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Tran Loc Hung; Phan Tri Kien; Nguyen Tan Nhut

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ON ASYMPTOTIC BEHAVIORS AND CONVERGENCE RATES RELATED TO WEAK LIMITING DISTRIBUTIONS OF GEOMETRIC RANDOM SUMS

Tran Loc Hung, Phan Tri Kien and Nguyen Tan Nhut

Geometric random sums arise in various applied problems like physics, biology, economics, risk processes, stochastic finance, queuing theory, reliability models, regenerative models, etc. Their asymptotic behaviors with convergence rates become a big subject of interest. The main purpose of this paper is to study the asymptotic behaviors of normalized geometric random sums of independent and identically distributed random variables via Gnedenko's Transfer Theorem. Moreover, using the Zolotarev probability metric, the rates of convergence in some weak limit theorems for geometric random sums are estimated.

Keywords: geometric random sums, Gnedenko's transfer theorem, Zolotarev probability metric

Classification: 60E07, 60F05, 60G50, 60F99

1. INTRODUCTION

Let $\{X_j, j \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables. Let ν_p be a geometric random variable with parameter $p \in (0,1)$, having probability mass function is given by

$$\mathbb{P}(\nu_p = k) = p(1-p)^{k-1}, \quad k \geq 1,\ p \in (0,1).$$

Assume that for each $p \in (0,1)$, the geometric random variable ν_p is independent of all $X_j, j \geq 1$. Then $S_{\nu_p} := \sum_{j=1}^{\nu_p} X_j$ is said to be a Geometric Random Sum (see for instance [17] and [16]). Many situations should be modelled as a geometric random sum. Applications include risk processes, ruin probability, queueing theory and reliability models. The following list of references contains useful survey: Asmusen (2003, 2010), Bon (2002), Brown (1990), Feller (1966), Gnedenko and Korolev (1996), Kalashnikov (1997), Klebabov (1984, 2003), Kruglov and Korolev (1990), Sandhya and Pillai (2003), Kotz et al. (2001), Daly (2016), etc. (see [1, 2, 4, 5, 6, 7, 9, 12, 16, 17, 18, 23, 26, 22] and the references given there).

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It is worth pointing out that Klebanov et al. (1984) (see [17] for more details) have shown that a geometric random sum is unique solution of problem posed by Zolotarev as follows: describing all random variables Y that for any $p \in (0,1)$, there exists a random variable X_p such that

$$Y \stackrel{D}{=} X_p + \epsilon_p Y,$$

where random variables Y, X_p and ϵ_p are independent, and Bernoulli random variable ϵ_p has probability mass function

$$\mathbb{P}(\epsilon_p = 0) = p \quad \text{and} \quad \mathbb{P}(\epsilon_p = 1) = 1 - p.$$

Furthermore, the concepts of Geometrically Infinitely Divisible (GID) and Geometrically Strictly Stable (GSS) distributions have been also introduced by Klebanov et al. (1984). From that, problems concerning to geometric random sums have attracted much attention of mathematicians such as Korolev and Kruglov (1990), Gnedenko and Korolev (1996), Kalashnikov (1997), Kotz et al. (2001), Sandhya and Pillai (1999, 2003), Daly (2016), Hung (2013, 2018), Korolev and Zeifman (2016, 2017), Korolev and Dorofeeva (2017), etc. (see [7, 12, 14, 15, 16, 19, 20, 21, 22, 23, 25, 26], and references therein).

However, in various situations the weak limit distributions of normalized geometric random sums should be different. Beside that, the convergence rates in weak limit theorems for geometric random sums have not been to estimated fully. These things are motivations of studying in this paper.

The main aim of this article is to establish some weak limit theorems for several normalized geometric random sums of i.i.d. random variables, using the Gnedenko's Transfer Theorem (see [10] and [12]). The rates of convergence in weak limit theorems for geometric random sums will be also estimated via Zolotarev probability metric (see [3, 29, 30, 31], and [24]). The received results are related to the class of heavy-tailed distributions which has been well-known such as exponential, Laplace and Linnik distributions (see [18]).

In order to apply the well-known Gnedenko's Transfer Theorem and without loss of generality, throughout this paper, we consider the geometric random variable ν_{p_n} with parameter $p_n = \theta/n$, where $n \geq 1$ and for any $\theta \in (0,1)$. It is worth saying that, the Zolotarev probability metric used in our paper is an ideal metric (see for instance [31] and [24]), so it is easy to estimate the approximations concerning with random sums of i.i.d. random variables. Moreover, this metric could be compared with well-known metrics like Kolmogorov metric, total variation metric, Lévy-Prokhorov metric and the probability metric based on Trotter operator, etc. (see [3, 13, 29, 30, 31], and [14]).

The rest of this paper is structured as follows. The fundamental concepts, Gnedenko's Transfer Theorem and Zolotarev probability metric with its properties will be recalled in Section 2. The Section 3 will present the main results of our paper. Some concluding remarks will be showed in the end of this paper. From now on, we will denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers, by $\mathbb{R} = (-\infty, +\infty)$ the set of real numbers. The symbols $\stackrel{D}{=}$, $\stackrel{D}{\longrightarrow}$ and $\stackrel{P}{\longrightarrow}$ denoted the equality in distribution, convergence in distribution and convergence in probability, respectively.

