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COMMON FIXED POINTS FOR FOUR NON-SELF MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. We formulate a common fixed point theorem for four non-self mappings in convex partial metric spaces. The result extends a fixed point theorem by Gajić and Rakočević (2007) proved for two non-self mappings in metric spaces with a Takahashi convex structure. We also provide an illustrative example on the use of the theorem.

Keywords: common fixed point; convex partial metric space; non-self mapping

MSC 2010: 47H10, 54H25

1. Introduction and preliminaries

Gajić and Rakočević [5] proved a common fixed point theorem for non-self mappings on a Takahashi convex metric space for a pair of mappings. In their work, they generalized the theorems by Jungck [7], Das and Naik [4], Ćirić et al. [3], Ćirić [2] and Imdad and Kumar [6]. In this study, we extend the theorem by Gajić and Rakočević to apply for two pairs of non-self mappings in convex partial metric spaces.

We now introduce the results which will be of use in this paper.

Definition 1.1 ([8]). A partial metric on a nonempty set X is a mapping $p\colon X\times X\to[0,\infty)$ such that for all $x,y,z\in X$,

(P0) $0\leqslant p(x,x)\leqslant p(x,y)$,
(P1) $x=y$ if and only if $p(x,x)=p(x,y)=p(y,y)$,
(P2) $p(x,y)=p(y,x)$ and
(P3) $p(x,y)\leqslant p(x,z)+p(z,y)-p(z,z)$.

A pair (X,p) is said to be a partial metric space.

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From Definition 1.1, we deduce that for all x, y, z in a partial metric space (X, p), we have:

$$p(x,y) = 0 \Rightarrow x = y, \tag{1.1}$$

$$p(x,y) \leqslant p(x,z) + p(z,y). \tag{1.2}$$

Proof. If $p(x,y) = 0$, then $p(x,x) = 0$ because $0 \leqslant p(x,x) \leqslant p(x,y)$ from (P0). Similarly, $p(x,y) = 0$ implies $p(y,y) = 0$ because $0 \leqslant p(y,y) \leqslant p(x,y)$. Hence $p(x,y) = 0$ implies $p(x,x) = p(x,y) = p(y,y) = 0$. From (P1) this means that $x = y$.

From (P3), we infer that

$$p(x,y) \leqslant p(x,z) + p(z,y).$$

□

As an example, let $X = \mathbb{R}^+$ and let $p\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon)\colon x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X\colon p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x,x_n).$$

Definition 1.2 ([8]). Let (X,p) be a partial metric space and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim\limits_{n\to\infty} p(x,x_n)$,

(ii) $\{x_n\}$ is called a *Cauchy sequence* if $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists and is finite,

(iii) a partial metric space (X,p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 1.3 ([8]). *If p is a partial metric on X, then the mapping $p^s\colon X \times X \to [0,\infty)$ given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric.

In this paper we will denote by p^s the metric derived from the partial metric p.

Lemma 1.4 ([8]). *Let (X,p) be a partial metric space and $\{x_n\}$ a sequence in X. Then*

(i) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),*

(ii) *(X,p) is complete if and only if (X,p^s) is complete. Furthermore, $\lim\limits_{n\to\infty} p(x_n,x)$ is zero if and only if $p(x,x) = \lim\limits_{n\to\infty} p(x_n,x) = \lim\limits_{n,m\to\infty} p(x_n,x_m)$.*

Definition 1.5 ([8]). Let (X,p) be a partial metric space and $\{x_n\}$ a sequence in X. Then

(i) the sequence $\{x_n\}$ is called 0-*Cauchy* if $\lim\limits_{n,m\to\infty} p(x_n,x_m) = 0$,

(ii) (X,p) is said to be 0-*complete* if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = 0$.

Definition 1.6 ([9]). Let (X,p) be a partial metric space and $I = [0,1]$ the closed unit interval. A mapping $W\colon X\times X\times I \to X$ is said to be a *convex* structure on X if for all $(x,y,t) \in X\times X\times I$,

$$p(u,W(x,y,t)) \leqslant tp(u,x) + (1-t)p(u,y)$$

for every $u \in X$. A partial metric space (X,p), together with the convex structure W, is called a *convex partial metric space.*

If (X,p) is a convex partial metric space, then for every $x,y\in X$, we define

$$\operatorname{seg}[x,y] := \{W(x,y,t)\colon\ t\in[0,1]\}. \tag{1.3}$$

In this study, we will use the following properties of a convex partial metric space with convex structure W.

Lemma 1.7. *Let $x,y \in X$ where (X,p) is a convex partial metric space with convex structure W. Let $w \in \operatorname{seg}[x,y]$. Then for all $u\in X$, we have*

(i) *$p(u,w) \leqslant \max\{p(u,x),p(u,y)\}$,*

(ii) *$p(x,w) \leqslant p(x,y)$.*

Proof. Suppose $\Gamma = \max\{p(u,x),p(u,y)\}$. Applying Definition 1.6, we have

$$p(u,w) \leqslant tp(u,x) + (1-t)p(u,y) \leqslant t\Gamma + (1-t)\Gamma = \Gamma = \max\{p(u,x),p(u,y)\}.$$

We have proved Lemma 1.7 (i). Now let us set $x=u$ in Lemma 1.7 (i). We get

$$p(x,w) \leqslant \max\{p(x,x),p(x,y)\} = p(x,y),$$

from (P0) of Definition 1.1, so we have proved Lemma 1.7 (ii). □

Definition 1.8 ([1]). Let (X,p) be a partial metric space and $B \subseteq X$. Then

(i) B is said to be *bounded* if there is a positive number M such that $p(x,y) \leqslant M$ for all $x,y \in B$,

(ii) if B is a bounded set, the *diameter* of B is defined as

$$\operatorname{diam}(B) = \sup_{u,v\in B}\{p(u,v)\}.$$

Let $f\colon C \to X$ be a mapping, where $C \subseteq X$. We say that f is a *self mapping* if $C = X$, otherwise f is called a *non-self mapping*. If there is an element $x \in C$ such that $fx = x$, we say that x is a *fixed point* of f in X.

