. Additionally, we have

$$\dim(F(n_{l+1},\dots,n_k)) + \dim(F(n_1,\dots,n_l,n_{l+1}+\dots+n_k)) = \dim(F(n_1,\dots,n_k)),$$

so from Theorem 4.1 and the fact that the upper bound for the cup-length is the dimension of the manifold, we obtain the following result:

(4.1) If $\mathrm{cup}(F(n_{l+1},\dots,n_k))$ and $\mathrm{cup}(F(n_1,\dots,n_l,n_{l+1}+\dots+n_k))$ are maximal, then $\mathrm{cup}(F(n_1,\dots,n_k))$ is also maximal.

We are ready to prove our main result.

P r o o f of Theorem 1.3. By Lemma 1.2 for $k=1$ it is enough to take $j=1$. So, we may assume that $k \geqslant 2$. We continue our proof by induction on k.

Base case $k=2$. Since $F(1^{\cdots j},n_1,n_2)$ is homeomorphic to $F(1^{\cdots j},n_2,n_1)$ (for any j), we may assume that $n_1 \leqslant n_2$. If $n_1 = n_2$, then the result follows from Lemma 3.4.

So, let us assume that $n_1 < n_2$. Furthermore, let j' be a positive integer such that $\mathrm{cup}(F(1^{\cdots j'},n_2,n_2))$ is maximal (j' exists by Lemma 3.4) and consider the following fiber bundle:

$$\begin{tikzcd} F(1^{\cdots(n_2-n_1)},n_1) \arrow[r, hook] & F(1^{\cdots j'},n_2,1^{\cdots(n_2-n_1)},n_1) \arrow[d] \\ & F(1^{\cdots j'},n_2,n_2). \end{tikzcd}$$

By Lemma 1.2, Lemma 3.4 and (4.1), we conclude that the cup-length of the flag manifold $F(1^{\cdots j'+n_2-n_1}, n_1, n_2)$ is maximal.

Inductive step. Suppose that the claim is true for all $l \in [k] \setminus \{1\}$ and let us prove it for $k+1$.

Let j' be a positive integer such that $\operatorname{cup}(F(1^{\cdots j'}, n_k, n_{k+1}))$ is maximal and j'' a positive integer such that $\operatorname{cup}(F(1^{\cdots j''}, n_1, \ldots, n_{k-1}, j' + n_k + n_{k+1}))$ is maximal (j' and j'' exist by inductional hypothesis). Now, using (4.1) for the fiber bundle

$$\begin{tikzcd} F(1^{\cdots j'}, n_k, n_{k+1}) \arrow[r, hook] & F(1^{\cdots j''}, n_1, \ldots, n_{k-1}, 1^{\cdots j'}, n_k, n_{k+1}) \arrow[d] \\ & F(1^{\cdots j''}, n_1, \ldots, n_{k-1}, j' + n_k + n_{k+1}) \end{tikzcd}$$

we conclude that $\operatorname{cup}(F(1^{\cdots j'+j''}, n_1, \ldots, n_{k+1}))$ is also maximal. $\square$

The number j constructed in the previous proof is quite large. We demonstrate this in cases $k=2$ and $k=3$ (we use the same notation as above).

Let $k=2$ and w.l.o.g. $n_1 \leqslant n_2$. Then $j = j' + n_2 - n_1$, where, by the remark after Lemma 3.4, $j' \leqslant (n_2-1)2^{n_2}$. So, $j \leqslant (n_2-1)2^{n_2} + n_2 - n_1 < n_2 \cdot 2^{n_2}$.

Now, let $k=3$ and w.l.o.g. $n_1 \geqslant n_2 \geqslant n_3$. Then $j = j' + j''$, and from the case $k=2$ one has $j' < n_2 \cdot 2^{n_2}$ and $j'' < \max\{(j' + n_2 + n_3)2^{j'+n_2+n_3}, n_1 \cdot 2^{n_1}\}$. So,

$$j < \max\{(n_2 \cdot 2^{n_2} + n_2 + n_3)2^{n_2 \cdot 2^{n_2}+n_2+n_3}, n_1 \cdot 2^{n_1}\}.$$

At the end of this section we show that if $\operatorname{cup}(F(1^{\cdots j}, n_1, \ldots, n_k))$ is maximal, then $\operatorname{cup}(F(1^{\cdots j'}, n_1, \ldots, n_k))$ is also maximal for all $j' \geqslant j$. Clearly, it is enough to consider the case $j' = j+1$. Then the proof follows from Lemma 1.2 and (4.1) applied to the following fiber bundle:

$$\begin{tikzcd} F(1^{\cdots j}, n_1, \ldots, n_k) \arrow[r, hook] & F(1^{\cdots j+1}, n_1, \ldots, n_k) \arrow[d] \\ & F(1, j + n_1 + \ldots + n_k). \end{tikzcd}$$

This construction implies that in order to obtain all flag manifolds with maximal cup-length, it is enough to find (for every $k \geqslant 2$ and $n_1, \ldots, n_k \in \mathbb{N}$) the minimal $j = j(n_1, \ldots, n_k)$ such that $F(1^{\cdots j}, n_1, \ldots, n_k)$ has maximal cup-length.

