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EXACT SIMULTANEOUS LOCATION-SCALE TESTS FOR TWO SHIFTED EXPONENTIAL SAMPLES

Amitava Mukherjee, Zhi Lin Chong and Marco Marozzi

The failure time distribution for various items often follows a shifted (two-parameter) exponential model and not the traditional (one-parameter) exponential model. The shifted exponential is very useful in practice, in particular in the engineering, biomedical sciences and industrial quality control when modeling time to event or survival data. The open problem of simultaneous testing for differences in origin and scale parameters of two shifted exponential distributions is addressed. Two exact tests are proposed using maximum likelihood estimators. They are based on the combination of two statistics following a maximum-type and a distance-type approach. The exact null distributions of the respective test statistics are derived analytically. Small sample type-one error rate and power of the tests are studied numerically. We showed that the test based on the maximum type combination (the Max test) should be preferred being generally more powerful than the test based on the distance type combination (the Distance test). An application to a biomedical experiment is discussed.

Keywords: hypothesis testing, failure time model, simultaneous testing, shifted exponential, type-one error rate, power

Classification: 62F03, 62N05

1. INTRODUCTION

The one-parameter exponential distribution is widely used in practice. For example, in the engineering sciences it is used to model the lifespan of electronic components, transmission, engine and mechanical equipment. Nevertheless, many researchers, including Huang, Mukherjee and Yang [8], Kao [10], Krishnamoorthy and Xia [11], Roy and Mathew [17], and Wu [27], have found that in various practical applications, such as measuring the reliability of a product, monitoring the high-voltage of current in certain metal oxide semiconductor transistor on a flash memory wafer, modelling the consumer lifetime, among others, the two-parameter (shifted) exponential distribution should be preferred to the one-parameter exponential distribution. The probability density function (pdf) of the two-parameter exponential distribution is

$$f(x;\theta,\lambda) = \frac{1}{\lambda}\exp^{-(x-\theta)/\lambda}; \qquad x > \theta > 0, \quad \lambda > 0. \tag{1}$$

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Note that θ can be interpreted as the minimum life or guaranteed period in which no failure should occur. This distribution is also known as the shifted exponential distribution. The mean of X is $\theta + \lambda$ and the variance is λ. If $\theta = 0$, equation (1) reduces to the one-parameter exponential distribution. One may conceptualize a two-parameter exponential distribution for $\theta \in (-\infty, \infty)$, see, for example, Johnson and Kotz[9]. Here, we consider $\theta \geq 0$ as we are mainly interested in the time to event data.

Inference problems about a single two-parameter exponential distribution, or a single parameter for two-parameter exponential distributions have been widely studied. In the later half of 20th century, Varde [23] studied life testing and reliability estimation for the two-parameter exponential distribution using a Bayesian approach. Cohen and Helm [4] studied how to estimate the parameters of the two-parameter exponential distribution using a modified method of moments. Engelhardt and Bain [6] studied the reliability tolerance limits and confidence limits. Ebrahimi [5] addressed the problem of hybrid life testing for the two-parameter exponential distribution.

In early 21^{st} century, Raqab [16] considered a multiple type II censoring scheme and derived approximate maximum likelihood predictors of forthcoming failure times of shifted (two-parameter) exponential distributions. Pal, Masoom, and Woo [15] studied the estimation and testing problem related to the stress-strength model under two-parameter exponential distribution, that is, the probability that the stress having a two-parameter exponential distribution exceeds strength variable, which also follows a two-parameter exponential model. Later, Chaturvedi and Sharma [3] provided a simpler method for unbiased estimation of stress-strength reliability. Roy and Mathew [17] proposed a generalized confidence interval for the reliability function of the two-parameter exponential distribution. Wu, Lee, and Lei [26] studied the lifetime performance index of products using the two-parameter exponential distribution. Krishnamoorthy, Mukherjee, and Guo [12] also deliberated over the interval estimation and hypothesis testing of reliability in stress-strength model with stress and strength having two-parameter exponential distributions. Ahmadi and MirMostafaee [1] examined the prediction intervals for future order statistics and records data where the underlying distribution is two-parameter exponential. Singh and Abebe [20] compared multiple exponential populations with more than one control. Kao [10] showed that the amount of current needed to break an insulator of the P-type high-voltage metal oxide semiconductor (MOS) transistor (HPM) on a non-volatile drive memory wafer follows the two-parameter exponential distribution and this article nicely demonstrates the importance of the two-parameter exponential distribution in the engineering sciences. In the same year, Wu [27] proposed an interval estimation technique for the scale parameter of a two-parameter exponential distribution based on Type-II progressive censoring.

In the current decade, Schenk, Burkschat, Cramer and Kamps [19] studied the Bayesian prediction and estimation of consecutive order statistics for Type-II censored samples taken from one- and two-parameter exponential distributions. Ganguly, Mitra, Samanta and Kundu [7] derived the exact distribution of the hybrid Type-II censoring scheme, where the two-parameter exponential distribution is used to model lifetimes. Based on generalized pivot variable, Baklizi [2] introduced bootstrap and Bayesian intervals estimation of stress-strength reliability for the two-parameter exponential distribution. Li, Song, and Shi [13] proposed a parametric bootstrap method for constructing simultaneous confidence intervals for all pairwise differences of means from several two-parameter exponential distributions. Sangnawakij and Niwitpong [18] proposed confidence intervals for the single coefficient of variation and the difference of coefficients of variation in the two-parameter exponential distributions. Krishnamoorthy and Xia [11] considered some problems related to estimating the confidence interval of the survival probability for a two-parameter exponential distribution. However, none of the existing works addressed combined testing of the two parameters in a comprehensive way.

In recent years, the two-parameter exponential distribution is widely used in statistical process monitoring and control. Mukherjee, McCracken and Chakraborti [14] proposed Shewhart-type control charts for simultaneous monitoring of the origin (location) and scale parameters of a two-parameter exponential distribution in the known parameter situation. Huang, Mukherjee and Yang [8] extended the proposals of Mukherjee, McCracken and Chakraborti [14] using the concept of cumulative sum (CUSUM) to introduce Phase-II monitoring schemes of parameters of the two-parameter exponential distribution with known standards. Very recently, van Zyl and van der Merwe [22] pointed out that the extension of the results of Mukherjee, McCracken and Chakraborti [14] to the unknown parameter situation is extremely difficult. The same may be said about the results of Huang, Mukherjee and Yang [8].

In this paper, we address the problem of simultaneous testing for different origin and scale parameters of two two-parameter exponential distributions that has not been solved yet. More precisely, we propose two tests respectively based on a maximum type and distance type combination of two statistics. The remainder of the paper is organized as follows. In Section 2, we prove some preliminary results. In Section 3, we describe the two tests and prove some theoretical results. In Section 4 we study the type-one error rate and power of the tests. The tests are illustrated with an application example in Section 5. Conclusions are drawn in Section 6.

