Michal Pešta; Barbora Peštová; Matúš Maciak

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CHANGEPOINT ESTIMATION FOR DEPENDENT AND NON-STATIONARY PANELS

Michal Pešta, Barbora Peštová, Matúš Maciak, Praha

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Abstract. The changepoint estimation problem of a common change in panel means for a very general panel data structure is considered. The observations within each panel are allowed to be generally dependent and non-stationary. Simultaneously, the panels are weakly dependent and non-stationary among each other. The follow up period can be extremely short and the changepoint magnitudes may differ across the panels accounting also for a specific situation that some magnitudes are equal to zero (thus, no jump is present in such case). We introduce a novel changepoint estimator without a boundary issue meaning that it can estimate the change close to the extremities of the studied time interval. The consistency of the nuisance-parameter-free estimator is proved regardless of the presence/absence of the change in panel means under relatively simple conditions. Empirical properties of the proposed estimator are investigated through a simulation study.

Keywords: panel data; changepoint; change in means; estimation; dependence; non-stationarity; call options; non-life insurance

MSC 2020: 62F10, 62H12, 62P05

1. Introduction and aims

A typical panel data structure in empirical economics usually assumes independent panels, which are used to represent a set of multiple units (e.g. countries, companies, or different financial markets). These units are a priori assumed to be independent among each other. The main focus is placed either on detection of a possible changepoint [5], [12], [2] or estimation of an unknown changepoint location [1], [13]. Sometimes, the assumption of independent units is not realistic. *Dependence* is then traditionally modeled by additional stochastic terms, which are implemented in a lin-

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ear form that is common across all panels. The key interest is, again, to detect or to estimate the potential changepoints [10], [6], [4], [8].

In more complex situations, however, the linear form of dependence across panels is quite limiting and other approaches are needed to properly take into account the complexity of the underlying mechanism, which generates the available data. Such situations are common, for instance, for various financial markets, where individual panels represent financial values for some sort of commodity with an implicit form of *natural ordering*. For example, a set of panels for time developments of the intrinsic values of options with specific strikes—option values. Such scenario with non-stationary panels is also considered in Maciak et al. [9], where the focus is on the changepoint detection. In this paper, we consider the same structure of the underlying model, although, the *main interest* is on the *consistent estimation* of the unknown common changepoint without any prior knowledge whether a changepoint is present in the panel data or not. On one hand, *dependent and non-stationary* panels with possibly an *extremely short follow-up period* allow for huge flexibility of the model. On the other hand, the presented estimation method is completely *nuisance parameter free*, which makes it effortlessly applicable. In addition, the model formulation also allows for a situation, where only some of the panels are subject to a change while there is no change in the remaining panels.

From the practical point of view, the proposed methodology can be applied to *call options*, where the option intrinsic values $Y_{i,t}$ for a specific strike (panel) $i = 1, \ldots, N$ are observed repeatedly over several trading days $t = 1, \ldots, T$. Thus, the panels are naturally ordered by the corresponding options' values. A standard approach with linearly dependent panels is not appropriate as it cannot truly capture the underlying panel complexity. Another application can be taken from *non-life insurance*, where associations in many countries unite several insurance companies. Claim amounts paid by every insurance company each year are collected into a common database. The total (cumulative) claim amount $Y_{i,t}$ paid by insurance company i (ordered with respect to the received premium) in year t can be viewed as panel data.

The changepoint model assumed for the scenarios described above is

$$Y_{i,t} = \mu_i + \delta_i \mathbb{1}\{t > \tau\} + \varepsilon_{i,t}, \quad i = 1, \ldots, N, \ t = 1, \ldots, T, \tag{1.1}$$

where $\mu_i \in \mathbb{R}$ are the panel-specific mean parameters, $\tau \in \{1, \ldots, T\}$ is some common *changepoint* time (same for all considered panels) with the corresponding jump magnitudes $\delta_i \in \mathbb{R}$. Thus, if there is some common changepoint in model (1.1) present at the time $\tau < T$, then the corresponding panel-specific means change from μ_i before the change to $\mu_i + \delta_i$ after the change. This formulation also allows for a specific case where $\delta_i = 0$, meaning no jump is present for some given panel i. The panel-specific disturbances $\{\varepsilon_{i,t}\}_t$ are general sequences of random errors. The length of the follow-up period, T, is supposed to be fixed and not depending on the number of panels N.

2. Changepoint estimator

Various consistent estimators of the changepoint in panel data are proposed in, e.g. [10], [1], or [2], but all under the circumstances that the change occurred for sure. Although in our case, we *do not know whether a change has occurred or not.* Therefore, we propose the following idea: If the panel means change somewhere inside $\{1,\ldots,T-1\}$, let the estimate select this break point; if there is no change in panel means, the estimator points out the very last time point T with probability going to one. In other words, the value of the changepoint estimator can be T, meaning no change. This is in contrast to several previous works, where T is not reachable. Peštová and Pešta [13] have already constructed a desirable changepoint estimator applicable without the knowledge whether the change occurred for sure. Their estimator indeed has not the boundary issue, but it assumes independence across the panels as well as a form of stationarity within the panels. Our intention here is to overcome such restrictive conditions.