In this section we give a necessary condition that a flag manifold with maximal cup-length needs to satisfy. This proof is based on a result from [7] (in fact, its mod 2 variant), where Gröbner bases for all flag manifolds were constructed.

Throughout this section, let F denote the flag manifold $F(n_1, n_2, \ldots, n_k)$ and t the unique integer such that $2^t < S_{k-1} \leqslant 2^{t+1}$. Also, we use the notation introduced in Section 2.

Lemma 5.1. *For every $f \in \mathbb{Z}_2[W_1, W_2, \ldots, W_{k-1}]$ there is a polynomial p such that $p = f$ in $H^*(F; \mathbb{Z}_2)$ and*

(i) *for each monomial $W_1^{\alpha(1)} \ldots W_{k-1}^{\alpha(k-1)}$ of p and $i \in [k-1]$ we have $|\alpha(i)| \leqslant S_{i-1}$;*

(ii) *if no monomial of f contains a variable from $W_1 \cup W_2 \cup \ldots \cup W_l$ for an $l \in [k-1]$, then the same holds for p.*

The following lemma will be the key for obtaining the main result of this section (this lemma generalizes [8], Corollary 3.1.4.).

Lemma 5.2. *Let $\alpha(i)$ for $i \in [k-1]$ be an arbitrary n_i-tuple of nonnegative integers. If $\sum\limits_{i=l}^{k-1} \|\alpha(i)\| > \sum\limits_{i=l}^{k-1} n_i S_{i-1}$ for an $l \in [k-1]$, then $W_1^{\alpha(1)} W_2^{\alpha(2)} \ldots W_{k-1}^{\alpha(k-1)} = 0$ in $H^*(F; \mathbb{Z}_2)$.*

Proof. It suffices to prove that $W_l^{\alpha(l)} \ldots W_{k-1}^{\alpha(k-1)} = 0$ in $H^*(F; \mathbb{Z}_2)$. We know that $W_l^{\alpha(l)} \ldots W_{k-1}^{\alpha(k-1)} \in H^q(F; \mathbb{Z}_2)$, where $q = \sum\limits_{i=l}^{k-1} \|\alpha(i)\|$. Let p be the polynomial from Lemma 5.1 such that $p = W_l^{\alpha(l)} \ldots W_{k-1}^{\alpha(k-1)}$ in $H^*(F; \mathbb{Z}_2)$. Suppose that p is nonzero. Then an arbitrary monomial in p is of the form $W_l^{\beta(l)} \ldots W_{k-1}^{\beta(k-1)}$, where $\beta(i) \in \mathbb{N}_0^{n_i}$ and $|\beta(i)| \leqslant S_{i-1}$ for all $i \in \{l, \ldots, k-1\}$. But the dimension of $W_l^{\beta(l)} \ldots W_{k-1}^{\beta(k-1)}$ (and also of p) is

$$\sum_{i=l}^{k-1} \|\beta(i)\| \leqslant \sum_{i=l}^{k-1} n_i |\beta(i)| \leqslant \sum_{i=l}^{k-1} n_i S_{i-1} < q,$$

which is a contradiction since $p = W_l^{\alpha(l)} \ldots W_{k-1}^{\alpha(k-1)} \in H^q(F; \mathbb{Z}_2)$. □

We are ready to prove the main result of this section.

Proposition 5.3. *Suppose that a flag manifold F has maximal cup-length and let t be as above. Then for every permutation π of the set $\{1, 2, \ldots, k\}$ and every*

$l \in [k-1]$ *we have*

$$n_{\pi(k)} \sum_{i=1}^{l} n_{\pi(i)} + \sum_{1 \leqslant i < i' \leqslant l} n_{\pi(i)} n_{\pi(i')} \leqslant \sum_{i=1}^{l} \mathrm{ht}(n_{\pi(i)}).$$