2. PRELIMINARY RESULTS

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m from a two-parameter exponential distribution with origin parameter θ_0 and scale parameter λ_0. Let $Y_1, Y_2, \ldots, Y_n$ be another random sample of size n from a two-parameter exponential distribution with the origin parameter θ_1 and scale parameter λ_1. We assume that the samples come from two mutually independent populations. We are interested in simultaneous testing for different origin and scale parameters of two two-parameter exponential distributions that corresponds to test

$$H_0 : \theta_0 = \theta_1 \cap \lambda_0 = \lambda_1, \text{ versus } H_1 : \theta_0 \neq \theta_1 \cup \lambda_0 \neq \lambda_1. \tag{2}$$

The maximum likelihood estimators of θ_0 and θ_1 are respectively $\hat{\theta}_0 = X_{(1)}$ and $\hat{\theta}_1 = Y_{(1)}$, see, for details, Johnson and Kotz [9]. $X_{(1)}$ and $Y_{(1)}$ denote the minimum or the first-order statistic of the first and second samples, respectively. The maximum likelihood estimators of λ_0 and λ_1 are respectively

$$\hat{\lambda}_0 = \frac{1}{m}\sum_{i=1}^{m}(X_i - \hat{\theta}_0) = \bar{X} - X_{(1)} \tag{3}$$

and

$$\hat{\lambda}_1 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\theta}_1) = \bar{Y} - Y_{(1)}, \tag{4}$$

where $\bar{X}$ and $\bar{Y}$ denote the mean of $\vec{X} = (X_1, X_2, \ldots, X_m)$ and $\vec{Y} = (Y_1, Y_2, \ldots, Y_n)$ respectively. $\hat{\theta}_0$ and $\hat{\lambda}_0$, as well as $\hat{\theta}_1$ and $\hat{\lambda}_1$ are independent, see Johnson and Kotz [9]. $\hat{\theta}_0$ follows a two-parameter exponential distribution with origin parameter θ_0 and scale parameter $\frac{\lambda_0}{m}$. Consequently,

$$E_X = \frac{2m(\hat{\theta}_0 - \theta_0)}{\lambda_0}$$

follows a chi-square distribution with two degrees of freedom. Similarly, $\hat{\theta}_1$ follows a two-parameter exponential distribution with origin parameter θ_1 and scale parameter $\frac{\lambda_1}{n}$ and

$$E_Y = \frac{2n(\hat{\theta}_1 - \theta_1)}{\lambda_1}$$

follows a chi-square distribution with two degrees of freedom. E_X and E_Y are independent. Moreover, nE_X and mE_Y follow an exponential distribution with rates $\frac{1}{2n}$ and $\frac{1}{2m}$, respectively.

To obtain the distribution of $\hat{\lambda}_0$ and $\hat{\lambda}_1$, first we consider

$$\sum_{i=1}^{m}\left(X_i - \hat{\theta}_0\right) = \sum_{i=1}^{m}\left(X_i - X_{(1)}\right) = \sum_{i=2}^{m}(m-i+1)(X_{(i)} - X_{(i-1)}).$$

It is easy to see that

$$F_X = \frac{2m\hat{\lambda}_0}{\lambda_0} = \frac{2}{\lambda_0}\sum_{i=2}^{m}(m-i+1)(X_{(i)} - X_{(i-1)})$$

follows a chi-square distribution with $2m-2$ degrees of freedom, see, for example, Tanis [21]. Similarly,

$$F_Y = \frac{2n\hat{\lambda}_1}{\lambda_1} = \frac{2}{\lambda_1}\sum_{i=2}^{n}(n-i+1)(Y_{(i)} - Y_{(i-1)})$$

follows a chi-square distribution with $2n-2$ degrees of freedom. Moreover, $(\hat{\theta}_0, \hat{\lambda}_0)$ and $(\hat{\theta}_1, \hat{\lambda}_1)$ are mutually independent. Then, it follows that (E_X, E_Y) and (F_X, F_Y) are also mutually independent.

Consider the statistics W_1 and W_2, defined as

$$W_1 = \frac{mE_Y - nE_X}{F_Y + F_X} = \frac{\frac{2mn(\hat{\theta}_1 - \theta_1)}{\lambda_1} - \frac{2mn(\hat{\theta}_0 - \theta_0)}{\lambda_0}}{\frac{2n\hat{\lambda}_1}{\lambda_1} + \frac{2m\hat{\lambda}_0}{\lambda_0}}, \tag{5}$$

and

$$W_2 = \frac{F_Y/(2n-2)}{F_X/(2m-2)} = \frac{n\hat{\lambda}_1\lambda_0(m-1)}{m\hat{\lambda}_0\lambda_1(n-1)}. \tag{6}$$

It is easy to see that under the null hypothesis, when $\theta_0 = \theta_1$ and $\lambda_0 = \lambda_1$, the statistics W_1 and W_2 reduce respectively to

$$W_1 = \frac{mE_Y - nE_X}{F_Y + F_X} = \frac{mn(\hat{\theta}_1 - \hat{\theta}_0)}{n\hat{\lambda}_1 + m\hat{\lambda}_0} \tag{7}$$

and

$$W_2 = \frac{F_Y/(2n-2)}{F_X/(2m-2)} = \frac{n\hat{\lambda}_1(m-1)}{m\hat{\lambda}_0(n-1)}. \tag{8}$$

Note that the two-sample t-statistic evolves from the problem of testing the equality of location parameters (means) of two independent normal distributions when scale parameters of the two distributions are equal. Similarly, W_1 evolves from the problem of testing the equality of location parameters (origins) of two-independent shifted exponential distributions when scale parameters of the two distributions are equal. In this sense, W_1 is somewhat analogous to the two-sample t statistic for testing the equality of means of two independent normal populations, even though their formulae are different. Likewise, W_2 resembles the Fisher's F ratio statistic for testing the equality of variances of two independent normal populations.

Theorem 2.1. The pdf of W_1 is

$$f_{W_1}(w_1) = \begin{cases} \frac{m+n-2}{m+n} \frac{1}{\left(1-\frac{w_1}{n}\right)^{m+n-1}}, & w_1 < 0 \\ \frac{m+n-2}{m+n} \frac{1}{\left(1+\frac{w_1}{m}\right)^{m+n-1}}, & w_1 \geq 0. \end{cases}$$

Proof. See Appendix A.1. □

Using Theorem 2.1 and by integration, it follows that the cumulative density function (cdf) of W_1 is

$$F_{W_1}(w_1) = \begin{cases} \frac{n}{m+n} \frac{1}{\left(1-\frac{w_1}{n}\right)^{m+n-2}}, & w_1 < 0 \\ \frac{n}{m+n} - \frac{m}{m+n}\left[\frac{1}{\left(1+\frac{w_1}{m}\right)^{m+n-2}} - 1\right], & w_1 \geq 0. \end{cases}$$

Let $u = F(W_1)$. The quantile (or inverse cdf) function of W_1 is

$$Q_{W_1}(u) = \begin{cases} n\left[1 - \left(\frac{n}{u(m+n)}\right)^{\frac{1}{m+n-2}}\right], & u < \frac{n}{m+n} \\ m\left[\left(\frac{m}{(1-u)(m+n)}\right)^{\frac{1}{m+n-2}} - 1\right], & u \geq \frac{n}{m+n}. \end{cases}$$

Theorem 2.2. W_2 follows an F distribution with $(2n-2)$ and $(2m-2)$ degrees of freedom independently of the distribution of W_1.