Suppose we have two mappings $f,g\colon C \to X$, with $C \subseteq X$. Let there be $x \in C$ such that $fx = gx = w$. We say that x is a *coincidence point* of f and g in X. If $x = w$, then we call x a *common fixed point* of f and g in X.

Suppose we have two mappings $f,g\colon C \to X$ with $C \subseteq X$. We say f and g are *coincidentally commuting* if for all $x \in C$ we have

$$fx = gx \Rightarrow fgx = gfx.$$

In this paper, we aim to extend the following theorem by Gajić and Rakočević (see [5]) which proves the existence of a common fixed point for non-self mappings in context of metric spaces under specified conditions.

Theorem 1.9 ([5]). *Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let C be a nonempty closed subset of X and ∂C the boundary of C. Let $f,g\colon C \to X$ and suppose $\partial C \neq \emptyset$. Let us assume that f and g satisfy the following conditions:*

(i) *For every $x,y \in C$, $d(gx,gy) \leqslant M_\omega(x,y)$ where*

$$M_\omega(x,y) = \max\{\omega_1[d(fx,fy)], \omega_2[d(fx,gx)], \omega_3[d(fy,gy)], \\ \omega_4[d(fx,gy)], \omega_5[d(gx,fy]\},$$

$\omega_i\colon [0,\infty) \to [0,\infty)$, $i \in \{1,2,3,4,5\}$ is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $\lim\limits_{r\to\infty}(r - \omega_i(r)) = \infty$,

(ii) *$\partial C \subseteq f(C)$,*

(iii) *$g(C) \cap C \subset f(C)$,*

(iv) *$fx \in \partial C \Rightarrow gx \in C$ and*

(v) *$f(C)$ is closed in X.*

Then there exists a coincidence point v in C. Moreover, if $\{f,g\}$ are coincidentally commuting, then v remains a unique common fixed point of f and g.

We now proceed to the main results.

2. Main results

In this section, we extend Theorem 1.9 to two pairs of non-self mappings. We prove the following theorem:

Theorem 2.1. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a closed subset of X with a nonempty boundary ∂C. Let $S, T, A, B\colon C \to X$. Let us assume that S, T, A and B satisfy the following conditions:*

(i) *For every $x, y \in C$, $p(Ax, By) \leqslant M_\omega(x, y)$ where*

$$M_\omega(x,y) = \max\{\omega_1[p(Sx,Ty)], \omega_2[p(Ax,Sx)], \omega_3[p(By,Ty)], \omega_4[p(Ax,Ty)], \omega_5[p(Sx,By)]\},$$

$\omega_i\colon [0,\infty) \to [0,\infty)$, $i = 1,2,3,4,5$, is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $\lim\limits_{r\to\infty}(r - 2\omega_i(r)) = \infty$.

(ii) *$\partial C \subseteq TC$, $\partial C \subseteq SC$,*

(iii) *$Sx \in \partial C \Rightarrow Ax \in C$; $Tx \in \partial C \Rightarrow Bx \in C$,*

(iv) *$AC \cap C \subset TC$, $BC \cap C \subset SC$ and*

(v) *SC, TC are closed in C.*

Then there exists a coincidence point $z \in C$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

P r o o f. Commencing with an arbitrary point $w \in \partial C$, we construct a sequence $\{x_n\}$ of points in C as follows:

By assumption (ii), there is a point $x_0 \in C$ such that $Sx_0 = w$. We find Ax_0. Then we proceed inductively as follows.

If $Ax_{2n} \in C$, then, by (iv), we choose $x_{2n+1} \in C$ such that $Tx_{2n+1} = Ax_{2n}$.

If however $Ax_{2n} \notin C$, because W is continuous in the third variable, it means that, by (iii), there is $\lambda_{2n,2n} \in (0,1)$ such that

$$W(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}) \in \partial C.$$

By (ii), this means we can choose $x_{n+1} \in C$ such that

$$Tx_{n+1} = W(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}) \in \partial C.$$

We then determine Bx_{2n+1}.

If $Bx_{2n+1} \in C$, then, by (iv), we choose $x_{2n+2} \in C$ such that $Sx_{2n+2} = Bx_{2n+1}$.

However if $Bx_{2n+1} \notin C$, because W is continuous in the third variable, this means there is $\lambda_{2n+1,2n+1} \in (0,1)$ such that

$$W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \partial C.$$

By (ii), this means we can choose $x_{n+2} \in C$ such that

$$Sx_{n+2} = W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \partial C.$$

We then determine Ax_{2n+2}.

We show that, for $n \geqslant 1$, we have

$$Ax_{2n} \neq Tx_{2n+1} \Rightarrow Bx_{2n-1} = Sx_{2n}. \tag{2.1}$$

Suppose we have $Bx_{2n-1} \neq Sx_{2n}$. Then we have $Sx_{2n} \in \partial C$, which by (iii) means $Ax_{2n} \in C$. This, by (iv), implies that $Ax_{2n} = Tx_{2n+1}$, which is a contradiction. Using a similar argument we have

$$Bx_{2n+1} \neq Sx_{2n+2} \Rightarrow Ax_{2n} = Tx_{2n+1}. \tag{2.2}$$

We now prove that the sequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ are bounded. For each $n \geqslant 1$ let

$$D_n = \bigcup_{i=0}^{n-1}\{Ax_{2i}\} \cup \bigcup_{i=0}^{n-1}\{Bx_{2i+1}\} \cup \bigcup_{i=0}^{n-1}\{Sx_{2i}\} \cup \bigcup_{i=0}^{n-1}\{Tx_{2i+1}\}, \quad n \geqslant 1.$$

Let $\alpha_n = \operatorname{diam}(D_n)$. We show that

$$\alpha_n \leqslant \max\{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1})\}, \quad 0 \leqslant j \leqslant n-1. \tag{2.3}$$

Let us consider the case where $\alpha_n = 0$, $n \geqslant 1$.