The common changepoint τ in panel data can be estimated as

$$\widehat{\tau}_N := \arg\max_{t=1,\ldots,T} \mathcal{U}_N(t), \tag{2.1}$$

where

$$\mathcal{U}_N(t) = \begin{cases} \dfrac{1}{t(T-t)} \displaystyle\sum_{i=1}^{N} \sum_{u=1}^{t} \sum_{v=t+1}^{T} (Y_{i,u} - Y_{i,v})^2, & t < T; \\ \dfrac{2}{(T-1)^2} \displaystyle\sum_{i=1}^{N} \sum_{v=2}^{T} \sum_{u=1}^{v-1} (Y_{i,u} - Y_{i,v})^2, & t = T. \end{cases} \tag{2.2}$$

Assumption A1. The vectors $\boldsymbol{\varepsilon}_i \equiv [\varepsilon_{i,1},\ldots,\varepsilon_{i,T}]^\top$ are zero mean α-mixing, where the mixing coefficients satisfy $\sum_{i=1}^{\infty}\{\alpha(\boldsymbol{\varepsilon}_\circ, i)\}^{\chi/(2+\chi)} < \infty$ for some $\chi > 0$, $\operatorname{Var}\varepsilon_{i,t} = \sigma_i^2 > 0$, and $\sup_{i\in\mathbb{N}} \mathsf{E}\,|\varepsilon_{i,t}|^{4+2\chi} < \infty$ for all $t \in \{1,\ldots,T\}$.

Note that the panels of errors $\boldsymbol{\varepsilon}_i$ are neither independent nor identically distributed. Moreover, there is no prescribed form of stationarity assumed within the panel. Such *universal* assumptions on the random errors strengthen our forthcoming result. The sequence $\{\varepsilon_{i,t}\}_{t=1}^{T}$ can be viewed as some complete fragment of a *non-stationary* but homoscedastic process. Furthermore, *non-stationarity* across the panels $\{\boldsymbol{\varepsilon}_i\}_{i=1}^{N}$ is allowed as well.

Assumption A2. Let $\sup_{i\in\mathbb{N}} |\delta_i| < \infty$.

Assumption A3. Let $\underline{\varsigma}_i = \min_{u,v\in\{1,\ldots,T\}} \mathsf{E}\, \varepsilon_{i,u}\varepsilon_{i,v}$ and $\overline{\varsigma}_i = \max_{u,v\in\{1,\ldots,T\}} \mathsf{E}\, \varepsilon_{i,u}\varepsilon_{i,v}$. For

$$\tau < T, \quad \lim_{N\to\infty} \frac{1}{\sqrt{N}} \left[\left\{ 1 - \frac{2\tau(T-\tau)}{(T-1)^2} \right\} \sum_{i=1}^{N} \delta_i^2 - \frac{2}{T-1} \sum_{i=1}^{N} (\sigma_i^2 - \underline{\varsigma}_i) \right] = \infty$$

and

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \delta_i^2 = \infty.$$

For

$$\tau = T, \quad \lim_{N\to\infty} \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{N} (\sigma_i^2 - \overline{\varsigma}_i) - \frac{T-1}{2} \sum_{i=1}^{N} \delta_i^2 \right) = \infty.$$

Assumptions A2 and A3 control trade-off between the size of breaks, the number of panels, and the variability of errors. They may be considered as *detectability assumptions*, because they basically specify the value of signal-noise ratio for finding the consistent estimator.

Theorem 2.1 (Consistency). *Under Assumptions* A1–A3, $\lim_{N\to\infty} \mathsf{P}[\widehat{\tau}_N = \tau] = 1$.

Proof. For $t < T$ and with respect to Assumption A1, let us calculate

$$\begin{aligned}
(2.3)\quad \mathsf{E}\,\mathcal{U}_N(t) &= \frac{1}{t(T-t)} \sum_{i=1}^{N} \sum_{u=1}^{t} \sum_{v=t+1}^{T} \mathsf{E}(\varepsilon_{i,u} + \delta_i \mathbb{1}\{u > \tau\} - \varepsilon_{i,v} - \delta_i \mathbb{1}\{v > \tau\})^2 \\
&= \frac{1}{t(T-t)} \sum_{i=1}^{N} \sum_{u=1}^{t} \sum_{v=t+1}^{T} (\mathsf{E}\,\varepsilon_{i,u}^2 + \mathsf{E}\,\varepsilon_{i,v}^2 - 2\,\mathsf{E}\,\varepsilon_{i,u}\varepsilon_{i,v} + \delta_i^2 \mathbb{1}\{v > \tau \geqslant u\}) \\
&= 2\sum_{i=1}^{N} (\sigma_i^2 - c_i) + \Delta_{t,\tau}^{T} \sum_{i=1}^{N} \delta_i^2,
\end{aligned}$$

where

$$\Delta_{t,\tau}^{T} = \begin{cases} \dfrac{T-\tau}{T-t}, & t \leqslant \tau \\ \dfrac{\tau}{t}, & t > \tau \end{cases} \quad \text{and} \quad c_i = \frac{1}{t(T-t)} \sum_{u=1}^{t} \sum_{v=t+1}^{T} \mathsf{E}\,\varepsilon_{i,u}\varepsilon_{i,v}.$$