Proof. See Appendix A.2. □

3. DESCRIPTION OF THE TESTS

We combine the W_1 and W_2 statistics to test simultaneously for different origin and scale parameters of two two-parameter exponential distributions. Two different combination methods are proposed.

3.1. The Max test

The first method is based on the maximum of W_1 and W_2. We refer to it as the Max test. Note that, we consider the two-sided location alternative. If $\theta_1 > \theta_0$ we expect that W_1 will be larger than the expected value of W_1 under H_0. On the other hand, if $\theta_1 < \theta_0$ we expect that $-W_1$ will be larger than the expected value of W_1 under H_0. Further, we expect that if $\lambda_1 > \lambda_0$ then W_2 will be larger than W_2 under H_0. Finally, we expect that if $\lambda_1 < \lambda_0$ then $\frac{1}{W_2}$ will be larger than $\frac{1}{W_2}$ under H_0.

Let M_1 and M_2 denote the cdfs of $\max\{W_1, -W_1\}$ and $\max\{W_2, \frac{1}{W_2}\}$, respectively.

Theorem 3.1. The cdfs of M_1 and M_2 are respectively

$$M_1(z) = \frac{n}{m+n}\left[1 - \frac{1}{\left(1+\frac{z}{n}\right)^{m+n-2}}\right] - \frac{m}{m+n}\left[\frac{1}{\left(1+\frac{z}{m}\right)^{m+n-2}} - 1\right], \qquad 0 \le z < \infty;$$

and

$$M_2(z) = F(z) - F\left(\frac{1}{z}\right)$$

where $F(\cdot)$ is the cdf of an F distribution with $(2n-2)$ and $(2m-2)$ degrees of freedom.

Proof. See Appendix A.3. □

The Max test is based on

$$M = \max\{M_1, M_2\}. \tag{9}$$

It is easy to see that under H_0, M_1 and M_2 are two independent Uniform$(0,1)$ variables. Therefore, the null distribution of the Max test statistic M is a Beta$(2,1)$ and H_0 is rejected if $M > M_\alpha$, where M_α is the $100(1-\alpha)$th percentile of a Beta$(2,1)$ distribution. The distribution function of M is $P[M \le x] = x^2$, for $0 \le x \le 1$. Therefore

$$M_\alpha = \sqrt{1-\alpha}. \tag{10}$$

The exact p-value of the Max test is

$$p_M = 1 - \hat{M}^2 \tag{11}$$

where $\hat{M}$ is the observed value of the Max test statistic.

3.2. The Distance test

The second method is based on the sum of squares of the M_1 and M_2 statistics. We refer to it as the Distance test. The test statistic S of the Distance test is defined as:

$$S = M_1^2 + M_2^2. \tag{12}$$

S is the square of the Euclidian distance of (M_1, M_2) from $(0,0)$. Since M_1 and M_2 are independent, S is also the square of the Mahalanobis distance of (M_1, M_2) from $(0,0)$. H_0 is rejected if $S > S_\alpha$, where S_α is the $100(1-\alpha)$th percentile of the distribution of S.

Theorem 3.2. The pdf of S under H_0 is given by

$$f_S(s) = \begin{cases} \frac{\pi}{4}, & 0 < s < 1 \\ \frac{1}{2}(\sin^{-1}\frac{1}{\sqrt{s}} - \sin^{-1}\sqrt{\frac{s-1}{s}}), & 1 \le s < 2. \end{cases}$$

Proof. See Appendix A.4. □

It follows from Theorem 3.2 that the cdf of S under H_0 is

$$F_S(s) = \begin{cases} \frac{\pi s}{4}, & 0 < s < 1; \\ \sqrt{s-1} + \frac{s}{2}\left(\sin^{-1}\frac{1}{\sqrt{s}} - \sin^{-1}\sqrt{\frac{s-1}{s}}\right), & 1 \le s < 2. \end{cases}$$

The distribution of M can also be verified from Weissman [25].
Consequently, if $\left(1 - \frac{\pi}{4}\right) \le \alpha \le 1$, we have $S_\alpha = \frac{4}{\pi}(1-\alpha)$. When $\alpha < \pi/4$, the critical point S_α is the solution of

$$\sin\left(1 - \alpha - (S_\alpha - 1)^{1/2}\right) = \frac{1}{S_\alpha\sqrt{S_\alpha}}\left(1 - (S_\alpha - 1)^{3/2}\right). \tag{13}$$

The second case, that is, $\alpha < \pi/4$, is more important from practical point of view as the nominal significance level is set to small values such as 0.01 or 0.05.
The exact p-value of the test is

$$p_S = \begin{cases} 1 - \frac{\pi\hat{S}}{4}, & 0 < \hat{S} < 1 \\ 1 - \sqrt{\hat{S}-1} - \frac{\hat{S}}{2}\left(\sin^{-1}\frac{1}{\sqrt{\hat{S}}} - \sin^{-1}\sqrt{\frac{\hat{S}-1}{\hat{S}}}\right), & 1 \le \hat{S} < 2. \end{cases} \tag{14}$$

where $\hat{S}$ is the observed value of the S test statistic.

4. TYPE-ONE ERROR RATE AND POWER COMPARISON STUDY

In this section, we compare type-one error rate and power of the proposed tests. The cut-off points for the Max test can be obtained from Equation 10. These are 0.9747 and 0.9950 for $\alpha = 0.05$ and $\alpha = 0.01$, respectively. The cut-off points for the Distance test are computed from the distribution of S or Equation 13 via a numerical method as 1.4355 and 1.7306 for $\alpha = 0.05$ and $\alpha = 0.01$, respectively.