This means $p(Sx_0, Ax_0) = p(Ax_0, Bx_1) = p(Bx_1, Tx_1) = 0$. Applying (1.2) this means

$$Sx_0 = Ax_0 = Bx_1 = Tx_1.$$

We shall show that Sx_0 is a common fixed point of S and A. As the mappings S and A are coincidentally commuting at the coincidence point x_0, we have

$$Sx_0 = Ax_0 \Rightarrow SSx_0 = SAx_0 = ASx_0. \tag{2.4}$$

From (i), we have

$$
\begin{aligned}
p(SSx_0, Sx_0) &= p(ASx_0, Bx_1) \leqslant M_\omega(Sx_0, x_1) \\
&= \max\{\omega_1[p(SSx_0, Tx_1)], \omega_2[p(ASx_0, SSx_0)], \omega_3[p(Bx_1, Tx_1)], \\
&\qquad \omega_4[p(ASx_0, Tx_1)], \omega_5[p(SSx_0, Bx_1)]\} \\
&= \max\{\omega_1[p(SSx_0, Sx_0)], \omega_2[p(SSx_0, SSx_0)], \omega_3[p(Sx_0, Sx_0)], \\
&\qquad \omega_4[p(SSx_0, Sx_0)], \omega_5[p(SSx_0, Sx_0)]\} \\
&\leqslant \omega_t[p(SSx_0, Sx_0)] \quad \text{for some } t \in \{1,2,3,4,5\}, \\
&< \tfrac{1}{2} p(SSx_0, Sx_0) \quad \text{for } p(SSx_0, Sx_0) > 0 \\
&\Rightarrow p(SSx_0, Sx_0) = 0.
\end{aligned}
$$

By (1.2), this implies

$$
SSx_0 = Sx_0. \tag{2.5}
$$

Hence Sx_0 is a fixed point of S. From (2.4) we have $SSx_0 = ASx_0$. Thus (2.5) implies $ASx_0 = Sx_0$, making Sx_0 a fixed point of A too.

Using a similar argument we have $Tx_1 = Sx_0$ being a common fixed point of T and B. Hence, $z = Sx_0$ is a common fixed point of all four mappings S, T, A and B.

To show the uniqueness of the fixed point, let z' be also a fixed point of S, T, A and B. Then we have

$$
\begin{aligned}
p(z, z') &= p(Az, Bz') \\
&\leqslant \max\{\omega_1[p(Sz, Tz')], \omega_2[p(Az, Sz)], \omega_3[p(Bz', Tz')], \\
&\qquad \omega_4[p(Az, Tz')], \omega_5[p(Sz, Bz')]\} \\
&= \max\{\omega_1[p(z, z')], \omega_2[p(z, z)], \omega_3[p(z', z')], \\
&\qquad \omega_4[p(z, z')], \omega_5[p(z, z')]\} \\
&\leqslant \omega_i[p(z, z')] \quad \text{for some } i \in \{1,2,3,4,5\} \\
&< \tfrac{1}{2} p(z, z') \quad \text{for } p(z, z') > 0, \\
&\Rightarrow p(z, z') = 0 \\
&\Rightarrow z = z, \quad \text{by (1.1)}.
\end{aligned}
$$

Hence when $\alpha_n = 0$, then $z = Sx_0$ is the unique common fixed point of S, T, A and B.

We now consider the cases when $\alpha_n > 0$.

Case 1: Consider the case where $\alpha_n = p(Sx_{2i}, Ax_{2j})$ for some $0 \leqslant i, j \leqslant n-1$.

Subcase 1.1: If $i \geqslant 1$ and $Sx_{2i} = Bx_{2i-1}$ we have

$$\begin{aligned}\alpha_n = p(Sx_{2i}, Ax_{2j}) &= p(Ax_{2j}, Bx_{2i-1}) \leqslant M_\omega(x_{2j}, x_{2i-1})\\ &\leqslant \omega_s(\alpha_n) \quad \text{for some } s \in \{1,2,3,4,5\}\\ &< \tfrac{1}{2}\alpha_n,\end{aligned}$$

which is a contradiction. Hence $i = 0$.

Subcase 1.2: If however $i \geqslant 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it follows that $Sx_{2i} \in \operatorname{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and hence, by Lemma 1.7 (i), we have

$$\alpha_n = p(Sx_{2i}, Ax_{2j}) \leqslant \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i-2}, Ax_{2j})\}.$$

Subcase 1.2.1: If $p(Ax_{2j}, Bx_{2i-1}) \geqslant p(Ax_{2i-2}, Ax_{2j})$, we have

$$\alpha_n = p(Sx_{2i}, Ax_{2j}) \leqslant p(Ax_{2j}, Bx_{2i-1}),$$

which leads to the contradiction in Case 1.1, meaning that $i = 0$.

Subcase 1.2.2: Otherwise, if $p(Ax_{2j}, Bx_{2i-1}) < p(Ax_{2i-2}, Ax_{2j})$, then for some k such that $2i-2 < 2k+1 < 2j$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned}\alpha_n = p(Sx_{2i}, Ax_{2j}) &\leqslant p(Ax_{2i-2}, Ax_{2j})\\ &\leqslant p(Ax_{2i-2}, Bx_{2k+1}) + p(Ax_{2j}, Bx_{2k+1}), \text{ by (1.2)}\\ &\leqslant M_\omega(x_{2i-2}, x_{2i-1}) + M_\omega(x_{2j}, x_{2i-1})\\ &\leqslant \omega_s(\alpha_n) + \omega_t(\alpha_n)\\ &< \tfrac{1}{2}\alpha_n + \tfrac{1}{2}\alpha_n \Rightarrow \alpha_n < \alpha_n,\end{aligned}$$

which is a contradiction. Hence $i = 0$.

Case 2: The case where $\alpha_n = p(Ax_{2i}, Bx_{2j+1})$ leads to contradiction by Case 1.1.