P r o o f. Since F has the maximal cup-length, so does the flag manifold $\widetilde{F} := F(n_{\pi(1)}, n_{\pi(2)}, \ldots, n_{\pi(k)})$. Let $\widetilde{w}_{i,1}$ for $i \in [k-1]$ be the first Stiefel-Whitney class of the ith tautological vector bundle over this manifold, and

$$\widetilde{w}_{1,1}^{a_1} \widetilde{w}_{2,1}^{a_2} \ldots \widetilde{w}_{k-1,1}^{a_{k-1}} \neq 0$$

a class in $H^*(\widetilde{F}; \mathbb{Z}_2)$ such that $\sum\limits_{i=1}^{k-1} a_i = \dim(\widetilde{F})$. Then, by Lemma 5.2, we have

$$\begin{aligned}\sum_{i=1}^{l} a_i = \dim(\widetilde{F}) - \sum_{i=l+1}^{k} a_i \geqslant \dim(\widetilde{F}) - \sum_{i=l+1}^{k} n_{\pi(i)} \Bigl(n_{\pi(k)} + \sum_{j=1}^{i-1} n_{\pi(j)} \Bigr) \\ = n_{\pi(k)} \sum_{i=1}^{l} n_{\pi(i)} + \sum_{1 \leqslant i < i' \leqslant l} n_{\pi(i)} n_{\pi(i')}.\end{aligned}$$

On the other hand, $a_i \leqslant \mathrm{ht}(n_{\pi(i)})$, for $i \in [k-1]$, which together with the previous inequality gives us the desired result. $\square$

Let us go back to the question from the previous sections. So, for the given positive integers $n_1, n_2, \ldots, n_k$, $k \geqslant 2$, we want to find j such that $\mathrm{cup} F(1^{\cdots j}, n_1, \ldots, n_k)$ is maximal. In what follows we show that if the numbers n_i are large enough, then Proposition 5.3 implies that j also must be large.

Suppose that $n_i \geqslant m$ for an $m \geqslant 4$ and all $i \in [k]$. As usual, let t be the unique integer such that $2^t < j + \sum\limits_{i=1}^{k} n_i \leqslant 2^{t+1}$. Then, by Proposition 2.1, $\mathrm{ht}(n_i) = 2^{t+1} - 1$, so applying Proposition 5.3 for the permutation $(\pi(1), \ldots, \pi(k+j)) = (n_1, \ldots, n_{k-1}, 1, \ldots, 1, n_k)$ and $l = k-1$, gives

$$(2^{t+1} - 1)(k-1) \geqslant \sum_{1 \leqslant i < i' \leqslant k} n_i n_{i'}.$$

Since $j \geqslant 2^t - \sum\limits_{i=1}^{k} n_i + 1$, one has $2j + 2\sum\limits_{i=1}^{k} n_i - 3 \geqslant 2^{t+1} - 1$, so by the previous inequality

$$2(k-1)j \geqslant \sum_{1 \leqslant i < i' \leqslant k} n_i n_{i'} - 2(k-1) \sum_{i=1}^{k} n_i + 3(k-1) = f(n_1, \ldots, n_k).$$

Note that f is a linear function in each n_t, $t \in [k]$. Furthermore, for $t \in [k]$

$$f(n_1,\dots,n_k) = \Bigl(\sum_{i'\neq t} n_{i'} - 2(k-1)\Bigr) n_t + \sum_{\substack{1\leqslant i<i'\leqslant k\\ i,i'\neq t}} n_i n_{i'} - 2(k-1)\sum_{i'\neq t} n_{i'} + 3(k-1),$$

and since $\sum\limits_{i'\neq t} n_{i'} - 2(k-1) \geqslant 2(k-1)$, this function is increasing in n_t for every $t \in [k]$. This implies $f(n_1,\dots,n_k) \geqslant f(m,\dots,m)$, and hence

$$j \geqslant \frac{f(m,\dots,m)}{2(k-1)} = \frac{1}{4}(km^2 - 4km + 6) > 1. \tag{5.1}$$

This inequality immediately implies Theorem 1.4. However, we note that to obtain this result one does not need Lemma 5.2, i.e. it follows from the case $l = k-1$ of Proposition 5.3, which is in fact (obvious) inequality $\dim(F) \leqslant \sum\limits_{i=1}^{k-1} \mathrm{ht}(n_i)$.

Remark 5.4. Of course, j obtained using the proofs of Lemma 3.4 and Theorem 1.3 is much larger than the lower bound from (5.1) (see the paragraphs after the proof of Theorem 1.3), i.e. there is quite a gap between the lower and the upper bound (that we obtain in this paper) for the minimal j with the desired property.

We finish this section with the following example.

Example 5.5. The cup-length of $F(1^{\cdots 17}, 3, 5, 7)$ is not maximal. To prove this, it is enough to apply Proposition 5.3 for the permutation $\pi = (3,5,1,\dots,1,7)$ and $l = 2$. Indeed, $\mathrm{ht}(3) = \mathrm{ht}(5) = 31$, but the left-hand side of the inequality from Proposition 5.3 is equal to 71.

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