Similarly for $t = T$ we have

$$(2.4)\qquad \mathsf{E}\,\mathcal{U}_N(t) = \frac{2T}{T-1} \sum_{i=1}^{N} (\sigma_i^2 - c_i) + \frac{2\tau(T-\tau)}{(T-1)^2} \sum_{i=1}^{N} \delta_i^2.$$

Let us define

$$X_i := \begin{cases} \dfrac{1}{t(T-t)} \displaystyle\sum_{u=1}^{t} \sum_{v=t+1}^{T} (Y_{i,u} - Y_{i,v})^2, & t < T; \\ \dfrac{2}{(T-1)^2} \displaystyle\sum_{v=2}^{T} \sum_{u=1}^{v-1} (Y_{i,u} - Y_{i,v})^2, & t = T. \end{cases}$$

Bradley [3], Theorem 5.2(a) provides $\alpha(X_\circ, i) \leqslant \alpha(\boldsymbol{\varepsilon}_\circ, i)$ and, thus by Assumption A1,

$$0 \leqslant \sum_{i=1}^{\infty} \{\alpha(X_\circ, i)\}^{\chi/(2+\chi)} \leqslant \sum_{i=1}^{\infty} \{\alpha(\boldsymbol{\varepsilon}_\circ, i)\}^{\chi/(2+\chi)} < \infty. \tag{2.5}$$

If $t < T$, then

$$\begin{aligned} \sup_{i\in\mathbb{N}} \mathsf{E}\,|X_i|^{2+\chi} &\leqslant \frac{1}{t(T-t)} \sum_{u=1}^{t} \sum_{v=t+1}^{T} \sup_{i\in\mathbb{N}} \mathsf{E}\,|(Y_{i,u} - Y_{i,v})^2|^{2+\chi} \\ &= \frac{1}{t(T-t)} \sum_{u=1}^{t} \sum_{v=t+1}^{T} \sup_{i\in\mathbb{N}} \mathsf{E}\,|(\varepsilon_{i,u} - \varepsilon_{i,v})^2 \\ &\quad - 2(\varepsilon_{i,u} - \varepsilon_{i,v})\delta_i \mathbb{1}\{v > \tau \geqslant u\} + \delta_i^2 \mathbb{1}\{v > \tau \geqslant u\}|^{2+\chi} \\ &\leqslant \frac{3^{1+\chi}}{t(T-t)} \sum_{u=1}^{t} \sum_{v=t+1}^{T} \Big\{ \sup_{i\in\mathbb{N}} \mathsf{E}\,|(\varepsilon_{i,u} - \varepsilon_{i,v})^2|^{2+\chi} \\ &\quad + 2^{2+\chi} \sup_{i\in\mathbb{N}} |\delta_i|^{2+\chi} \sup_{i\in\mathbb{N}} \mathsf{E}\,|\varepsilon_{i,u} - \varepsilon_{i,v}|^{2+\chi} + \sup_{i\in\mathbb{N}} |\delta_i|^{4+2\chi} \Big\} < \infty \end{aligned} \tag{2.6}$$

by Assumptions A1, A2, and by Pešta [11], Lemma A.3. If $t = T$, then

$$\begin{aligned} \sup_{i\in\mathbb{N}} \mathsf{E}\,|X_i|^{2+\chi} &\leqslant \Big(\frac{T}{T-1}\Big)^{2+\chi} \frac{2}{T(T-1)} \sum_{v=2}^{T} \sum_{u=1}^{v-1} \mathsf{E}\,|(Y_{i,u} - Y_{i,v})^2|^{2+\chi} \\ &= \frac{2T^{1+\chi}}{(T-1)^{3+\chi}} \sum_{v=2}^{T} \sum_{u=1}^{v-1} \sup_{i\in\mathbb{N}} \mathsf{E}\,|(\varepsilon_{i,u} - \varepsilon_{i,v})^2 \\ &\quad - 2(\varepsilon_{i,u} - \varepsilon_{i,v})\delta_i \mathbb{1}\{v > \tau \geqslant u\} + \delta_i^2 \mathbb{1}\{v > \tau \geqslant u\}|^{2+\chi} \\ &\leqslant \frac{2(3T)^{1+\chi}}{(T-1)^{3+\chi}} \sum_{v=2}^{T} \sum_{u=1}^{v-1} \Big\{ \sup_{i\in\mathbb{N}} \mathsf{E}\,|(\varepsilon_{i,u} - \varepsilon_{i,v})^2|^{2+\chi} \\ &\quad + 2^{2+\chi} \sup_{i\in\mathbb{N}} |\delta_i|^{2+\chi} \sup_{i\in\mathbb{N}} \mathsf{E}\,|\varepsilon_{i,u} - \varepsilon_{i,v}|^{2+\chi} + \sup_{i\in\mathbb{N}} |\delta_i|^{4+2\chi} \Big\} < \infty \end{aligned} \tag{2.7}$$

again by Assumptions A1, A2, and by Pešta [11], Lemma A.3.