2. PRELIMINARIES

Throughout the forthcoming, unless otherwise specified, we will denote by ν_{p_n} the *geometric distributed random variable*, shortly $\nu_{p_n} \sim \mathrm{Geo}(p_n)$, with parameters $p_n = \theta/n,\ \theta \in (0,1),\ n \geq 1$, and its probability mass function is given by

$$\mathbb{P}(\nu_{p_n} = k) = p_n(1-p_n)^{k-1}, \quad k = 1, 2, \dots .$$

It is easily seen that, for $\nu_{p_n} \sim \mathrm{Geo}(p_n)$, the probability generating function, characteristic function and expectation of geometric random variable ν_{p_n}, respectively are defined by

$$\psi_{\nu_{p_n}}(t) := \mathbb{E}(t^{\nu_{p_n}}) = \frac{p_n t}{1-(1-p_n)t}, \quad \text{for} \quad |\, t \,| < (1-p_n)^{-1},$$

$$\varphi_{\nu_{p_n}}(t) := \mathbb{E}(e^{i\nu_{p_n} t}) = \frac{p_n \mathrm{e}^{it}}{1-(1-p_n)\mathrm{e}^{it}}, \quad \text{for} \quad t \in \mathbb{R},$$

and

$$\mathbb{E}(\nu_{p_n}) = \frac{1}{p_n}.$$

In the notation of [22], a random variable $\mathcal{L}$ is said to be *Laplace distributed random variable* with parameters μ and $\sigma > 0$, denoted by $\mathcal{L} \sim \mathrm{Laplace}(\mu, \sigma)$, if its characteristic function given as

$$\varphi_{\mathcal{L}}(t) = \frac{\mathrm{e}^{\mu i t}}{1 + \frac{\sigma^2}{2} t^2}, \quad \text{for} \quad t \in \mathbb{R}.$$

It is obvious that, for $\mathcal{L} \sim \mathrm{Laplace}(0, \sigma)$,

$$\mathbb{E}(\mathcal{L}) = 0, \quad \mathbb{E}(\mathcal{L}^2) = \sigma^2 \quad \text{and} \quad \mathbb{E}(|\mathcal{L}|^3) = \frac{3\sigma^3}{\sqrt{2}}.$$

We follow the notation of [22], a random variable Λ is said to be a *symmetric Linnik distributed random variable* with parameters $\alpha \in (0,2]$ and $\sigma > 0$, denoted by $\Lambda \sim \mathrm{Linnik}(\alpha, \sigma)$, if its characteristic function is given by

$$\varphi_{\Lambda}(t) = \frac{1}{1 + \sigma^\alpha |t|^\alpha}, \quad t \in \mathbb{R}.$$

It is clear that, when $\alpha = 2$, the Linnik distribution reduces to Laplace distribution, so the Linnik distribution should be also known as $\alpha-$Laplace. Moreover, Linnik distribution is special case of geometric strictly stable (GSS) distributions, introduced by Klebanov et al. (see [17, 18] and [22] for more details).

We shall recall a version of Gnedenko's Transfer Theorem originated by Gnedenko and Fahim (1969) in [10] and [12]. It will be used to prove our main results in next section.

Theorem 2.1. (Gnedenko's Transfer Theorem) Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables and $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables, independent of X_j for $j \geq 1$. Assume that there exist sequences $\{a_n\}$ and $\{b_n\}$, such that $b_n > 0$ and $b_n \to \infty$ as $n \to \infty$, one has

1. $b_n^{-1}\sum_{j=1}^{n}(X_j - a_n) \xrightarrow{D} \mathcal{F}$ as $n \to \infty$;
2. $\dfrac{N_n}{n} \xrightarrow{D} \mathcal{A}$ as $n \to \infty$,

where $\mathcal{F}$ and $\mathcal{A}$ are random variables with characteristic function $\varphi_{\mathcal{F}}(t)$ and distribution function $A(x)$, respectively.
Then, as $n \to \infty$,

$$b_n^{-1}\sum_{j=1}^{N_n}(X_j - a_n) \xrightarrow{D} \mathcal{Y},$$

where $\mathcal{Y}$ is a random variable whose characteristic function is defined by

$$\varphi_{\mathcal{Y}}(t) = \int_0^{+\infty} [\varphi_{\mathcal{F}}(t)]^z \mathrm{d}A(z).$$

We shall denote by $C(\mathbb{R})$ the set of all real-valued, bounded, uniformly continuous functions defined on $\mathbb{R}$ with the norm $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$. Furthermore, for $r \in \mathbb{N}$, $\beta \in (0,1]$ and $s = r + \beta$, let us set

$$C^r(\mathbb{R}) = \Big\{ f \in C(\mathbb{R}):\ f^{(\kappa)} \in C(\mathbb{R}),\ 1 \le \kappa \le r \Big\},$$

and

$$\mathfrak{D}_s = \Big\{ f \in C^r(\mathbb{R}):\ \big|f^{(r)}(x) - f^{(r)}(y)\big| \le \big|x - y\big|^{\beta} \Big\},$$

where $f^{(\kappa)}$ is the derivative function of order κ of f.