Parameters		(m, n)								
θ_1	λ_1	(5,5)	(5,10)	(5,15)	(10,5)	(10,10)	(10,15)	(15,5)	(15,10)	(15,15)
0	0.25	0.383	0.539	0.588	0.488	0.760	0.845	0.536	0.840	0.922
0.5	0.25	0.536	0.784	0.839	0.677	0.987	0.995	0.675	0.998	1
1	0.25	0.917	0.982	0.987	0.988	1	1	0.994	1	1
1.5	0.25	0.992	0.999	0.999	1	1	1	1	1	1
2	0.25	0.999	1	1	1	1	1	1	1	1
3	0.25	1	1	1	1	1	1	1	1	1
0	0.5	0.124	0.168	0.177	0.141	0.238	0.298	0.152	0.270	0.358
0.5	0.5	0.264	0.385	0.406	0.386	0.907	0.951	0.378	0.975	0.995
1	0.5	0.788	0.927	0.940	0.951	0.999	1	0.973	1	1
1.5	0.5	0.971	0.994	0.995	0.999	1	1	1	1	1
2	0.5	0.997	1	1	1	1	1	1	1	1
3	0.5	1	1	1	1	1	1	1	1	1
0	0.75	0.062	0.069	0.072	0.062	0.077	0.089	0.063	0.079	0.096
0.5	0.75	0.188	0.197	0.155	0.314	0.821	0.892	0.318	0.950	0.989
1	0.75	0.678	0.844	0.862	0.920	0.999	0.999	0.957	1	1
1.5	0.75	0.944	0.986	0.989	0.999	1	1	1	1	1
2	0.75	0.993	0.999	0.999	1	1	1	1	1	1
3	0.75	1	1	1	1	1	1	1	1	1
0	1	0.050	0.050	0.049	0.050	0.051	0.050	0.049	0.050	0.050
0.5	1	0.166	0.128	0.077	0.299	0.743	0.820	0.315	0.917	0.981
1	1	0.600	0.730	0.723	0.888	0.998	0.999	0.938	1	1
1.5	1	0.906	0.970	0.974	0.997	1	1	1	1	1
2	1	0.986	0.997	0.998	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1
0	2	0.124	0.141	0.156	0.167	0.237	0.267	0.177	0.295	0.359
0.5	2	0.210	0.167	0.157	0.368	0.579	0.585	0.414	0.812	0.935
1	2	0.443	0.387	0.291	0.784	0.984	0.991	0.876	1	1
1.5	2	0.732	0.766	0.703	0.976	1	1	0.996	1	1
2	2	0.904	0.951	0.944	0.999	1	1	1	1	1
3	2	0.994	0.999	0.999	1	1	1	1	1	1
0	5	0.502	0.633	0.685	0.677	0.876	0.934	0.723	0.934	0.978
0.5	5	0.545	0.638	0.687	0.749	0.915	0.949	0.807	0.970	0.992
1	5	0.609	0.658	0.690	0.856	0.977	0.986	0.922	0.999	1
1.5	5	0.698	0.711	0.708	0.954	0.999	0.999	0.989	1	1
2	5	0.793	0.789	0.754	0.991	1	1	0.999	1	1
3	5	0.931	0.939	0.910	1	1	1	1	1	1
0	10	0.831	0.936	0.964	0.939	0.995	0.999	0.957	0.999	1
0.5	10	0.848	0.937	0.964	0.951	0.997	0.999	0.970	0.999	1
1	10	0.864	0.940	0.965	0.970	0.999	1	0.986	1	1
1.5	10	0.882	0.945	0.965	0.986	1	1	0.997	1	1
2	10	0.905	0.950	0.966	0.996	1	1	1	1	1
3	10	0.951	0.969	0.973	1	1	1	1	1	1

Type-one error rates correspond to $\theta_1 = 0$ and $\lambda_1 = 1$ and are underlined.

Tab. 1. Power and type-one error rate of the Max test for $\theta_0 = 0$ and $\lambda_0 = 1$.

We consider all combinations of $m = 5, 10, 15$ and $n = 5, 10, 15$ as the sample size settings. To obtain a precise estimate of the type-I error rate power, we consider 100000 Monte Carlo simulations, as in Wang and Ng [24]. Under H_0, the samples are simulated with $\theta_0 = \theta_1$ and $\lambda_0 = \lambda_1$, and the proportion of simulations where test p-value is less than or equal to α is the estimated type-I error rate. Under H_1, two shifted exponentially distributed samples are simulated with $\theta_0 \neq \theta_1$ or $\lambda_0 \neq \lambda_1$ or both. The power is estimated by the proportion of simulations where test p-value is less than or equal to α. The resulting root mean squared error ($RMSE$) when estimating a rejection probability of ξ is

$$RMSE = \sqrt{\xi(1-\xi)/100000}. \qquad (15)$$

Parameters		(m, n)								
θ_1	λ_1	(5,5)	(5,10)	(5,15)	(10,5)	(10,10)	(10,15)	(15,5)	(15,10)	(15,15)
0	0.25	0.256	0.440	0.511	0.134	0.374	0.496	0.063	0.277	0.411
0.5	0.25	0.664	0.792	0.818	0.876	0.963	0.976	0.927	0.989	0.995
1	0.25	0.810	0.871	0.887	0.923	0.973	0.982	0.949	0.990	0.996
1.5	0.25	0.825	0.876	0.891	0.924	0.973	0.983	0.950	0.990	0.996
2	0.25	0.827	0.878	0.893	0.922	0.972	0.983	0.949	0.990	0.996
3	0.25	0.828	0.878	0.893	0.924	0.972	0.982	0.950	0.990	0.996
0	0.5	0.098	0.164	0.195	0.066	0.161	0.227	0.052	0.136	0.212
0.5	0.5	0.343	0.431	0.454	0.516	0.677	0.722	0.566	0.750	0.808
1	0.5	0.492	0.552	0.570	0.599	0.698	0.737	0.632	0.758	0.809
1.5	0.5	0.519	0.564	0.582	0.604	0.701	0.738	0.638	0.757	0.808
2	0.5	0.524	0.565	0.587	0.608	0.699	0.737	0.642	0.759	0.810
3	0.5	0.528	0.569	0.583	0.607	0.696	0.735	0.641	0.754	0.811
0	0.75	0.059	0.072	0.077	0.050	0.069	0.083	0.047	0.064	0.082
0.5	0.75	0.214	0.238	0.238	0.311	0.394	0.413	0.336	0.431	0.461
1	0.75	0.335	0.356	0.362	0.386	0.421	0.434	0.402	0.446	0.465
1.5	0.75	0.362	0.381	0.382	0.397	0.420	0.434	0.407	0.442	0.466
2	0.75	0.374	0.385	0.385	0.398	0.422	0.436	0.408	0.445	0.464
3	0.75	0.376	0.384	0.388	0.401	0.423	0.434	0.407	0.444	0.465
0	1	0.050	0.051	0.050	0.051	0.051	0.050	0.049	0.051	0.051
0.5	1	0.182	0.189	0.176	0.250	0.306	0.311	0.264	0.325	0.334
1	1	0.291	0.303	0.302	0.323	0.336	0.341	0.327	0.339	0.339
1.5	1	0.326	0.331	0.330	0.334	0.341	0.342	0.337	0.339	0.340
2	1	0.334	0.339	0.337	0.340	0.342	0.343	0.340	0.339	0.341
3	1	0.340	0.338	0.342	0.340	0.342	0.341	0.341	0.340	0.341
0	2	0.099	0.066	0.051	0.164	0.162	0.134	0.192	0.228	0.211
0.5	2	0.271	0.294	0.269	0.451	0.634	0.699	0.499	0.717	0.798
1	2	0.434	0.518	0.529	0.546	0.695	0.754	0.572	0.735	0.809
1.5	2	0.493	0.581	0.607	0.562	0.698	0.757	0.582	0.735	0.809
2	2	0.511	0.597	0.628	0.568	0.699	0.759	0.583	0.736	0.808
3	2	0.523	0.606	0.641	0.572	0.695	0.756	0.586	0.737	0.809
0	5	0.316	0.163	0.070	0.519	0.422	0.301	0.591	0.540	0.438
0.5	5	0.514	0.416	0.278	0.824	0.930	0.939	0.902	0.990	0.995
1	5	0.736	0.798	0.760	0.921	0.990	0.997	0.936	0.995	0.999
1.5	5	0.848	0.931	0.940	0.927	0.991	0.998	0.939	0.995	0.999
2	5	0.877	0.955	0.970	0.929	0.991	0.998	0.939	0.995	0.999
3	5	0.887	0.962	0.979	0.930	0.990	0.998	0.940	0.995	0.999
0	10	0.468	0.238	0.103	0.667	0.501	0.358	0.735	0.618	0.503
0.5	10	0.595	0.390	0.216	0.851	0.813	0.745	0.930	0.953	0.955
1	10	0.749	0.622	0.454	0.966	0.988	0.990	0.990	1	1
1.5	10	0.876	0.852	0.772	0.989	1	1	0.992	1	1
2	10	0.942	0.962	0.948	0.991	1	1	0.993	1	1
3	10	0.978	0.998	0.999	0.991	1	1	0.993	1	1

Type-one error rates correspond to $\theta_1 = 0$ and $\lambda_1 = 1$ and are underlined.