Relations (2.5), (2.6) and (2.7) together with Lin and Lu [7], Lemma 1.2.4 imply

$$
\begin{aligned}
(2.8)\qquad |\mathrm{Cov}(X_k, X_{k+i})| &\leqslant 10\|X_k\|_{2+\chi}\|X_{k+i}\|_{2+\chi}\{\alpha(X_\circ, i)\}^{\chi/(2+\chi)} \\
&\leqslant 10\Big(\sup_{i\in\mathbb{N}} \mathsf{E}\,|X_i|^{2+\chi}\Big)^{2/(2+\chi)}\{\alpha(\boldsymbol{\varepsilon}_\circ, i)\}^{\chi/(2+\chi)}
\end{aligned}
$$

for all $i, k \in \mathbb{N}$. The Jensen's inequality yields $\mathsf{E}\,X_i^2 \leqslant [\mathsf{E}\,|X_i|^{2+\chi}]^{2/(2+\chi)}$ for all $i \in \mathbb{N}$. According to (2.5), (2.6), (2.7) and (2.8), we get

$$
\begin{aligned}
(2.9)\ \mathrm{Var}\sum_{i=1}^N X_i &= \sum_{i=1}^N \mathrm{Var}\, X_i + \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \mathrm{Cov}(X_i, X_j) \\
&\leqslant N\sup_{i\in\mathbb{N}} \mathsf{E}\,X_i^2 + 10\Big(\sup_{i\in\mathbb{N}} \mathsf{E}\,|X_i|^{2+\chi}\Big)^{2/(2+\chi)} \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \{\alpha(X_\circ, |i-j|)\}^{\chi/(2+\chi)} \\
&\leqslant N\Big(\sup_{i\in\mathbb{N}} \mathsf{E}\,|X_i|^{2+\chi}\Big)^{2/(2+\chi)} + 20N\Big(\sup_{i\in\mathbb{N}} \mathsf{E}\,|X_i|^{2+\chi}\Big)^{2/(2+\chi)} \sum_{i=1}^\infty \{\alpha(\boldsymbol{\varepsilon}_\circ, i)\}^{\chi/(2+\chi)} \\
&\leqslant N\kappa(\chi, t, \tau, T),
\end{aligned}
$$

where $\kappa(\chi, t, \tau, T)$ is a constant independent of N.

The Chebyshev inequality provides $\mathcal{U}_N(t) - \mathsf{E}\,\mathcal{U}_N(t) = \mathcal{O}_{\mathsf{P}}\big(\sqrt{\mathrm{Var}\,\mathcal{U}_N(t)}\big)$ as $N \to \infty$. Since the index set $\{1, \ldots, T\}$ is finite and τ is finite as well, then (2.9) implies

$$
\max_{1\leqslant t\leqslant T} \mathrm{Var}\,\mathcal{U}_N(t) \leqslant NK(\chi, T),
$$

where $K(\chi, T)$ is a constant not depending on N. Thus, we also have uniform stochastic boundedness, i.e.,

$$
\max_{1\leqslant t\leqslant T} |\mathcal{U}_N(t) - \mathsf{E}\,\mathcal{U}_N(t)| = \mathcal{O}_{\mathsf{P}}(\sqrt{N}), \quad N \to \infty.
$$

With respect to (2.3) and (2.4), one has

$$
\begin{aligned}
\mathcal{U}_N(\tau) - \mathcal{U}_N(t) &= \mathcal{U}_N(\tau) - \mathsf{E}\,\mathcal{U}_N(\tau) - [\mathcal{U}_N(t) - \mathsf{E}\,\mathcal{U}_N(t)] + \mathsf{E}\,\mathcal{U}_N(\tau) - \mathsf{E}\,\mathcal{U}_N(t) \\
&\geqslant -2\max_{1\leqslant r\leqslant T} |\mathcal{U}_N(r) - \mathsf{E}\,\mathcal{U}_N(r)| + \mathsf{E}\,\mathcal{U}_N(\tau) - \mathsf{E}\,\mathcal{U}_N(t) \\
&= -2\max_{1\leqslant r\leqslant T} |\mathcal{U}_N(r) - \mathsf{E}\,\mathcal{U}_N(r)| \\
&\quad + \frac{2[\mathbb{1}\{\tau = T, t < T\} - \mathbb{1}\{\tau < T, t = T\}]}{T-1} \sum_{i=1}^N (\sigma_i^2 - c_i) \\
&\quad + \Big[\mathbb{1}\{\tau < T\} - \Delta_{t,\tau}^T \mathbb{1}\{t < T\} - \frac{2\tau(T-\tau)}{(T-1)^2}\mathbb{1}\{t = T\}\Big] \sum_{i=1}^N \delta_i^2
\end{aligned}
$$

for each $t \in \{1,\ldots,T\}$. Particularly, previous inequality holds for $\widehat{\tau}_N$. Note that $\widehat{\tau}_N = \arg\max_t \mathcal{U}_N(t)$. Hence, $\mathcal{U}_N(\tau) - \mathcal{U}_N(\widehat{\tau}_N) \leqslant 0$. Therefore,