We denote by $\mathfrak{X}$ the set of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The definition and properties of Zolotarev probability metric will be recalled as follows.

Definition 2.2. (Zolotarev [29] Let $X, Y \in \mathfrak{X}$. The Zolotarev probability metric on $\mathfrak{X}$ between two random variables X and Y, denoted by $d_s(X, Y)$, is defined by

$$d_s(X,Y) = \sup_{f\in\mathfrak{D}_s}\Big|\mathbb{E}\big[f(X) - f(Y)\big]\Big|.$$

Remark 2.3. (Bobkov [3], Manou-Abi [24], Zolotarev [29] and Zolotarev [31])

1. The Zolotarev probability metric $d_s(X, Y)$ on $\mathfrak{X}$ is an ideal metric of order s, i. e., for any $c \neq 0$, and for $X, Y, Z \in \mathfrak{X}$, we have

   $$d_s(X + Z, Y + Z) \le d_s(X, Y);$$

   and

   $$d_s(cX, cY) = |c|^s d_s(X, Y),$$

   where Z is independent of X and Y.

2. Let $d_s(X_n, X_0) \longrightarrow 0$ as $n \to \infty$. Then $X_n \xrightarrow{D} X_0$ as $n \to \infty$. (see [29], p. 424).

3. Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two sequences of i.i.d. random variables. Then,

$$d_s\Big(\sum_{j=1}^{n} X_j, \sum_{j=1}^{n} Y_j\Big) \leq n \times d_s\big(X_1, Y_1\big).$$

4. When $r = 1, \beta = 1$ and $s = r + \beta = 2$, we get

$$d_2(X,Y) = \sup_{f \in \mathfrak{D}_2} \Big|\mathbb{E}\big[f(X) - f(Y)\big]\Big|,$$

where

$$\mathfrak{D}_2 = \Big\{f \in C^1(\mathbb{R}) : \ \big|f'(x) - f'(y)\big| \leq \big|x - y\big|\Big\}.$$

5. When $r = 2, \beta = 1$ and $s = r + \beta = 3$, we get

$$d_3(X,Y) = \sup_{f \in \mathfrak{D}_3} \Big|\mathbb{E}\big[f(X) - f(Y)\big]\Big|,$$

where

$$\mathfrak{D}_3 = \Big\{f \in C^2(\mathbb{R}) : \ \big|f''(x) - f''(y)\big| \leq \big|x - y\big|\Big\}.$$

3. MAIN RESULTS

Before stating the main results we recall that the sum $S_{\nu_{p_n}} = \sum_{j=1}^{\nu_{p_n}} X_j$ is a *geometric random sum*, where $\{X_j, j \geq 1\}$ is a sequence of i.i.d. random variables, $\nu_{p_n} \sim \text{Geo}(p_n)$ with $p_n = \theta/n$ for any fixed $\theta \in (0,1)$ and $n \geq 1$. Assume that ν_{p_n} is independent of all X_j for $j \geq 1$. The following theorem will be considered as the version of Gnedenko's Transfer Theorem (see [10] and [11]) for geometric sums.

Theorem 3.1. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables. Let $\nu_{p_n} \sim \text{Geo}(p_n)$ and it is independent of all X_j for $j \geq 1$. Assume that $\{a_n\}$, $\{b_n\}$ be two sequences of real numbers such that $b_n > 0$, $b_n \to \infty$ as $n \to \infty$ and

$$b_n^{-1} \sum_{j=1}^{n} (X_j - a_n) \xrightarrow{D} \mathcal{F}, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{F}$ is a random variable with characteristic function $\varphi_{\mathcal{F}}(t)$. Then,

$$b_n^{-1} \sum_{j=1}^{\nu_{p_n}} (X_j - a_n) \xrightarrow{D} \mathcal{W}, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{W}$ is a random variable whose characteristic function defined by

$$\varphi_{\mathcal{W}}(t) = \frac{\theta}{\theta - \ln \varphi_{\mathcal{F}}(t)}, \quad t \in \mathbb{R},$$

here $\varphi_{\mathcal{F}}(t)$ is characteristic function of an infinitely divisible (ID) random variable.

P r o o f . By method of characteristic functions, it is easy to be concluded (or to refer from [11]) that

$$\frac{\nu_{p_n}}{n} \xrightarrow{D} \mathcal{E}_\theta, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{E}_\theta \sim \text{Exp}(\theta)$ is an exponential distributed random variable with parameter $\theta \in (0,1)$ and the distribution function of $\mathcal{E}_\theta$ is given in following form

$$F_{\mathcal{E}_\theta}(x) = 1 - \mathrm{e}^{-\theta x}, \quad x \in (0, \infty).$$

On account of the Theorem 2.1, we have

$$b_n^{-1} \sum_{j=1}^{\nu_{p_n}} \left(X_j - a_n \right) \xrightarrow{D} \mathcal{W}, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{W}$ is a random variable whose characteristic function defined by

$$\begin{aligned} \varphi_{\mathcal{W}}(t) &= \int_0^{+\infty} \left[\varphi_{\mathcal{F}}(t) \right]^z \theta \mathrm{e}^{-\theta z} \mathrm{d}z = \theta \int_0^{+\infty} \mathrm{e}^{-\left[\theta - \ln \varphi_{\mathcal{F}}(t) \right] z} \mathrm{d}z \\ &= \frac{\theta}{\theta - \ln \varphi_{\mathcal{F}}(t)}, \quad \text{for} \quad t \in \mathbb{R}. \end{aligned}$$

The proof is complete. □

Remark 3.2. According to Klebanov et al. (1984), the $\mathcal{W}$ is belonging to the class of geometric infinitely divisible (GID) random variables (see [17] and [23] for more details).