Tab. 2. Power and type-one error rate of the Distance test for $\theta_0 = 0$ and $\lambda_0 = 1$.

When H_0 is true, we are estimating $\xi = 0.05$ therefore $RMSE = 0.00069$. The $RMSE$ is maximum when estimating $\xi = 0.5$, i. e.,

$$\max\{RMSE\} = \sqrt{0.5(1-0.5)/100000} = 0.00158. \tag{16}$$

In this power comparison, α is set to 0.05. First we consider $\theta_0 = 0$ and $\lambda_0 = 1$. This is an important problem where a standard exponential distribution is compared to a shifted exponential distribution with a possible different rate parameter as well. We present the results for various values of θ_1 and λ_1 in Tables 1 and 2 for the Max and Distance test respectively. Note that in Tables 1 and 2, the probability of rejecting the null hypothesis for $\theta_1 = 0$ and $\lambda_1 = 1$ corresponds to the type-one error rate. In these tables, we observe that the type-one error rate of the tests is very close to $\alpha = 0.05$ and this indicates that the tests control the type-one error rate very well as expected. Tables 1 and 2 show that the Max test is generally more powerful than the Distance test, in particular when n or m are larger than 5.

Tables 3 and 4 present the results with $\theta_0 = 10$ and $\lambda_0 = 10$ for various values of θ_1 and λ_1. Note that in Tables 3 and 4, the probability of rejecting the null hypothesis for $\theta_1 = 10$ and $\lambda_1 = 10$ corresponds to the type-one error rate. Tables 3 and 4 confirm the results reported in Tables 1 and 2.

Parameters		(m,n)								
θ_1	λ_1	(5,5)	(5,10)	(5,15)	(10,5)	(10,10)	(10,15)	(15,5)	(15,10)	(15,15)
5	6	0.356	0.661	0.751	0.297	0.958	0.998	0.212	0.975	1
5	8	0.235	0.447	0.495	0.188	0.866	0.977	0.120	0.911	0.995
5	10	0.165	0.299	0.315	0.126	0.739	0.918	0.078	0.820	0.981
5	12	0.129	0.214	0.213	0.095	0.610	0.820	0.060	0.718	0.953
5	14	0.112	0.167	0.166	0.087	0.507	0.703	0.058	0.622	0.911
7	6	0.181	0.331	0.371	0.129	0.589	0.831	0.106	0.615	0.949
7	8	0.115	0.197	0.214	0.069	0.386	0.605	0.052	0.403	0.827
7	10	0.083	0.134	0.136	0.046	0.260	0.423	0.035	0.270	0.678
7	12	0.072	0.105	0.105	0.044	0.194	0.303	0.035	0.202	0.533
7	14	0.073	0.098	0.100	0.054	0.168	0.246	0.046	0.175	0.441
10	6	0.088	0.112	0.120	0.093	0.147	0.176	0.101	0.160	0.206
10	8	0.057	0.063	0.064	0.056	0.066	0.073	0.057	0.065	0.076
10	10	0.050	0.050	0.051	0.049	0.050	0.050	0.050	0.049	0.051
10	12	0.056	0.053	0.054	0.059	0.061	0.060	0.060	0.068	0.067
10	14	0.066	0.067	0.069	0.079	0.089	0.092	0.079	0.103	0.112
13	6	0.108	0.095	0.076	0.152	0.415	0.538	0.150	0.576	0.849
13	8	0.084	0.054	0.035	0.128	0.307	0.369	0.127	0.472	0.758
13	10	0.082	0.047	0.034	0.133	0.258	0.271	0.135	0.422	0.678
13	12	0.092	0.056	0.046	0.147	0.245	0.232	0.155	0.400	0.611
13	14	0.106	0.071	0.064	0.169	0.256	0.228	0.183	0.401	0.572
15	6	0.221	0.285	0.275	0.345	0.873	0.930	0.340	0.966	0.992
15	8	0.183	0.175	0.131	0.309	0.805	0.881	0.316	0.942	0.987
15	10	0.165	0.126	0.079	0.298	0.739	0.817	0.317	0.919	0.982
15	12	0.165	0.110	0.068	0.300	0.678	0.749	0.329	0.888	0.974
15	14	0.169	0.109	0.078	0.310	0.629	0.679	0.345	0.862	0.965

Type-one error rates correspond to $\theta_1 = 10$ and $\lambda_1 = 10$ and are underlined.

Tab. 3. Power and type-one error rate of the Max test for $\theta_0 = 10$ and $\lambda_0 = 10$.

5. PRACTICAL EXAMPLE

To illustrate the proposed two-sample tests, we analyse survival data from Krishnamoorthy and Xia [11]. This dataset consists of the number of survival days for patients with incurable lung cancer. As in Krishnamoorthy and Xia [11], we consider two categories of patients' cancer type, i. e. squamous and small cells. There are $m = 9$ patients with squamous cell cancer type, and $n = 9$ patients with small cell cancer type, see Table 5. Survival times are sorted in the increasing order. For the first sample, that is, for the patients with squamous cell, the maximum likelihood estimates of the location (origin) θ_0 and scale λ_0 parameters are 8 and 43, respectively. For the second sample, that is,

Parameters		(m, n)								
θ_1	λ_1	(5,5)	(5,10)	(5,15)	(10,5)	(10,10)	(10,15)	(15,5)	(15,10)	(15,15)
5	6	0.339	0.435	0.455	0.393	0.558	0.597	0.399	0.606	0.661
5	8	0.226	0.299	0.314	0.246	0.370	0.394	0.239	0.388	0.414
5	10	0.183	0.254	0.266	0.187	0.303	0.322	0.175	0.313	0.335
5	12	0.165	0.248	0.269	0.174	0.322	0.364	0.165	0.333	0.383
5	14	0.156	0.262	0.291	0.180	0.371	0.441	0.173	0.401	0.487
7	6	0.227	0.360	0.396	0.252	0.507	0.576	0.234	0.560	0.651
7	8	0.144	0.219	0.241	0.140	0.302	0.353	0.121	0.325	0.390
7	10	0.110	0.168	0.185	0.099	0.227	0.272	0.083	0.234	0.296
7	12	0.097	0.156	0.174	0.091	0.226	0.285	0.078	0.239	0.328
7	14	0.092	0.154	0.181	0.095	0.252	0.341	0.087	0.281	0.410
10	6	0.075	0.112	0.130	0.058	0.112	0.150	0.049	0.098	0.144
10	8	0.055	0.064	0.069	0.048	0.061	0.071	0.047	0.058	0.068
10	10	0.051	0.050	0.050	0.050	0.049	0.050	0.051	0.050	0.050
10	12	0.053	0.047	0.046	0.062	0.057	0.053	0.063	0.064	0.064
10	14	0.061	0.050	0.048	0.081	0.076	0.069	0.088	0.095	0.091
13	6	0.157	0.202	0.204	0.280	0.450	0.504	0.328	0.551	0.628
13	8	0.119	0.122	0.110	0.197	0.280	0.298	0.221	0.339	0.380
13	10	0.108	0.101	0.084	0.169	0.226	0.237	0.185	0.271	0.300
13	12	0.111	0.102	0.083	0.179	0.249	0.261	0.198	0.304	0.343
13	14	0.123	0.113	0.089	0.206	0.304	0.331	0.231	0.381	0.445
15	6	0.277	0.333	0.344	0.410	0.542	0.582	0.451	0.607	0.657
15	8	0.202	0.221	0.215	0.291	0.361	0.377	0.307	0.394	0.414
15	10	0.182	0.189	0.175	0.255	0.306	0.312	0.266	0.325	0.335
15	12	0.185	0.193	0.181	0.261	0.333	0.350	0.276	0.359	0.389
15	14	0.199	0.213	0.200	0.297	0.402	0.433	0.321	0.446	0.496