Case 3: The case where $\alpha_n = p(Ax_{2i}, Ax_{2j})$ leads to contradiction by Case 1.2.2.

Case 4: If $\alpha_n = p(Bx_{2i+1}, Bx_{2j+1})$ then for k such that $2i+1 < 2k < 2j+1$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned}\alpha_n &= p(Bx_{2i+1}, Bx_{2j+1})\\ &\leqslant p(Ax_{2k}, Bx_{2i+1}) + p(Ax_{2k}, Bx_{2j+1})\\ &\leqslant M_\omega(x_{2k}, x_{2i+1}) + M_\omega(x_{2k}, x_{2j+1})\\ &\leqslant \omega_s(\alpha_n) + \omega_t(\alpha_n)\\ &< \tfrac{1}{2}\alpha_n + \tfrac{1}{2}\alpha_n \Rightarrow \alpha_n < \alpha_n,\end{aligned}$$

which is a contradiction.

Case 5: If $\alpha_n = p(Tx_{2i+1}, Bx_{2j+1})$ for some $0 \leqslant i, j \leqslant n-1$, then:

Subcase 5.1: If $Tx_{2i+1} = Ax_{2i}$, then we have

$$\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) = p(Ax_{2i}, Bx_{2j+1}),$$

which is a contradiction by Case 1.1.

Subcase 5.2: Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \operatorname{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence by Lemma 1.7 (i) we have

$$\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) \leqslant \max\{p(Bx_{2i-1}, Bx_{2j+1}), p(Ax_{2i}, Bx_{2j+1})\}.$$

This means we have either $p(Tx_{2i+1}, Bx_{2j+1}) \leqslant p(Bx_{2i-1}, Bx_{2j+1})$, which is a contradiction by Case 4 or $p(Tx_{2i+1}, Bx_{2j+1}) \leqslant p(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by Case 1.1.

Case 6: If $\alpha_n = p(Tx_{2i+1}, Ax_{2j})$ for some $0 \leqslant i, j \leqslant n-1$, then:

Subcase 6.1: If $Tx_{2i+1} = Ax_{2i}$, then we have

$$\alpha_n = p(Tx_{2i+1}, Ax_{2j}) = p(Ax_{2i}, Ax_{2j})$$

which is not possible by Case 1.2.2.

Subcase 6.2: Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$, then $Tx_{2i+1} \in \operatorname{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence $\alpha_n = p(Tx_{2i+1}, Ax_{2j}) \leqslant \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i}, Ax_{2j})\}$. This implies we have either $p(Tx_{2i+1}, Ax_{2j}) \leqslant p(Ax_{2j}, Bx_{2i-1})$, which is a contradiction by Case 1.1 or else we have $p(Tx_{2i+1}, Ax_{2j}) \leqslant p(Ax_{2i}, Ax_{2j})$, which is a contradiction by Case 1.2.2.

Case 7: Suppose $\alpha_n = p(Tx_{2i+1}, Tx_{2j+1})$ for some $0 \leqslant i, j \leqslant n-1$.

Subcase 7.1: If $Tx_{2j+1} = Ax_{2j}$, we have

$$\alpha_n = p(Tx_{2i+1}, Tx_{2j+1}) = p(Tx_{2i+1}, Ax_{2j}),$$

which is a contradiction by Case 6.

Subcase 7.2: Otherwise, if $Tx_{2j+1} \neq Ax_{2j}$, then $Tx_{2j+1} \in \operatorname{seg}[Bx_{2j-1}, Ax_{2j}]$ and hence $\alpha_n = p(Tx_{2i+1}, Tx_{j+1}) \leqslant \max\{p(Tx_{2i+1}, Bx_{2j-1}), p(Tx_{2i+1}, Ax_{2j})\}$.

This implies we have either $p(Tx_{2i+1}, Tx_{2j+1}) \leqslant p(Tx_{2i+1}, Bx_{2j-1})$, which results in a contradiction by Case 5, or else we have $p(Tx_{2i+1}, Tx_{2j+1}) \leqslant p(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

Case 8: Let $\alpha_n = p(Sx_{2i}, Bx_{2j+1})$ for some $0 \leqslant i, j \leqslant n-1$.

Subcase 8.1: If $i \geqslant 1$ and $Sx_{2i} = Bx_{2i-1}$, then $\alpha_n = p(Sx_{2i}, Bx_{2j+1}) = p(Bx_{2i-1}, Bx_{2j+1})$, which is not possible as per Case 4. Hence $i = 0$.

Subcase 8.2: If however $i \geqslant 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it means that $Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$. This implies that $\alpha_n = p(Sx_{2i}, Bx_{2j+1}) \leqslant \max\{p(Ax_{2i-2}, Bx_{2j+1}), p(Bx_{2i-1}, Bx_{2j+1})\}$. This leads to a contradiction by Case 1.1 when $p(Sx_{2i}, Bx_{2j+1}) \leqslant p(Ax_{2i-2}, Bx_{2j+1})$ and a contradiction by Case 4 when it happens that $p(Sx_{2i}, Bx_{2j+1}) \leqslant p(Bx_{2i-1}, Bx_{2j+1})$. Hence $i = 0$.

Case 9: Let us consider the case when $\alpha_n = p(Sx_{2i}, Sx_{2j})$ for some $0 \leqslant i < j \leqslant n-1$.

Subcase 9.1: If $i \geqslant 1$ and $Sx_{2j} = Bx_{2j-1}$, then we have $\alpha_n = p(Sx_{2i}, Sx_{2j}) = p(Sx_{2i}, Bx_{2j-1})$, which leads to a contradiction according to Case 8. Hence $i = 0$.

Subcase 9.2: If $i \geqslant 1$ and $Sx_{2j} \neq Bx_{2j-1}$, it follows that $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$ and $p(Sx_{2i}, Sx_{2j}) \leqslant \max\{p(Sx_{2i}, Ax_{2j-2}), p(Sx_{2i}, Bx_{2j-1})\}$.