In the sequel, as direct results from Theorem 3.1, some corollaries on asymptotic behaviors of several normalized geometric random sums are provided. Moreover, these results can be obtained from papers ([8, 10, 11, 16] and [22]). Therefore, the following corollaries are given without proofs.

Corollary 3.3. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{D}(X_1) = 1$. Let $\nu_{p_n} \sim \text{Geo}(p_n)$, independent of all X_j for $j \geq 1$. Then,

$$p_n^{1/2} \sum_{j=1}^{\nu_{p_n}} X_j \xrightarrow{D} \mathcal{L} \sim \text{Laplace}(0,1), \quad \text{as} \quad n \to \infty.$$

Corollary 3.4. (A Rényi-type limit theorem) Let $\{X_j, j \geq 1\}$ be a sequence of non-negative and i.i.d. random variables with $\mathbb{E}(X_1) = 1$. Let $\nu_{p_n} \sim \text{Geo}(p_n)$ and independent of all X_j for $j \geq 1$. Then,

$$p_n \sum_{j=1}^{\nu_{p_n}} X_j \xrightarrow{D} \mathcal{E}_1 \quad \text{as} \quad n \to \infty,$$

where $\mathcal{E}_1 \sim \text{Exp}(1)$ is an exponential distributed random variable with parameter 1.

Corollary 3.5. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. symmetric random variables. Let $\nu_{p_n} \sim \text{Geo}(p_n)$ and be independent of all X_j for $j \geq 1$. Then

$$n^{-1/\alpha} \sum_{j=1}^{\nu_{p_n}} X_j \xrightarrow{D} \Lambda \sim \text{Linnik}(\alpha, \sigma), \quad \text{as} \quad n \to \infty,$$

where $\alpha \in (0, 2]$ and $\sigma > 0$.

P r o o f. For every $\epsilon > 0$, we have

$$\begin{aligned}\mathbb{P}(|X_j| \geq \epsilon.n^{1/\alpha}) &= 1 - \mathbb{P}(|X_j| < \epsilon.n^{1/\alpha}) \\ &= 1 - \mathbb{P}(-\epsilon.n^{1/\alpha} < X_j < \epsilon.n^{1/\alpha}) \to 0, \quad \text{as} \quad n \to \infty.\end{aligned}$$

According to Petrov's result (Theorem 3.7 in [27], p. 104), the limit distributions of sum $n^{-1/\alpha} \sum_{j=1}^{n} X_j$ belongs to the set of stable distributions. On the other hand, since X_j are symmetric random variables for $j \geq 1$, the sum $n^{-1/\alpha} \sum_{j=1}^{n} X_j$ converges to symmetric stable distributed random variable with characteristic function

$$\varphi_{\mathcal{S}}(t) = \exp\{-s^\alpha |t|^\alpha\}, \text{with } s > 0 \text{ and } \alpha \in (0, 2].$$

Then, on account of Theorem 3.1, the geometric sums $n^{-1/\alpha} \sum_{j=1}^{\nu_{p_n}} X_j$ converges in distribution to symmetric Linnik distributed random variable Λ with characteristic function given by

$$\frac{\theta}{\theta - \ln \varphi_{\mathcal{S}}(t)} = \frac{\theta}{\theta + s^\alpha |t|^\alpha} = \frac{1}{1 + \sigma^\alpha |t|^\alpha} = \varphi_\Lambda(t),$$

where $\sigma^\alpha = s^\alpha \theta^{-1}$, $\theta \in (0, 1)$.
The proof is immediate. □

We shall provide the results related to compound random sums in the sequel. Let $\{\eta_j, j \geq 1\}$ be a sequence of i.i.d. non-negative integer-valued random variables with $\mathbb{E}(\eta_j) = \delta \in (0, \infty)$, for all $j \geq 1$. Assume that η_j and X_j are independent for all $j \geq 1$. Let us define the extended random sums as follows

$$S_{W_n} = b_n^{-1} \sum_{j=1}^{W_n} \big(X_j - a_n\big),$$

and it is called the *compound random sums*, where $W_n = \eta_1 + \eta_2 + \ldots + \eta_n$. Then, we have the following theorem.

Theorem 3.6. Assume that

$$b_n^{-1} \sum_{j=1}^{n} (X_j - a_n) \xrightarrow{D} \mathcal{F}, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{F}$ is a limiting random variable having characteristic function $\varphi_{\mathcal{F}}(t)$. Then,

$$b_n^{-1}\sum_{j=1}^{W_n}(X_j - a_n) \xrightarrow{D} \mathcal{U}, \quad \text{as} \quad n \to \infty,$$

where the characteristic function of $\mathcal{U}$ defined by $\varphi_{\mathcal{U}}(t) = [\varphi_{\mathcal{F}}(t)]^{\delta}$.