Type-one error rates correspond to $\theta_1 = 10$ and $\lambda_1 = 10$ and are underlined.

Tab. 4. Power and type-one error rate of the Distance test for $\theta_0 = 10$ and $\lambda_0 = 10$.

for the patients with small cancer cell, the maximum likelihood estimates of the origin θ_1 and scale λ_1 parameters are 13 and 9.111, respectively, and α is set to 0.05.

Squamous Cell									
Sample-1	8	10	11	25	42	72	81	100	110
Small Cell									
Sample-2	13	16	18	20	21	23	27	30	31

Tab. 5. Number of survival days for patients with squamous and small cell lung cancer.

We would like to simultaneously test for different origin and scale parameters of the populations behind sample 1 and 2. Here $W_1 = 0.8635$ and $W_2 = 0.2119$. The Max test should be preferred according to our simulation study. Then, we use the maximum combining function to combine W_1 and W_2, and obtain $\max\{W_1, -W_1\} = 0.8635$ and $\max\{W_2, \frac{1}{W_2}\} = 4.7195$. By considering the cdfs of $\max\{W_1, -W_1\}$ and $\max\{W_2, \frac{1}{W_2}\}$ as in Theorem 3, M_1 and M_2 are respectively computed as 0.7691 and 0.9965. Therefore, $M = \max\{M_1, M_2\} = 0.9965$. The cut-off value M_α is 0.9747 for $\alpha = 0.05$. Since $M > M_\alpha$, we reject the null hypothesis that the parent population distributions are equal. The p-value of the Max test is 0.00694.

We also apply the Distance test and obtain $S = \{M_1^2 + M_2^2\} = 1.5846$. The cut-off value S_α from the cdf of S is 1.4355. Since $S > S_\alpha$ then we reject H_0. The p-value of the Distance test is 0.02525. Note that this p-value is larger than the p-value of the Max test that finds more evidence than the Distance test against the null hypothesis, confirming the results of our simulation study. We conclude that both tests lead to the same conclusion that $\theta_0 \neq \theta_1$ and $\lambda_0 \neq \lambda_1$.

6. CONCLUSION

The open problem of simultaneous testing for different origin and scale parameters of two-parameter (shifted) exponential distributions has been addressed. Two combined tests have been proposed, studied and compared. We showed that the test based on a maximum type combination should be preferred being generally more powerful than the other test based on a distance type combination. Since two-parameter exponential distribution is very useful in the engineering sciences and industrial quality control, a possible direction for future research is to use the Max test to design a Phase-II control chart for shifted exponential processes with unknown parameters.

APPENDIX

A.1. Proof of Theorem 2.1

We obtain the result following the line of Pal, Masoom, and Woo [15]. First, consider the joint density of E_X and E_Y

$$f_{(E_X,E_Y)}(e_X, e_Y) = \frac{1}{4} e^{-\frac{1}{2}(e_X+e_Y)}, \quad e_X > 0, e_Y > 0.$$

Define, $Z_1 = mE_Y - nE_X$ and $D_X = E_X$. Under this transformation,

$$E_X = D_X, \quad \text{and} \quad E_Y = \frac{Z_1 + nD_X}{m}.$$

Unconditionally, $d_X > 0, z_1 \in (-\infty, \infty)$ but conditionally, $d_X > 0, z_1 > -nd_X$. The Jacobian of the transformation is

$$\left| J\left(\frac{e_X, e_Y}{d_X, z_1}\right) \right| = \left\| \begin{matrix} 1 & 0 \\ \frac{n}{m} & \frac{1}{m} \end{matrix} \right\| = \frac{1}{m}.$$

The joint density of Z_1 and D_X is

$$f_{(D_X,Z_1)}(d_X, z_1) = \frac{1}{4m} e^{-\frac{1}{2}\left(\frac{z_1+nd_X}{m}+d_X\right)}, \quad d_X > 0, z_1 > -nd_X.$$

Therefore, the marginal distribution of Z_1 is

$$f_{Z_1}(z_1) \quad = \quad \frac{1}{4m} e^{-\frac{z_1}{2m}} \int_{\max\{0,-\frac{z_1}{n}\}}^{\infty} e^{-\frac{d_X}{2}(1+\frac{n}{m})} \, \mathrm{d}d_X, \quad z_1 \in (-\infty, \infty),$$

$$= \begin{cases} \frac{1}{4m} e^{-\frac{z_1}{2m}} \left[-\frac{2}{\left(1+\frac{n}{m}\right)} e^{-\frac{d_X}{2}\left(1+\frac{n}{m}\right)} \right]_{-\frac{z_1}{n}}^{\infty}, & z_1 < 0; \\ \frac{1}{4m} e^{-\frac{z_1}{2m}} \left[-\frac{2}{\left(1+\frac{n}{m}\right)} e^{-\frac{d_X}{2}\left(1+\frac{n}{m}\right)} \right]_{0}^{\infty}, & z_1 \geq 0; \end{cases}$$

$$= \begin{cases} \frac{1}{4m} e^{-\frac{z_1}{2m}} \frac{2}{(1+\frac{n}{m})} e^{\frac{z_1}{2n}(1+\frac{n}{m})}, & z_1 < 0; \\ \frac{1}{2m(1+\frac{n}{m})} e^{-\frac{z_1}{2m}}, & z_1 \geq 0; \end{cases}$$

$$= \begin{cases} \frac{1}{2(m+n)} e^{\frac{z_1}{2n}}, & z_1 < 0; \\ \frac{1}{2(m+n)} e^{-\frac{z_1}{2m}}, & z_1 \geq 0. \end{cases}$$

Note that the mean of Z_1 is $2(m-n)$ and that Z_1 follows a Laplace distribution if $m = n$. Moreover, $F_Y + F_X$ follows a chi-square distribution with $2m+2n-4$ degrees of freedom independently of Z_1. We need the distribution of $W_1 = \frac{Z_1}{Z_S}$ where $Z_S = F_Y + F_X$.