If it happens $p(Sx_{2i}, Sx_{2j}) \leqslant p(Sx_{2i}, Ax_{2j-2})$, we get a contradiction by Case 1. However if it happens that $p(Sx_{2i}, Sx_{2j}) \leqslant p(Sx_{2i}, Bx_{2j-1})$, then we get a contradiction by Case 8. Hence $i = 0$.

Case 10: Suppose $\alpha_n = p(Sx_{2i}, Tx_{2j+1})$ for some $0 \leqslant i, j \leqslant n-1$.

Subcase 10.1: If $i \geqslant 1$ and $Sx_{2i} = Bx_{2i-1}$, then we have $\alpha_n = p(Sx_{2i}, Tx_{2j+1}) = p(Tx_{2j+1}, Bx_{2i-1})$, which is not possible as per Case 8. Hence $i = 0$.

Subcase 10.2: If however $i \geqslant 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it follows that $Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and

$$p(Sx_{2i}, Tx_{2j+1}) \leqslant \max\{p(Tx_{2j+1}, Ax_{2i-2}), p(Tx_{2j+1}, Bx_{2i-1})\}.$$

This leads to contradictions by Case 6 and Case 5. Hence $i = 0$.

We have considered ten possible cases for α_n and conclude from Cases 1, 8, 9 and 10 that for some $0 \leqslant j \leqslant n-1$ we have

$$\alpha_n \in \{p(Sx_0, Sx_{2j}), p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1}), p(Sx_0, Tx_{2j+1})\}. \tag{2.6}$$

Note that, from the construction of the sequence, $Sx_{2j} \in C$ implies $Sx_{2j} = Bx_{2j-1}$. This leads to

$$p(Sx_0, Sx_{2j}) = p(Sx_0, Bx_{2j-1}). \tag{2.7}$$

For $Sx_{2j} \notin C$, we have $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$. By Lemma 1.7 (i), this implies

$$p(Sx_0, Sx_{2j}) \leqslant \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}. \tag{2.8}$$

From (2.7) and (2.8) we conclude that

$$p(Sx_0, Sx_{2j}) \leqslant \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}. \tag{2.9}$$

Using a similar argument we also have

$$p(Sx_0, Tx_{2j+1}) \leqslant \max\{p(Sx_0, Bx_{2j-1}), p(Sx_0, Ax_{2j})\}. \tag{2.10}$$

Applying (2.9) and (2.10) to (2.6) we get

$$\alpha_n \leqslant \max\{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1})\}, \quad 0 \leqslant j \leqslant n-1.$$

We have proved (2.3).

Consider the case where $\max\{p(Sx_0, Ax_{2j})\} \leqslant \max\{p(Sx_0, Bx_{2j+1})\}$, $0 \leqslant j \leqslant n-1$. Then, for some $u \in \{1, 2, \ldots, 5\}$, (2.3) implies

$$\begin{aligned}
\alpha_n &\leqslant p(Sx_0, Bx_{2j+1}) \\
&\leqslant p(Sx_0, Ax_0) + p(Ax_0, Bx_{2j+1}), \quad \text{by (1.2)} \\
&\leqslant p(Sx_0, Ax_0) + \omega_u[\alpha_n] \\
&\leqslant p(Sx_0, Ax_0) + 2\omega_u[\alpha_n] \\
&\Rightarrow \alpha_n - 2\omega_u[\alpha_n] \leqslant p(Sx_0, Ax_0).
\end{aligned}$$

Consider now the case when $\max\{p(Sx_0, Ax_{2j})\} > \max\{p(Sx_0, Bx_{2j+1})\}$, $0 \leqslant j \leqslant n-1$. Then for some $v \in \{1, 2, \ldots, 5\}$, (2.3) implies

$$\begin{aligned}
\alpha_n &\leqslant p(Sx_0, Ax_{2j}) \\
&\leqslant p(Sx_0, Ax_0) + p(Ax_0, Ax_{2j}), \text{ by (1.2)} \\
&\leqslant p(Sx_0, Ax_0) + 2\omega_v[\alpha_n], \quad \text{by Case 1.2.2} \\
&\Rightarrow \alpha_n - 2\omega_v[\alpha_n] \leqslant p(Sx_0, Ax_0).
\end{aligned}$$

Thus in both cases, we have for some $s \in \{1, 2, \ldots, 5\}$

$$\alpha_n - 2\omega_s[\alpha_n] \leqslant p(Sx_0, Ax_0). \tag{2.11}$$

By assumption (i) there is an $r_0 \in [0, \infty)$ such that for each $s \in \{1, 2, \ldots, 5\}$ we have $r - 2\omega_s(r) > p(Sx_0, Ax_0)$ for $r > r_0$.

There is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ and $s \in \{1, 2, \ldots, 5\}$ such that for each n, we have

$$a_n - 2\omega_s[a_n] \leqslant p(Sx_0, Ax_0).$$

Thus by (2.11), $a_n \leqslant r_0$. Thus we have

$$a = \lim_{n\to\infty} a_n = \operatorname{diam}(D) \leqslant r_0.$$

We have hence proved that $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ are bounded sequences.