P r o o f. According to the Weak Law of Large Numbers for sequence $\{\eta_j, j \geq 1\}$ of i.i.d. random variables with finite mean $\delta \in (0, \infty)$, we can assert that

$$\frac{W_n}{n} \xrightarrow{P} \delta, \quad \text{as} \quad n \to \infty.$$

It can be inferred that

$$\frac{W_n}{n} \xrightarrow{D} \mathcal{D}_{\delta}, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{D}_{\delta}$ is a random variable degenerated at point $\delta \in (0, \infty)$ with distribution function defined as follows

$$D_{\delta}(x) = \begin{cases} 1, & \text{if} \quad x \geq \delta; \\ 0, & \text{if} \quad x < \delta. \end{cases}$$

The partition should be chosen such that

$$0 = z_0 < z_1 < z_2 < \ldots < z_k = \delta < z_{k+1} < \ldots$$

Then, we can deduce that

$$D_{\delta}(z_i) - D_{\delta}(z_{i-1}) = \begin{cases} 1, & \text{if} \quad i = k; \\ 0, & \text{if} \quad i \neq k, \end{cases} \quad \text{for} \quad i = 1, 2, \ldots$$

According to Theorem 2.1 and using the definition of Stieltjes integral (see [28], p. 120), as $n \to \infty$, the characteristic function of limit distribution of compound random sums $b_n^{-1}\sum_{j=1}^{W_n}(X_j - a_n)$ is defined by

$$\int_0^{\infty} [\varphi_{\mathcal{F}}(t)]^z \mathrm{d}D(z) = \sum_{i=1}^{\infty} [\varphi_{\mathcal{F}}(t)]^{z_i} [D_{\delta}(z_i) - D_{\delta}(z_{i-1})] = [\varphi_{\mathcal{F}}(t)]^{\delta}.$$

The proof is straight-forward. □

Corollary 3.7. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{D}(X_1) = 1$. Let $\{\eta_j, j \geq 1\}$ be a sequence of i.i.d. non-negative integer-valued random variables with $\mathbb{E}(\eta_1) = \delta \in (0, \infty)$ and $W_n = \eta_1 + \eta_2 + \ldots + \eta_n$. Assume that the random variables from two sequences $\{\eta_j, j \geq 1\}$ and $\{X_j, j \geq 1\}$ are independent. Then,

$$n^{-1/2}\sum_{j=1}^{W_n} X_j \xrightarrow{D} \mathcal{N}, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{N}$ is a normal distributed random variable with characteristic function

$$\varphi_{\mathcal{N}}(t) = \exp\Big\{ -\frac{\delta}{2}t^2 \Big\}.$$

Remark 3.8. Assume that $\nu_1,\ \nu_2,\ldots$ be a sequence of independent and identically geometric distributed random variables with parameter $p \in (0,1)$. Then, for each $n \in \mathbb{N}$, the sum

$$\nu_1 + \nu_2 + \ldots + \nu_n$$

has a negative-binomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0,1)$. Clearly, the characteristic function of $\sum_{i=1}^{n} \nu_i$, defined by

$$\varphi_{\sum\limits_{i=1}^{n} \nu_i}(t) = \mathbb{E}\big[\mathrm{e}^{it(\nu_1+\nu_2+\ldots+\nu_n)}\big] = \Big\{\mathbb{E}\big[\mathrm{e}^{it\nu_1}\big]\Big\}^n = \left\{\frac{p\mathrm{e}^{it}}{1-(1-p)\mathrm{e}^{it}}\right\}^n,$$

is the characteristic function of negative-binomial distribution with parameters n and p. Hence, the negative-binomial sums should be defined as an extension of geometric random sums. Moreover, according to Theorem 3.6, the limiting distributions of the negative-binomial sums of i.i.d. random variables would be established (see for instance [15]).

Using the Zolotarev probability metric, we will discuss on convergence rates in weak limit theorems for normalized geometric random sums which have been presented in corollaries 3.3, 3.4 and 3.5. Firstly, we wish to provide the general theorem as follows.

Theorem 3.9. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(|X_1|^s) < +\infty$ and $\{Y_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(|Y_1|^s) < +\infty$. Let $\nu_{p_n} \sim \mathrm{Geo}(p_n)$ be a geometric random variable with parameter $p_n \in (0,1)$, assumed independent of X_j and Y_j for all $j \geq 1$. Moreover, suppose that for $r \in \mathbb{N}$, the condition

$$\mathbb{E}(X_1^i) = \mathbb{E}(Y_1^i), \tag{1}$$

holds for $i = 1, 2, \ldots, r$. Then,

$$d_s\Bigg(c(p_n)\sum_{j=1}^{\nu_{p_n}} X_j, c(p_n)\sum_{j=1}^{\nu_{p_n}} Y_j\Bigg) \leq \frac{[c(p_n)]^s}{p_n}.\frac{1}{r!}\big[\mathbb{E}(|X_1|^s) + \mathbb{E}(|Y_1|^s)\big]$$

where $\beta \in (0,1],\ s = r + \beta,\ c(p_n) > 0,\ c(p_n) \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} \dfrac{[c(p_n)]^s}{p_n} = 0$.