The joint density of Z_1 and Z_S is

$$f_{(Z_1,Z_S)}(z_1, z_S) = \begin{cases} \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} e^{-\left(-\frac{z_1}{2n}+\frac{z_S}{2}\right)} (z_S)^{m+n-3}, & z_1 < 0, z_S > 0 \\ \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} e^{-\left(\frac{z_1}{2m}+\frac{z_S}{2}\right)} (z_S)^{m+n-3}, & z_1 \geq 0, z_S > 0 \end{cases}$$

Define $W_1 = \frac{Z_1}{Z_S}$ and $Z_X = Z_S$. Under this transformation, $Z_1 = W_1 Z_X$ and $Z_S = Z_X$ with $Z_X > 0, W_1 \in (-\infty, \infty)$. The Jacobian of the transformation is

$$\left| J\left(\frac{z_1, z_S}{w_1, z_X} \right) \right| = \left\| \begin{matrix} z_X & w_1 \\ 0 & 1 \end{matrix} \right\| = z_X.$$

The joint density of W_1 and Z_X is

$$f_{(W_1,Z_X)}(w_1, z_X) = \begin{cases} \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} e^{-\left(-\frac{w_1 z_X}{2n}+\frac{z_X}{2}\right)} (z_X)^{m+n-2}, & w_1 < 0, z_X > 0 \\ \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} e^{-\left(\frac{w_1 z_X}{2m}+\frac{z_X}{2}\right)} (z_X)^{m+n-2}, & w_1 \geq 0, z_X > 0. \end{cases}$$

Therefore, the marginal distribution of W_1 is

$$f_{W_1}(w_1) = \begin{cases} \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} \int_0^\infty e^{-\left(\frac{1}{2}-\frac{w_1}{2n}\right)z_X} (z_X)^{m+n-2}\, \mathrm{d}z_X, & w_1 < 0 \\ \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} \int_0^\infty e^{-\left(\frac{1}{2}+\frac{w_1}{2m}\right)z_X} (z_X)^{m+n-2}\, \mathrm{d}z_X, & w_1 \geq 0. \end{cases}$$

The integrals can be easily computed using the following property of the Gamma distribution

$$\frac{\Gamma(b+1)}{a^{b+1}} = \int_0^\infty x^b e^{-ax}\, \mathrm{d}x.$$

Consequently,

$$f_{W_1}(w_1) = \begin{cases} \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} \frac{\Gamma(m+n-1)}{\left(\frac{1}{2}-\frac{w_1}{2n}\right)^{m+n-1}}, & w_1 < 0; \\ \frac{1}{2^{m+n-1}(m+n)\Gamma(m+n-2)} \frac{\Gamma(m+n-1)}{\left(\frac{1}{2}+\frac{w_1}{2m}\right)^{m+n-1}}, & w_1 \geq 0; \end{cases}$$

$$= \begin{cases} \frac{m+n-2}{(m+n)} \frac{1}{\left(1-\frac{w_1}{n}\right)^{m+n-1}}, & w_1 < 0; \\ \frac{m+n-2}{(m+n)} \frac{1}{\left(1+\frac{w_1}{m}\right)^{m+n-1}}, & w_1 \geq 0. \end{cases}$$

A.2. Proof of Theorem 2.2

We need to show that

1. W_1 and W_2 are mutually independent;
2. W_2 follows an F distribution with $(2n-2)$ and $(2m-2)$ degrees of freedom.

Consider the transformation

$$Z_1 = mE_Y - nE_X, Z_2 = F_X, \quad \text{and} \quad Z_3 = F_Y;$$

where Z_1, Z_2 and Z_3 are mutually independent. The joint density of Z_1, Z_2 and Z_3 is

$$f_{(Z_1,Z_2,Z_3)}(z_1,z_2,z_3) = \begin{cases} \frac{1}{2^{m+n-1}(m+n)\Gamma(m-1)\Gamma(n-1)} e^{-\left(-\frac{z_1}{2n}+\frac{z_2}{2}+\frac{z_3}{2}\right)} z_2^{m-2} z_3^{n-2}, & z_1 < 0, \quad z_2, \quad z_3 > 0 \\ \frac{1}{2^{m+n-1}(m+n)\Gamma(m-1)\Gamma(n-1)} e^{-\left(\frac{z_1}{2m}+\frac{z_2}{2}+\frac{z_3}{2}\right)} z_2^{m-2} z_3^{n-2}, & z_1 \geq 0, \quad z_2, \quad z_3 > 0. \end{cases}$$

Further, consider the transformation

$$\begin{aligned} W_1 &= \frac{Z_1}{Z_2+Z_3}, \\ W_2 &= \left(\frac{Z_3}{Z_2}\right)\frac{m-1}{n-1}, \\ W_3 &= Z_2+Z_3. \end{aligned}$$

Under this transformation,

$$\begin{aligned} Z_1 &= W_1W_3, \\ Z_3 &= W_3 - Z_2 \\ &\Rightarrow W_2 = \left(\frac{W_3-Z_2}{Z_2}\right)\frac{m-1}{n-1} \Rightarrow \frac{W_3}{Z_2} = 1+W_2\frac{n-1}{m-1} \Rightarrow Z_2 = \frac{W_3}{1+W_2\frac{n-1}{m-1}}. \end{aligned}$$

Finally,

$$Z_3 = W_3 - \frac{W_3}{1+W_2\frac{n-1}{m-1}} = W_3\left(\frac{W_2\frac{n-1}{m-1}}{1+W_2\frac{n-1}{m-1}}\right).$$

The Jacobian of the transformation is

$$\begin{aligned} \left|J\left(\frac{z_1,z_2,z_3}{w_1,w_2,w_3}\right)\right| &= \left\|\begin{matrix} w_3 & 0 & w_1 \\ 0 & -w_3\frac{n-1}{m-1}\left(1+w_2\frac{n-1}{m-1}\right)^{-2} & \left(1+w_2\frac{n-1}{m-1}\right)^{-1} \\ 0 & w_3\frac{n-1}{m-1}\left(1+w_2\frac{n-1}{m-1}\right)^{-2} & 1-\left(1+w_2\frac{n-1}{m-1}\right)^{-1} \end{matrix}\right\| \\ &= \frac{n-1}{m-1}\left(\frac{w_3}{1+w_2\frac{n-1}{m-1}}\right)^2. \end{aligned}$$