$$
\begin{aligned}
&\frac{2}{\sqrt{N}} \max_{1\leqslant r\leqslant T} |\mathcal{U}_N(r) - \mathsf{E}\mathcal{U}_N(r)| \\
&\quad\geqslant \frac{2[\mathbb{1}\{\tau = T, \widehat{\tau}_N < T\} - \mathbb{1}\{\tau < T, \widehat{\tau}_N = T\}]}{T-1} \frac{1}{\sqrt{N}} \sum_{i=1}^N (\sigma_i^2 - c_i) \\
&\qquad + \left[\mathbb{1}\{\tau < T\} - \Delta^T_{\widehat{\tau}_N,\tau} \mathbb{1}\{\widehat{\tau}_N < T\} - \frac{2\tau(T-\tau)}{(T-1)^2}\mathbb{1}\{\widehat{\tau}_N = T\}\right] \frac{1}{\sqrt{N}} \sum_{i=1}^N \delta_i^2.
\end{aligned}
$$

Note that $0 \leqslant \Delta^T_{t,\tau} \leqslant 1$ for all $t,\tau \in \{1,\ldots,T-1\}$ and $\Delta^T_{t,\tau} = 1$ if and only if $t = \tau$. Moreover, $2\tau(T-\tau)/(T-1)^2 < 1$ for all $\tau \in \{1,\ldots,T\}$. Firstly, if $\tau < T$, then

$$
\begin{aligned}
(2.10)\quad &\frac{2}{\sqrt{N}} \max_{1\leqslant r\leqslant T} |\mathcal{U}_N(r) - \mathsf{E}\mathcal{U}_N(r)| \\
&\quad\geqslant -\frac{2}{T-1}\mathbb{1}\{\widehat{\tau}_N = T\} \frac{1}{\sqrt{N}} \sum_{i=1}^N (\sigma_i^2 - c_i) \\
&\qquad + \left[1 - \Delta^T_{\widehat{\tau}_N,\tau} \mathbb{1}\{\widehat{\tau}_N < T\} - \frac{2\tau(T-\tau)}{(T-1)^2}\mathbb{1}\{\widehat{\tau}_N = T\}\right] \frac{1}{\sqrt{N}} \sum_{i=1}^N \delta_i^2 \\
&\quad= \mathbb{1}\{\tau \geqslant \widehat{\tau}_N\}\left[(1 - \Delta^T_{\widehat{\tau}_N,\tau}\mathbb{1}\{\widehat{\tau}_N < T\}) \frac{1}{\sqrt{N}} \sum_{i=1}^N \delta_i^2\right] \\
&\qquad + \mathbb{1}\{\tau < \widehat{\tau}_N\}\left[\mathbb{1}\{\widehat{\tau}_N < T\}(1 - \Delta^T_{\widehat{\tau}_N,\tau}) \frac{1}{\sqrt{N}} \sum_{i=1}^N \delta_i^2\right. \\
&\qquad \left. + \mathbb{1}\{\widehat{\tau}_N = T\}\left(\left\{1 - \frac{2\tau(T-\tau)}{(T-1)^2}\right\} \frac{1}{\sqrt{N}} \sum_{i=1}^N \delta_i^2 - \frac{2}{T-1}\frac{1}{\sqrt{N}} \sum_{i=1}^N (\sigma_i^2 - c_i)\right)\right].
\end{aligned}
$$

The left-hand side of (2.10) is $\mathcal{O}_{\mathsf{P}}(1)$ as $N \to \infty$. Thus, the right-hand side of (2.10) and Assumption A3 give $\mathsf{P}[\widehat{\tau}_N = \tau] \to 1$ as $N \to \infty$. Secondly, if $\tau = T$, then

$$
(2.11)\quad \frac{2}{\sqrt{N}} \max_{1\leqslant r\leqslant T} |\mathcal{U}_N(r) - \mathsf{E}\mathcal{U}_N(r)| \geqslant \mathbb{1}\{\widehat{\tau}_N < T\} \frac{1}{\sqrt{N}} \left(\frac{2}{T-1} \sum_{i=1}^N (\sigma_i^2 - c_i) - \sum_{i=1}^N \delta_i^2\right).
$$

Since expression in (2.11) is $\mathcal{O}_{\mathsf{P}}(1)$ as $N \to \infty$, we have $\mathsf{P}[\widehat{\tau}_N = \tau] \to 1$ as $N \to \infty$ due to Assumption A3. $\square$

3. Simulation experiment

For a common abrupt change in panel means, a simulation study is performed in order to study the *finite sample* properties of the proposed consistent changepoint estimator. Particularly, the estimator's *empirical distributions*—visualized through histograms—are of interest. A similar simulation scenario setup to Peštová and Pešta [13] is chosen. Random samples of panel data (2000 each time) are generated from the changepoint model (1.1). In order to demonstrate the performance of the estimator in the case of small panel size, the panel length is set to $T = 10$. The number of panels is $N = 2, 5, 10, 20, 50$.