Proof. According to Remark 2.3, we have

$$\begin{aligned}
&d_s\Bigg(c(p_n)\sum_{j=1}^{\nu_{p_n}} X_j, c(p_n)\sum_{j=1}^{\nu_{p_n}} Y_j\Bigg) = [c(p_n)]^s d_s\Bigg(\sum_{j=1}^{\nu_{p_n}} X_j, \sum_{j=1}^{\nu_{p_n}} Y_j\Bigg) \\
&= [c(p_n)]^s \sum_{k=1}^{\infty}\Bigg\{\mathbb{P}(\nu_{p_n} = k) d_s\Bigg(\sum_{j=1}^{k} X_j, \sum_{j=1}^{k} Y_j\Bigg)\Bigg\} \\
&\leq [c(p_n)]^s \sum_{k=1}^{\infty}\Big\{\mathbb{P}(\nu_{p_n} = k)k.d_s(X_1, Y_1)\Big\} \\
&= [c(p_n)]^s \mathbb{E}(\nu_{p_n}).d_s(X_1, Y_1) = \frac{[c(p_n)]^s}{p_n}.d_s(X_1, Y_1).
\end{aligned}$$

By Taylor series expansion for function $f \in \mathfrak{D}_s$ and for all $x,\ y \in \mathbb{R}$, it follows that

$$f(x) = f(0) + \sum_{i=1}^{r} \frac{f^{(i)}(0)}{i!} x^i + \frac{x^r}{r!}\big[f^{(r)}(\eta_1 x) - f^{(r)}(0)\big];$$
$$f(y) = f(0) + \sum_{i=1}^{r} \frac{f^{(i)}(0)}{i!} y^i + \frac{y^r}{r!}\big[f^{(r)}(\eta_2 y) - f^{(r)}(0)\big],$$

where $0 < \eta_1 < 1$ and $0 < \eta_2 < 1$. Hence, for any $f \in \mathfrak{D}_s$, it may be concluded that

$$\begin{aligned} f(x) - f(y) &= \sum_{i=1}^{r} \frac{f^{(i)}(0)}{i!}(x^i - y^i) + \frac{x^r}{r!}\big[f^{(r)}(\eta_1 x) - f^{(r)}(0)\big] - \frac{y^r}{r!}\big[f^{(r)}(\eta_2 y) - f^{(r)}(0)\big] \\ &\le \sum_{i=1}^{r} \frac{f^{(i)}(0)}{i!}(x^i - y^i) + \frac{|x|^r}{r!}\big|f^{(r)}(\eta_1 x) - f^{(r)}(0)\big| + \frac{|y|^r}{r!}\big|f^{(r)}(\eta_2 y) - f^{(r)}(0)\big| \\ &\le \sum_{i=1}^{r} \frac{f^{(i)}(0)}{i!}(x^i - y^i) + \frac{1}{r!}\big(|x|^{r+\beta}\eta_1^{\beta} + |y|^{r+\beta}\eta_2^{\beta}\big) \\ &\le \sum_{i=1}^{r} \frac{f^{(i)}(0)}{i!}(x^i - y^i) + \frac{1}{r!}(|x|^s + |y|^s). \end{aligned}$$

On account of Zolotarev probability metric, using the condition (1), one has

$$d_s(X_1, Y_1) \le \frac{1}{r!}\big[\mathbb{E}(|X_1|^s) + \mathbb{E}(|Y_1|^s)\big].$$

Therefore,

$$d_s\left(c(p_n)\sum_{j=1}^{\nu_{p_n}} X_j, c(p_n)\sum_{j=1}^{\nu_{p_n}} Y_j\right) \le \frac{[c(p_n)]^s}{p_n}.\frac{1}{r!}\big[\mathbb{E}(|X_1|^s) + \mathbb{E}(|Y_1|^s)\big].$$

The proof is straightforward. □

Theorem 3.10. Let $\{X_j, j \ge 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = 1$ and $\mathbb{E}(|X_1|^3) = \rho < +\infty$. Let $\nu_{p_n} \sim \mathrm{Geo}(p_n)$ be a geometric random variables with parameter $p_n \in (0,1)$, assumed independent of all X_j for $j \ge 1$. Then,

$$d_3\left(p_n^{1/2}\sum_{j=1}^{\nu_{p_n}} X_j, \mathcal{L}\right) \le n^{-1/2}\theta^{1/2}\left(\frac{\rho}{2} + \frac{3}{2\sqrt{2}}\right),$$

where $\mathcal{L} \sim \mathrm{Laplace}(0,1)$.