The joint distribution of W_1, W_2 and W_3 is

$$f_{(W_1,W_2,W_3)}(w_1,w_2,w_3)$$
$$= \begin{cases} \frac{(n-1)e^{-(-\frac{w_1w_3}{2n}+\frac{w_3}{2})}(\frac{w_3}{1+w_2\frac{n-1}{m-1}})^m(w_3(\frac{w_2\frac{n-1}{m-1}}{1+w_2\frac{n-1}{m-1}}))^{n-2}}{(m-1)2^{m+n-1}(m+n)\Gamma(m-1)\Gamma(n-1)}, & w_1<0, w_2, w_3>0; \\ \frac{(n-1)e^{-(\frac{w_1w_3}{2m}+\frac{w_3}{2})}(\frac{w_3}{1+w_2\frac{n-1}{m-1}})^m(w_3(\frac{w_2\frac{n-1}{m-1}}{1+w_2\frac{n-1}{m-1}}))^{n-2}}{(m-1)2^{m+n-1}(m+n)\Gamma(m-1)\Gamma(n-1)}, & w_1\geq 0, w_2, w_3>0; \end{cases}$$
$$= \begin{cases} \frac{(n-1)e^{-\frac{w_3}{2}(1-\frac{w_1}{n})}(w_3)^{m+n-2}(w_2\frac{n-1}{m-1})^{n-2}}{(m-1)2^{m+n-1}(m+n)\Gamma(m-1)\Gamma(n-1)(1+w_2\frac{n-1}{m-1})^{m+n-2}}, & w_1<0, w_2, w_3>0; \\ \frac{(n-1)e^{-\frac{w_3}{2}(1+\frac{w_1}{m})}(w_3)^{m+n-2}(w_2\frac{n-1}{m-1})^{n-2}}{(m-1)2^{m+n-1}(m+n)\Gamma(m-1)\Gamma(n-1)(1+w_2\frac{n-1}{m-1})^{m+n-2}}, & w_1\geq 0, w_2, w_3>0. \end{cases}$$

Therefore, using the same property of the Gamma distribution as before, we obtain the joint distribution of W_1 and W_2

$$f_{(W_1,W_2)}(w_1,w_2) = \int_0^\infty f_{(W_1,W_2,W_3)}(w_1,w_2,w_3)\,\mathrm{d}w_3$$
$$= \begin{cases} \frac{(n-1)\Gamma(m+n-1)(w_2\frac{n-1}{m-1})^{n-2}}{(m-1)(m+n)\Gamma(m-1)\Gamma(n-1)(1+w_2\frac{n-1}{m-1})^{m+n-2}(1-\frac{w_1}{n})^{m+n-1}}, & w_1<0, w_2>0 \\ \frac{(n-1)\Gamma(m+n-1)(w_2\frac{n-1}{m-1})^{n-2}}{(m-1)(m+n)\Gamma(m-1)\Gamma(n-1)(1+w_2\frac{n-1}{m-1})^{m+n-2}(1+\frac{w_1}{m})^{m+n-1}}, & w_1\geq 0, w_2>0. \end{cases}$$

Since $f_{(W_1,W_2)}(w_1,w_2)$ is the product of $f_{W_1}(w_1)$ and the pdf of an F distribution with $(2n-2)$ and $(2m-2)$ degrees of freedom, we conclude that W_1 and W_2 are independent and that

$$f_{W_2}(w_2) = \frac{\Gamma(m+n-2)}{\Gamma(n-1)\Gamma(m-1)}\left(\frac{n-1}{m-1}\right)^{n-1} w_2^{n-2}\left(1+w_2\frac{n-1}{m-1}\right)^{-(n+m-2)}, \quad w_2>0.$$

A.3. Proof of Theorem 3.1

The distribution of $W_1^* = \max\{W_1, -W_1\}$ is

$$\begin{aligned} G_{W_1^*}(w) &= Prob\,[W_1^*\leq w] = Prob\,[\max\{W_1,-W_1\}\leq w] \\ &= Prob\,[W_1\geq -w\cap W_1\leq 0] + Prob\,[W_1\leq w\cap W_1>0] \\ &= \left[\int_{-w}^0 \frac{m+n-2}{m+n}\frac{1}{(1-\frac{w_1}{n})^{m+n-1}}\mathrm{d}w_1\right] + \left[\int_0^w \frac{m+n-2}{m+n}\frac{1}{(1+\frac{w_1}{m})^{m+n-1}}\mathrm{d}w_1\right] \\ &= \frac{m+n-2}{m+n}\left[\frac{(1-\frac{w_1}{n})^{2-m-n}}{(2-m-n)(-\frac{1}{n})}\right]_{-w}^0 + \frac{m+n-2}{m+n}\left[\frac{(1+\frac{w_1}{m})^{2-m-n}}{(2-m-n)(\frac{1}{m})}\right]_0^w \\ &= \frac{n}{m+n}\left[1-\frac{1}{(1+\frac{w}{n})^{m+n-2}}\right] - \frac{m}{m+n}\left[\frac{1}{(1+\frac{w}{m})^{m+n-2}}-1\right], \end{aligned}$$

where $0\leq w<\infty$. Similarly, the distribution of $W_2^* = \max\{W_2, \frac{1}{W_2}\}$ is

$$\begin{aligned} G_{W_2^*}(w) &= Prob(W_2^* \leq w) = Prob\left(\max\{W_2, \frac{1}{W_2}\} \leq w\right) \\ &= Prob\,(W_2 \geq 1/w \cap W_2 < 1) + Prob\,(W_2 \leq w \cap W_2 \geq 1) \\ &= F(1) - F(1/w) + F(w) - F(1) = F(w) - F(1/w). \end{aligned}$$

A.4. Proof of Theorem 3.2

Let $S = M_1^2 + M_2^2$ and $T = M_1$. Under this transformation, $M_1 = T$ and $M_2 = \sqrt{S - T^2}$. The Jacobian of transformation is

$$\left| J\left(\frac{m_1, m_2}{s, t}\right)\right| = \left\| \begin{matrix} 0 & 1 \\ \frac{1}{2\sqrt{s-t^2}} & -\frac{t}{\sqrt{s-t^2}} \end{matrix} \right\| = \left| -\frac{1}{2\sqrt{s-t^2}} \right| = \frac{1}{2\sqrt{s-t^2}}.$$

Note that, being cdfs, M_1 and M_2 are two independent uniformly distributed random variables in $(0,1)$, i. e. $M_1, M_2 \sim U(0,1)$. Therefore, $f_{(M_1,M_2)}(m_1,m_2) = 1$ for all $(m_1, m_2) \in (0,1) \times (0,1)$, and

$$f_{(S,T)}(s,t) = f_{(M_1,M_2)}(m_1,m_2) \left| J\left(\frac{m_1, m_2}{s, t}\right)\right| = \frac{1}{2\sqrt{s-t^2}},$$

For $0 < s < 1$, we have $0 < t < \sqrt{s}$, therefore

$$f_S(s) = \int_0^{\sqrt{s}} f_{(S,T)}(s,t)\,\mathrm{d}t = \int_0^{\sqrt{s}} \frac{1}{2\sqrt{s-t^2}}\,\mathrm{d}t = \left[\frac{1}{2}\sin^{-1}\frac{t}{\sqrt{s}}\right]_0^{\sqrt{s}} = \frac{1}{2}\sin^{-1} 1 = \frac{\pi}{4}.$$

For $1 \leq s < 2$, we have $\sqrt{s-1} < t < 1$, therefore

$$\begin{aligned} f_S(s) &= \int_{\sqrt{s-1}}^1 f_{(S,T)}(s,t)dt = \int_{\sqrt{s-1}}^1 \frac{1}{2\sqrt{s-t^2}}\,\mathrm{d}t = \left[\frac{1}{2}\sin^{-1}\frac{t}{\sqrt{s}}\right]_{\sqrt{s-1}}^1 \\ &= \frac{1}{2}\left(\sin^{-1}\frac{1}{\sqrt{s}} - \sin^{-1}\sqrt{\frac{s-1}{s}}\right). \end{aligned}$$

Hence the result follows.

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