The random row error vectors are simulated in a non-stationary way as $\boldsymbol{\varepsilon}_i = [\varepsilon_{i,1}, \ldots, \varepsilon_{i,T}] = 0.3\boldsymbol{\varepsilon}_{i-1}\mathbb{1}\{i \leqslant N/2\} + 0.7\boldsymbol{\varepsilon}_{i-1}\mathbb{1}\{i > N/2\} + \boldsymbol{\varepsilon}_i$ with a burn-in period of 50 row vectors (thrown away). Here, the row innovations $\boldsymbol{\varepsilon}_i = [\varepsilon_{i,1}, \ldots, \varepsilon_{i,T}]$ are generated as three time series: iid, AR(1), and GARCH(1,1) sequences. The coefficient of the considered AR(1) process is $\varphi = 0.3$. The GARCH(1,1) process has coefficients $\alpha_0 = 1$, $\alpha_1 = 0.1$ and $\beta_1 = 0.2$. In all three sequences, the innovations of $\varepsilon_{i,t}$'s (i.e., innovations of elements of the row innovations) are obtained as iid random variables from a standard normal $N(0, 1)$ or Student t_5 distribution multiplied by a suitable constant in order that the errors possess common variance for all panels, i.e., $\sigma_i = \sigma$ for all i. For the AR(1) and GARCH(1,1), we throw away again a burn-in period having length 50. All possible combinations of the above mentioned settings are produced as Monte Carlo simulation scenarios. A selection of the results is provided below.

At first, the impact of the dependence structure and the errors' distribution on the changepoint estimator is examined. Figure 1 contains six different structures of model disturbances, where $N = 20$, $\tau = 8$ (depicted by the dotted vertical line), $\sigma = 0.2$, and 100% of the panels are subject to change of the size $\delta_i \sim U[0, 2]$ (i.e., the breaks are uniformly and independently distributed on $[0, 2]$).

One may conclude that the precision of our changepoint estimator is satisfactory even for relatively small number of panels regardless of the errors' structure. Innovations with lighter tails yield more precise estimators than innovations with heavier tails (i.e., Subfigures 1a, 1c, 1e vs. Subfigures 1b, 1d, 1f). It can be noticed that the GARCH(1,1) errors' model gives the worst estimator's precision from three dependence structures.

The proposed estimator works reasonably for various locations of the unknown changepoint as demonstrated in Figure 2. Being particular, various values of the changepoint (again depicted by the dotted vertical line) are chosen ($\tau = 1, 2, 5, 8, 9, 10$). Now, $N = 20$, $\sigma = 0.2$, 75% of the panels have a change of the size $\delta_i \sim U[0, 2]$, and the panel disturbances' innovations are AR(1) with $N(0, 1)$

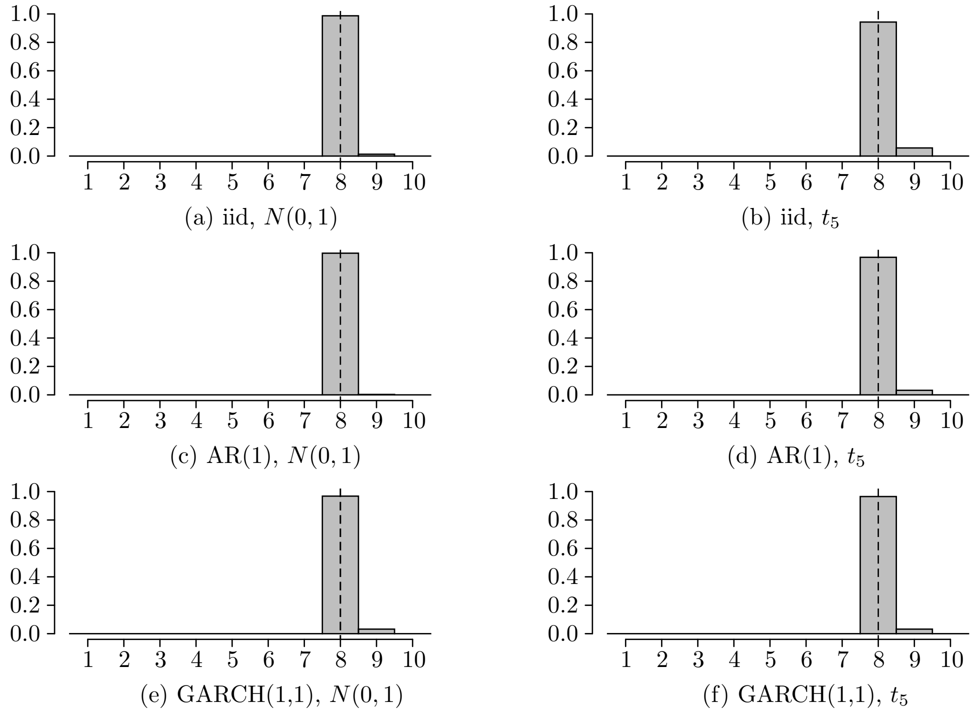

Figure 1. Histograms of $\widehat{\tau}_N$—various panel disturbances' distributions and structures ($\tau = 8$, $T = 10$, $N = 20$, $\sigma = 0.2$, 100% of the panels with the change size $\delta_i \sim U[0,2]$).

innovations. Let us recall that $\tau = 10$ corresponds to the 'no change' situation. And indeed, the empirical distribution of the estimator coherently concentrates at the last time point T.