Proof. Since $\mathcal{L} \sim \mathrm{Laplace}(0,1)$, we have

$$\mathbb{E}(\mathcal{L}) = 0, \quad \mathbb{E}(\mathcal{L}^2) = 1 \quad \text{and} \quad \mathbb{E}(|\mathcal{L}|^3) = \frac{3}{\sqrt{2}}.$$

Moreover, according to ([22], Proposition 2.2.7, p. 27) we obtain the following presentation

$$\mathcal{L} \overset{D}{=} p_n^{1/2} \sum_{j=1}^{\nu_{p_n}} \mathcal{L}_j, \tag{2}$$

where $\mathcal{L}_1,\ \mathcal{L}_2, \dots$ are independent copies of $\mathcal{L}$ and they are independent of ν_{p_n} for $p_n \in (0,1)$. Therefore, by Theorem 3.9 and Remark 2.3, one has

$$\begin{aligned} d_3\left(p_n^{1/2} \sum_{j=1}^{\nu_{p_n}} X_j, \mathcal{L}\right) &= d_3\left(p_n^{1/2} \sum_{j=1}^{\nu_{p_n}} X_j, p_n^{1/2} \sum_{j=1}^{\nu_{p_n}} \mathcal{L}_j\right) \\ &\le \frac{p_n^{3/2}}{p_n} \cdot \frac{1}{2!} \left[\mathbb{E}(|X_1|^3) + \mathbb{E}(|\mathcal{L}_1|^3)\right] \le p_n^{1/2} \Big(\frac{\rho}{2} + \frac{3}{2\sqrt{2}}\Big). \end{aligned}$$

Finally, with $p_n = \theta/n$, we obtain

$$d_3\left(p_n^{1/2} \sum_{j=1}^{\nu_{p_n}} X_j, \mathcal{L}\right) \le n^{-1/2} \theta^{1/2} \left(\frac{\rho}{2} + \frac{3}{2\sqrt{2}}\right).$$

This finishes the proof. □

Theorem 3.11. Let $\{X_j, j \ge 1\}$ be a sequence of non-negative i.i.d. random variables with $\mathbb{E}(X_1) = 1$ and $\mathbb{E}(X_1^2) = \varrho < +\infty$. Suppose that $\nu_{p_n} \sim \text{Geo}(p_n)$ and ν_{p_n} is independent of all X_j for $j \ge 1$. Then,

$$d_2\left(p_n \sum_{j=1}^{\nu_{p_n}} X_j, \mathcal{E}_1\right) \le n^{-1} \theta(\varrho + 2),$$

where $\mathcal{E}_1 \sim \text{Exp}(1)$.

P r o o f. Since $\mathcal{E}_1 \sim \text{Exp}(1)$, one has

$$\mathbb{E}(\mathcal{E}_1) = 1, \quad \mathbb{E}(\mathcal{E}_1)^2 = 2,$$

and by an argument analogous to ([14], Lemma 3.1) we get

$$\mathcal{E}_1 \overset{D}{=} p_n \sum_{j=1}^{\nu_{p_n}} \mathcal{E}_1(j), \tag{3}$$

where $\mathcal{E}_1(1),\ \mathcal{E}_1(2), \dots$ are i.i.d. random variables, copied from $\mathcal{E}_1$. According to Theorem 3.9 and with respect to Remark 2.3, we have

$$\begin{aligned} d_2\left(p_n \sum_{j=1}^{\nu_{p_n}} X_j, \mathcal{E}_1\right) &= d_2\left(p_n \sum_{j=1}^{\nu_{p_n}} X_j, p_n \sum_{j=1}^{\nu_{p_n}} \mathcal{E}_1(j)\right) \\ &\le p_n \left[\mathbb{E}(X_1^2) + \mathbb{E}(\mathcal{E}_1^2)\right] \le n^{-1} \theta(\varrho + 2). \end{aligned}$$

The proof is immediate. □

The following theorem deals with the rate of convergence in Corollary 3.5. It is worth pointing out that the received limit distribution in this corollary will be Linnik distributions which have not finite moments in general case (see [22], p. 212 for more details). Hence, here we can be able to estimate the rate convergence with $\alpha \in (1, 2)$. It is to be noticed that, when $\alpha = 2$, the Linnik distributions reduce to Laplace distribution, that has been established in Theorem 3.10.

Theorem 3.12. Let $\{X_j, j \geq 1\}$ be a sequence of symmetric i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}|X_1| = \tau < +\infty$. Let $\nu_{p_n} \sim \text{Geo}(p_n)$ be a geometric random variable with parameter $p_n \in (0, 1)$, assumed independent of all X_j for $j \geq 1$. Then, for $f \in C^1(\mathbb{R})$,

$$d_2\bigg(p_n^{1/\alpha}\sum_{j=1}^{\nu_{p_n}} X_j, \Lambda\bigg) \leq 2n^{\frac{\alpha-2}{\alpha}}\theta^{\frac{2-\alpha}{\alpha}} \sup_{f\in\mathfrak{D}_2} \|f'\| \bigg(\tau + \frac{2\sigma}{\alpha\sin\frac{\pi}{\alpha}}\bigg),$$

where $\|f'\| = \sup_{t\in\mathbb{R}} |f'(t)|, \Lambda \sim \text{Linnik}(\alpha, \sigma),\ \alpha \in (1, 2),\ \sigma > 0$ and $p_n = \theta/n$ with $\theta \in (0, 1)$.