To prove that $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge in C, we consider the set

$$E_n = \bigcup_{i=n}^{\infty}\{Ax_{2i}\} \cup \bigcup_{i=n}^{\infty}\{Bx_{2i+1}\} \cup \bigcup_{i=n}^{\infty}\{Sx_{2i}\} \cup \bigcup_{i=n}^{\infty}\{Tx_{2i+1}\}, \tag{2.12}$$

$n = 2, 3, \ldots$

By (2.3) we have for $n = 2, 3, \ldots$

$$e_n := \operatorname{diam}(E_n) \leqslant \sup_{j\geqslant n}\{p(Sx_{2n}, Ax_{2j}), p(Sx_{2n}, Bx_{2j+1})\}. \tag{2.13}$$

If $Sx_{2n} = Bx_{2n-1}$ we have as in Case 1 and Case 8, for each $j \geqslant n$ and for some $u \in \{1, 2, \ldots, 5\}$

$$\begin{aligned} e_n &\leqslant \sup_{j\geqslant n}\{p(Ax_{2j}, Bx_{2n-1}), p(Bx_{2j+1}, Bx_{2n-1})\}, \quad n = 2, 3 \ldots \\ &\leqslant 2\omega_u[e_{n-1}]. \end{aligned} \tag{2.14}$$

If however $Sx_{2n} \neq Bx_{2n-1}$, it follows that $Sx_{2n} \in \operatorname{seg}[Ax_{2n-2}, Bx_{2n-1}]$. Hence, as in Case 1 and Case 8, for each $j \geqslant n$ and for some $v \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned} e_n \leqslant \sup_{j\geqslant n}\{&p(Ax_{2n-2}, Ax_{2j}), p(Bx_{2n-1}, Ax_{2j}), \\ &p(Ax_{2n-2}, Bx_{2j+1}), p(Bx_{2n-1}, Bx_{2j+1})\}, \quad n = 2, 3, \ldots \\ \leqslant 2\omega_v[e_{n-2}].& \end{aligned} \tag{2.15}$$

By (2.14) and (2.15), there is a subsequence $\{\varepsilon_n\}$ of $\{e_n\}$ and $s \in \{1, 2, \ldots, 5\}$ such that for each n

$$\begin{aligned} \varepsilon_n &\leqslant 2\omega_s[\varepsilon_{n-2}], \quad n = 2, 3, \ldots \\ &< \varepsilon_{n-2}. \end{aligned} \tag{2.16}$$

We note that $e_n \geqslant e_{n+1}$ for every n. Let $\lim\limits_{n\to\infty} e_n = \lim\limits_{n\to\infty} \varepsilon_n = e$. We claim that $e = 0$.

If $e > 0$, then by (2.16) and assumption (i) of Theorem 2.1 we have

$$\lim_{n\to\infty} \varepsilon_n < \lim_{n\to\infty} \varepsilon_{n-2} \Rightarrow e < e,$$

which is a contradiction. Hence $e = 0$.

We recall from (2.13) that $e_n = \mathrm{Diam}(E_n)$. Taking $n, m \to \infty$ in (2.12), we get

$$\lim_{n,m\to\infty} p(Ax_{2n}, Ax_{2m}) = \lim_{n,m\to\infty} p(Bx_{2n+1}, Bx_{2m+1}) = 0. \tag{2.17}$$

This means both $\{A_{2n}\}$ and $\{B_{2n+1}\}$ are Cauchy sequences.

Because X is a complete partial metric space, this means there is $z \in X$ such that

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = z \quad \text{and} \quad p(z,z) = 0. \tag{2.18}$$

Consider the subsequence Sx_{2n_k} of Sx_{2n} such that $Sx_{2n_k} = Bx_{2n_k-1}$. Taking $n_k \to \infty$ we have

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Sx_{2n_k} = \lim_{n\to\infty} Bx_{2n_k-1} = z, \quad \text{with } p(z,z) = 0. \tag{2.19}$$

Using a similar argument we have

$$\lim_{n\to\infty} Tx_{2n+1} = z, \quad \text{with } p(z,z) = 0. \tag{2.20}$$

But both SC, TC are 0-complete. This implies $z \in SC$ and $z \in TC$.

As $z \in SC$, there is a point $u \in C$ such that $Su = z$. We show that u is a coincidence point of A, B and S:

$$\begin{aligned} p(Au, Bx_{2n+1}) &\leqslant \max\{\omega_1[p(Su, Tx_{2n+1})], \omega_2[p(Au, Su)], \omega_3[p(Bx_{2n+1}, Tx_{2n+1})], \\ &\qquad \omega_4[p(Au, Tx_{2n+1})], \omega_5[p(Su, Bx_{2n+1})]\} \\ &= \max\{\omega_1[p(z, Tx_{2n+1})], \omega_2[p(Au, z)], \omega_3[p(Bx_{2n+1}, Tx_{2n+1})], \\ &\qquad \omega_4[p(Au, Tx_{2n+1})], \omega_5[p(z, Bx_{2n+1})]\}. \end{aligned}$$

Taking $n \to \infty$ and applying (2.19) and (2.20), we get

$$\begin{aligned} p(Au, z) &\leqslant \max\{\omega_1[p(z,z)], \omega_2[p(Au,z)], \omega_3[p(z,z], \omega_4[p(Au,z)], \omega_5[p(z,z)]\} \\ &\leqslant \omega_i[p(Au,z)] \quad \text{for some } i \in \{1,2,\dots,5\} \\ &< p(Au,z) \quad \text{for } p(Au,z) > 0 \\ &\Rightarrow p(Au,z) = 0 \\ &\Rightarrow Au = z, \quad \text{from (1.1).} \end{aligned} \tag{2.21}$$

Using a similar procedure, when we expand $p(Ax_{2n}, Bu)$, we get $Bu = z$, making u a coincidence point of A, B and S. By the coincidental commutativity of S and A, we have $SAu = ASu \Rightarrow Sz = Az$.