The impact of the number of panels ($N = 2, 5, 10, 20$) is investigated in Figure 3 We set the panel disturbances' innovations as AR(1) with t_5 innovations, $\tau = 9$, $\sigma = 0.2$, and 50% of the panels have a change $\delta_i \sim U[0,2]$. The precision of $\widehat{\tau}_N$ improves markedly as N increases. If a higher number of panels (i.e., $N = 50$) is considered, then 100% precision is achieved. Longer panels were generated as well (e.g. $T = 25$; not presented here). Then as expected, the estimator's precision increases as the panel size gets bigger.

The effect of panel variability on the estimator's performance is shown in Figure 4. Various values of the variance parameter are taken into account ($\sigma = 0.1, 0.2, 0.5, 1.0$). Here, $\tau = 1$, $N = 10$, all of the panels have a change $\delta_i \sim U[0,2]$, and the panel disturbances' innovations come from GARCH(1,1) with $N(0,1)$ innovations. It is noticeable that the more volatile observations, the less precise change-point estimator. If the panel's variability (under the assumed dependency) is too

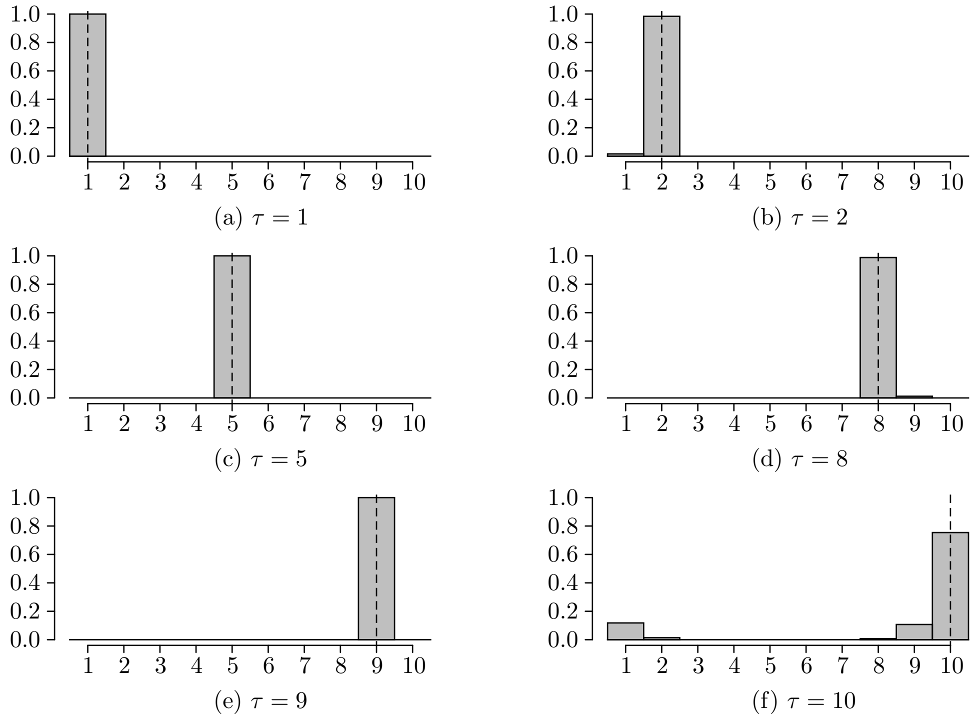

Figure 2. Histograms of $\widehat{\tau}_N$—various values of the changepoint τ ($T = 10$, $N = 20$, $\sigma = 0.2$, 75% of the panels with the change size $\delta_i \sim U[0, 2]$, AR(1) errors with $N(0, 1)$ innovations).

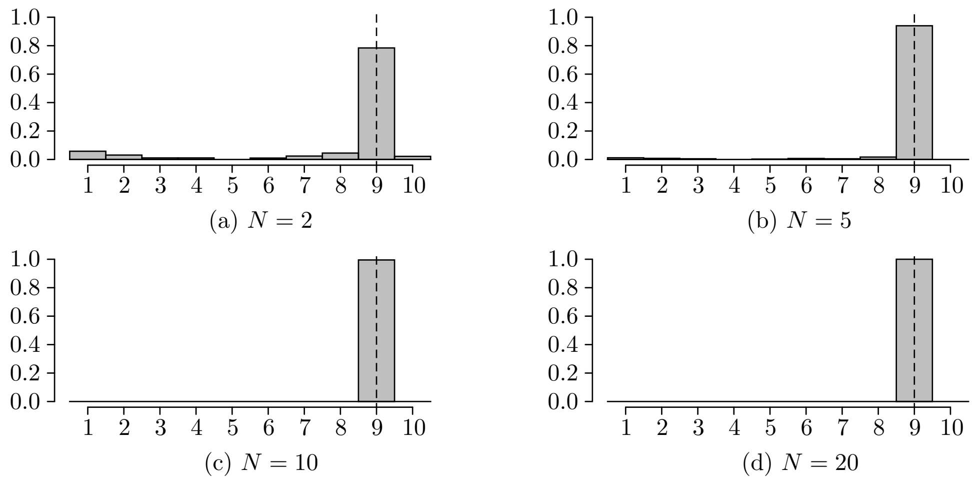

Figure 3. Histograms of $\widehat{\tau}_N$—various values of N ($\tau = 9$, $T = 10$, $\sigma = 0.2$, 50% of the panels with the change size $\delta_i \sim U[0, 2]$, AR(1) errors with t_5 innovations).