P r o o f. On account of Proposition 4.3.2 ([22], p. 201), we have

$$\Lambda \stackrel{D}{=} p_n^{1/\alpha}\sum_{j=1}^{\nu_{p_n}} \Lambda_j, \tag{4}$$

where $\Lambda_1,\ \Lambda_2, \ldots$ are independent copies of Λ. According to Remark 2.3, one has

$$\begin{aligned}
d_2\bigg(p_n^{1/\alpha}\sum_{j=1}^{\nu_{p_n}} X_j, \Lambda\bigg) &= d_2\bigg(p_n^{1/\alpha}\sum_{j=1}^{\nu_{p_n}} X_j, p_n^{1/\alpha}\sum_{j=1}^{\nu_{p_n}} \Lambda_j\bigg) = p_n^{2/\alpha} d_2\bigg(\sum_{j=1}^{\nu_{p_n}} X_j, \sum_{j=1}^{\nu_{p_n}} \Lambda_j\bigg)\\
&= p_n^{2/\alpha}\sum_{k=1}^{\infty}\left\{\mathbb{P}(\nu_{p_n} = k) d_2\bigg(\sum_{j=1}^{k} X_j, \sum_{j=1}^{k} \Lambda_j\bigg)\right\} \leq p_n^{2/\alpha}\sum_{k=1}^{\infty}\Big\{\mathbb{P}(\nu_{p_n} = k) k.d_2(X_1, \Lambda_1)\Big\}\\
&= p_n^{\frac{2-\alpha}{\alpha}} d_2(X_1, \Lambda) = n^{\frac{\alpha-2}{\alpha}}\theta^{\frac{2-\alpha}{\alpha}} d_2(X_1, \Lambda).
\end{aligned}$$

Furthermore, since $\Lambda \sim \text{Linnik}(\alpha, \sigma)$ with $\alpha \in (1, 2)$ and $\sigma > 0$. According to Kotz et al. ([22], p. 212), we obtain

$$\mathbb{E}(\Lambda) = 0 \quad \text{and} \quad \mathbb{E}|\Lambda| = \frac{2\sigma}{\alpha\sin\frac{\pi}{\alpha}} < +\infty.$$

For any $f \in C^1(\mathbb{R})$, by the Mean Value Theorem (see [28] for more details), for z is between x and y, we have

$$f(x) - f(y) = (x - y)f'(z) = (x - y)f'(0) + (x - y)\big[f'(z) - f'(0)\big].$$

Since $f \in C^1(\mathbb{R})$ and for any $z \in \mathbb{R}$, one has

$$\big|f'(z) - f'(0)\big| \leq \big|f'(z)\big| + \big|f'(0)\big| \leq \sup_{z\in\mathbb{R}}\big|f'(z)\big| + \sup_{t\in\mathbb{R}}\big|f'(t)\big| = 2\|f'\|.$$

Then, we infer that, for $f \in C^1(\mathbb{R})$,

$$\begin{aligned} f(x) - f(y) &\le (x-y)f'(0) + |x-y|\big|f'(z) - f'(0)\big| \\ &\le (x-y)f'(0) + 2\|f'\||x-y| \le (x-y)f'(0) + 2\|f'\|(|x| + |y|). \end{aligned}$$

Using hypothesis that $\mathbb{E}(X_1) = 0$ and $\mathbb{E}|X_1| = \tau < +\infty$, then

$$\begin{aligned} \mathbb{E}\big[f(X_1) - f(\Lambda)\big] &\le \mathbb{E}\big[(X_1 - \Lambda)f'(0) + 2\|f'\|(|X_1| + |\Lambda|)\big] \\ &= 2\|f'\|\left(\tau + \frac{2\sigma}{\alpha \sin\frac{\pi}{\alpha}}\right). \end{aligned}$$

Therefore,

$$d_2(X_1, \Lambda) = \sup_{f \in \mathfrak{D}_2} \Big|\mathbb{E}\big[f(X_1) - f(\Lambda)\big]\Big| \le 2 \sup_{f \in \mathfrak{D}_2} \|f'\|\left(\tau + \frac{2\sigma}{\alpha \sin\frac{\pi}{\alpha}}\right).$$

This concludes the proof. □

CONCLUDING REMARKS

We conclude this paper with the following comments.

1. Theorems 3.1 and 3.6 are consequences of Gnedenko's Transfer Theorem.

2. Based on Gnedenko's Transfer Theorem, the weak limit theorems for negative-binomial random sums would be studied (see Remark 3.8).

3. Using Zolotarev probability metric and by an argument analogous to this paper, the convergence rates of distributions of negative-binomial random sums would be also estimated.

4. The Laplace distribution, exponential distribution and symmetric Linnik distribution are special cases of geometrically strictly stable distributions, introduced in Klebanov et al. (1984) (see [17]). Therefore, the presentations (2), (3) and (4) in Theorems 3.10, 3.11 and 3.12, respectively may be obtained from Klebanov et al. (1984).

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Tran Loc Hung, University of Finance and Marketing, 2/4 Tran Xuan Soan Street, District No. 7, Ho Chi Minh City. Vietnam.
e-mail: tlhung@ufm.edu.vn

Phan Tri Kien, University of Finance and Marketing, 2/4 Tran Xuan Soan Street, District No. 7, Ho Chi Minh City University of Science, 227 Nguyen Van Cu Street, District N. 5, Ho Chi Minh City. Vietnam.
e-mail: phankien@ufm.edu.vn

Nguyen Tan Nhut, DongThap Province. Vietnam.
e-mail: ntn.nhut@gmail.com