In the same vein, $z \in TC$ means there is $v \in C$ such that $Tv = z$. We show that $Bv = z$:

$$\begin{aligned}
p(z, Bv) &= p(Au, Bv) \\
&\leqslant \max\{\omega_1[p(Su, Tv)], \omega_2[p(Su, Au)], \omega_3[p(Tv, Bv)], \\
&\qquad \omega_4[p(Au, Tv)], \omega_5[p(Su, Bv)]\} \\
&= \max\{\omega_1[p(z, z)], \omega_2[p(z, z)], \omega_3[p(z, Bv)], \\
&\qquad \omega_4[p(z, z)], \omega_5[p(z, Bv)]\} \\
&\leqslant \omega_j[p(z, Bv)] \quad \text{for some } j \in \{1, 2, \ldots, 5\} \\
&< p(z, Bv) \quad \text{for } p(z, Bv) > 0 \\
&\Rightarrow p(z, Bv) = 0 \\
&\Rightarrow Bv = z, \quad \text{from (1.1).}
\end{aligned}$$

Thus v is a coincidence point of B and T. By the coincidental commutativity property, we have

$$BTv = TBv \Rightarrow Bz = Tz.$$

Now the following holds:

$$\begin{aligned}
p(Az, Bz) &\leqslant \max\{\omega_1[p(Sz, Tz)], \omega_2[p(Sz, Az)], \omega_3[p(Tz, Bz)], \\
&\qquad \omega_4[p(Az, Tz)], \omega_5[p(Sz, Bz)]\} \\
&= \max\{\omega_1[p(Az, Bz)], \omega_2[p(Az, Az)], \omega_3[p(Bz, Bz)], \\
&\qquad \omega_4[p(Az, Bz)], \omega_5[p(Az, Bz)]\} \\
&\leqslant \omega_i[p(Az, Bz)] \quad \text{for } i \in \{1, 2, \ldots, 5\} \\
&< p(Az, Bz) \quad \text{for } p(Az, Bz) > 0 \\
&\Rightarrow p(Az, Bz) = 0 \\
&\Rightarrow Az = Bz.
\end{aligned}$$

Hence we have

$$Az = Bz = Sz = Tz. \tag{2.22}$$

Now the following holds:

$$\begin{aligned}
p(z, Bz) &= p(Au, Bz), \quad \text{from (2.21)} \\
&\leqslant \max\{\omega_1[p(Su, Tz)], \omega_2[p(Au, Su)], \omega_3[p(Bz, Tz)], \\
&\qquad \omega_4[p(Au, Tz)], \omega_5[p(Su, Bz)]\} \\
&= \max\{\omega_1[p(z, Bz)], \omega_2[p(z.z)], \omega_3[p(Bz, Bz)], \omega_4[p(z, Bz)], \omega_5[p(z, Bz)]\}.
\end{aligned}$$

This implies

$$\begin{aligned}
p(z,Bz) &\leqslant \omega_j[p(z,Bz)] \quad \text{for some } j\in\{1,2,\ldots,5\}\\
&< p(z,Bz) \quad \text{for } p(z,Bz)>0\\
&\Rightarrow p(z,Bz)=0\\
&\Rightarrow Bz=z, \quad \text{by (1.1)}.
\end{aligned}$$

From (2.22) we conclude that $Az = Bz = Sz = Tz = z$, meaning that z is a common fixed point of A, B, S and T.

We now show that z is unique. Suppose z' is also a common fixed point of A, B, S and T. We get

$$\begin{aligned}
p(z,z') &= p(Az,Bz')\\
&\leqslant \max\{\omega_1[p(Sz,Tz')],\omega_2[p(Az,Sz)],\omega_3[p(Bz',Tz')],\\
&\qquad \omega_4[p(Az,Tz')],\omega_5[p(Sz,Bz')]\}\\
&= \max\{\omega_1[p(z,z')],\omega_2[p(z,z)],\omega_3[p(z',z')],\\
&\qquad \omega_4[p(z,z')],\omega_5[p(z,z')]\}.
\end{aligned}$$

This implies

$$\begin{aligned}
p(z,z') &\leqslant \omega_k[p(z,z'] \quad \text{for } k\in\{1,2,\ldots,5\}\\
&< p(z,z') \quad \text{for } p(z,z')>0\\
&\Rightarrow p(z,z')=0\\
&\Rightarrow z=z'.
\end{aligned}$$

This proves that the common fixed point of A, B, S and T is unique. □

R e m a r k 2.2. Theorem 2.1 leads to corollaries if we consider the following cases:

(i) $A = B$;
(ii) $A = B$, $S = T$, we get Theorem 1.9;
(iii) $A = B$, $S = T = I$, we get an extension of a theorem proved by Ćirić (see [2]) into partial metric spaces;
(iv) $A = I$;
(v) $S = T$;
(vi) $x = y$.

The proof given above works even when we define C as closed in (X, p^s). This leads to the following theorem.

Theorem 2.3. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a closed subset* *of X, the closure being taken with respect to (X, p^s). Let ∂C, the boundary of C in (X, p^s), be nonempty. Let $A, B, S, T\colon C \to X$. Let us assume that A, B, S and T satisfy the following conditions:*

(i) *For every $x, y \in C, p(Ax, By) \leqslant M_\omega(x, y)$ where $M_\omega(x, y) = \max\{\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(By, Ty)], \omega_4[p(Ax, Ty)], \omega_5[p(Sx, By)]\}$, $\omega_i\colon \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2, 3, 4, 5$, is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $\lim\limits_{r\to\infty}(r - 2\omega_i(r)) = \infty$.*

(ii) *$\partial C \subseteq SC$, $\partial C \subseteq TC$,*

(iii) *$Sx \in \partial C \Rightarrow Ax \in C$; $Tx \in \partial C \Rightarrow Bx \in C$,*

(iv) *$AC \cap C \subset TC$, $BC \cap C \subset SC$ and*

(v) *SC, TC are closed in C.*

Then there exists a coincidence point $z \in C$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

Here we give an example on the use of Theorem 2.3, as it is better suited for the partial metric that we will use.

Example 2.4. Consider the partial metric space $(\mathbb{R}_+, p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$. Let $C = [0, 2]$. We note that C is closed in the derived metric $p^s(x, y) = |x - y|$ and $\partial C = \{0, 2\}$.

We define the mappings $A, B, S, T\colon C \to \mathbb{R}_+$ as follows:

$$Ax = \begin{cases} 3^x - 1, & x \in [0, 1], \\ 1, & x \in (1, 2], \end{cases} \qquad Bx = \begin{cases} 4^x - 1, & x \in [0, 1], \\ 2, & x \in (1, 2], \end{cases}$$
$$Sx = \begin{cases} 27^x - 1, & x \in [0, 1], \\ 6, & x \in (1, 2], \end{cases} \qquad Tx = \begin{cases} 64^x - 1, & x \in [0, 1], \\ 8, & x \in (1, 2]. \end{cases}$$

We have $AC = [0, 2]$ and $BC = [0, 3]$. We also have $TC = [0, 63]$ and $SC = [0, 26]$, both of which are closed in (X, p^s).