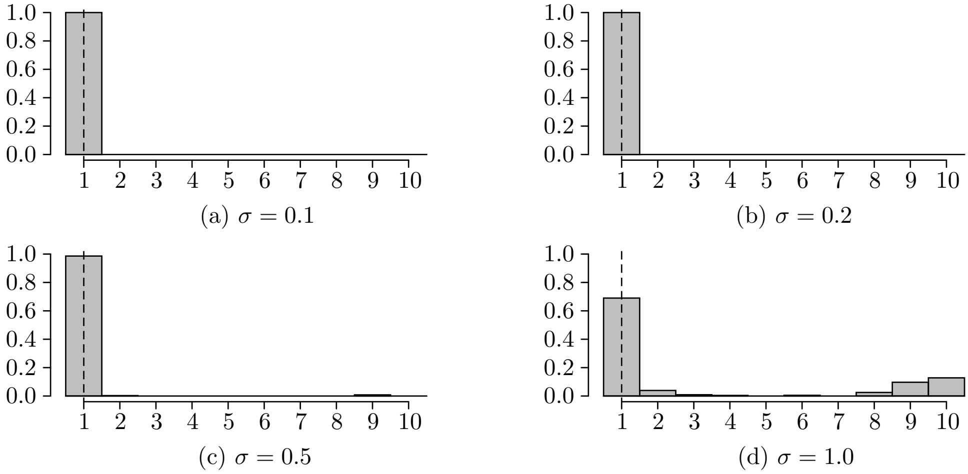

Figure 4. Histograms of $\widehat{\tau}_N$—various values of σ ($\tau = 1$, $T = 10$, $N = 10$, 100% of the panels with the change size $\delta_i \sim U[0, 2]$, GARCH(1,1) errors with $N(0, 1)$ innovations).

high compared to the size of the change, then it would be rather difficult to estimate a possible change.

At last, we elaborate how different portions of the panels with a change in mean influence the estimator's precision in Figure 5. Four cases were considered: 25%, 50%, 75%, and 100% of the panels have a change $\delta_i \sim U[0, 2]$. The panel disturbances' innovations are GARCH(1,1) with t_5 innovations, $\tau = 5$, and $N = 20$. If a relatively high number of panels contain a change, then the changepoint estimator performs reasonably in terms of precision.

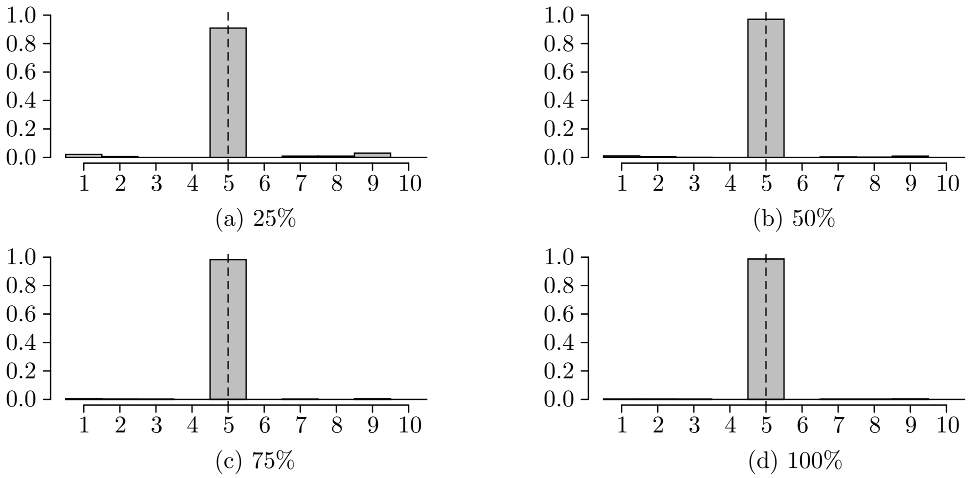

Figure 5. Histograms of $\widehat{\tau}_N$—various portion of the panels with the change size $\delta_i \sim U[0, 2]$ ($\tau = 5$, $T = 10$, $N = 20$, GARCH(1,1) errors with t_5 innovations).

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Authors' addresses: *Michal Pešta* (corresponding author), Charles University, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `Michal.Pesta@mff.cuni.cz`; *Barbora Peštová*, The Czech Academy of Sciences, Institute of Computer Science, Department of Medical Informatics and Biostatistics, Pod Vodárenskou věží 2, 182 07 Praha 8, Czech Republic, e-mail: `Pestova@cs.cas.cz`; *Matúš Maciak*, Charles University, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `Matus.Maciak@mff.cuni.cz`.