$Sx \in \partial C$ implies $z \in \{0, \frac{1}{3}\} \subset C$. Similarly $Tx \in \partial C$ implies $x \in \{0, \frac{\ln 3}{\ln 64}\} \subset C$. We also have $\partial K = \{1, 3\} \subseteq SC, TC$.

We note that both $\{S, A\}$ and $\{T, B\}$ are coincidentally commuting at $x = 0$, that is, $SA(0) = AS(0)$ and $TB(0) = BT(0)$. We also note that all four mappings are discontinuous at 1.

Let us define functions $g, h\colon C \to \mathbb{R}_+$ as

$$g(x) = \frac{27^x - 1}{4^x - 1}, \quad h(x) = \frac{64^x - 1}{4^x - 1}.$$

Both g and h are increasing functions. Using L'Hôpital rule, we can show that, as $x \to 0$, we have $h(x) \to 3$ and

$$g(x) \to \frac{\log 27}{\log 4} = \frac{1}{0.42062}. \tag{2.23}$$

Hence, for $x \in [0,1]$, we have

$$4^x - 1 \leqslant \frac{1}{3}(64^x - 1) \leqslant 0.43(64^x - 1). \tag{2.24}$$

We also have, from (2.23)

$$4^x - 1 \leqslant 0.42062(27^x - 1) \Rightarrow 4^x - 1 \leqslant 0.43(27^x - 1). \tag{2.25}$$

When $x, y \in [0,1]$ with $x \leqslant y$, we have

$$\begin{aligned} p(Ax, By) &= p(3^x - 1, 4^y - 1) \\ &= \max\{3^x - 1, 4^y - 1\} \\ &= 4^y - 1, \quad \text{because } x \leqslant y \\ &\leqslant 0.43(64^y - 1), \quad \text{by (2.24)} \\ &= 0.43Ty \\ &\leqslant 0.43 \max\{Sx, Ty\} \\ &= 0.43p(Sx, Ty). \end{aligned}$$

When $x, y \in [0,1]$ with $x > y$, we have

$$\begin{aligned} p(Ax, By) &= p(3^x - 1, 4^y - 1) \\ &= \max\{4^x - 1, 4^y - 1\}, \quad \text{because } 3^x - 1 \leqslant 4^x - 1 \\ &= 4^x - 1, \text{ because } x > y \\ &\leqslant 0.43(27^x - 1), \quad \text{by (2.25)} \\ &= 0.43Sx \\ &\leqslant 0.43 \max\{Sx, Ty\} \\ &= 0.43p(Sx, Ty). \end{aligned}$$

For $x, y \in (1, 2]$ we have

$$\begin{aligned} p(Ax, By) &= p(1, 2) \\ &= \max\{1, 2\} \\ &= 2 \\ &< 0.43 \times 6 \\ &\leqslant 0.43 \max\{6, Ty\} \\ &= 0.43 \max\{Sx, Ty\}, \quad \text{because } Sx = 6 \\ &= 0.43 p(Sx, Ty). \end{aligned}$$

Considering $x \in [0, 1]$, $y \in (1, 2]$ we get

$$\begin{aligned} p(Ax, By) &= p(3^x - 1, 2) \\ &= \max\{3^x - 1, 2\} \\ &= 2, \quad \text{because } 3^x - 1 \leqslant 2 \text{ for } x \in [0, 1] \\ &< 0.43 \times 8 \\ &= 0.43 Ty \quad \text{because } Ty = 8 \\ &\leqslant 0.43 \max\{Sx, Ty\} \\ &= 0.43 p(Sx, Ty). \end{aligned}$$

Considering $x \in (1, 2]$, $y \in (0, 1)$.

$$p(Ax, By) = p(1, 4^y - 1) = \max\{1, 4^y - 1\} = \begin{cases} 1, & \\ 6, & y \in (\frac{1}{2}, 1]. \end{cases}$$

For $y \in [0, \frac{1}{2}]$ we have

$$\begin{aligned} p(Ax, By) &= p(1, 4^y - 1) \\ &= 1 \\ &< 0.43 \times 6 \\ &= 0.43 Sx \quad \text{because } Sx = 6 \\ &\leqslant 0.43 \max\{Sx, Ty\} \\ &= 0.43 p(Sx, Ty). \end{aligned}$$

For $y \in (\frac{1}{2}, 1]$ we have

$$\begin{aligned} p(Ax, By) &= p(1, 4^y - 1) \\ &= 4^y - 1 \end{aligned}$$

$$
\begin{aligned}
&\leqslant 0.43(64^y - 1), \quad \text{by } (2.24)\\
&= 0.43Ty\\
&\leqslant 0.43\max\{Sx, Ty\}\\
&= 0.43p(Sx, Ty).
\end{aligned}
$$

Thus in all cases, for every $x, y \in C$ we have

$$p(Ax, By) \leqslant 0.43p(Sx, Ty) \leqslant M_\omega(x, y),$$

where

$$
\begin{aligned}
M_\omega(x, y) = \max\{&\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(Ay, Ty)],\\
&\omega_4[p(Ax, Ty)], \omega_5[p(Sx, Ay)]\},
\end{aligned}
$$

with $\omega_i(r) = 0.43r$. We note that $\omega_i(r) = 0.43r < \frac{1}{2}r$. As $r - 2\omega_i(r) = r - 2\times 0.43r = 0.14r$, we also have $\lim\limits_{r\to\infty} r - 2\omega_i(r) = \lim\limits_{r\to\infty} 0.14r = \infty$.

Thus all the conditions of Theorem 2.3 are satisfied and 0 is the unique common fixed point of A, B, S and T.

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