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FINITE p-NILPOTENT GROUPS WITH SOME SUBGROUPS WEAKLY $\mathcal{M}$-SUPPLEMENTED

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The paper is dedicated to Professor Shaoxue Liu for his 80th birthday.

Abstract. Suppose that G is a finite group and H is a subgroup of G. Subgroup H is said to be weakly $\mathcal{M}$-supplemented in G if there exists a subgroup B of G such that (1) $G = HB$, and (2) if H_1/H_G is a maximal subgroup of H/H_G, then $H_1B = BH_1 < G$, where H_G is the largest normal subgroup of G contained in H. We fix in every noncyclic Sylow subgroup P of G a subgroup D satisfying $1 < |D| < |P|$ and study the p-nilpotency of G under the assumption that every subgroup H of P with $|H| = |D|$ is weakly $\mathcal{M}$-supplemented in G. Some recent results are generalized.

Keywords: p-nilpotent group; weakly $\mathcal{M}$-supplemented subgroup; finite group

MSC 2010: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation. G always means a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$.

A subgroup H of G is called $\mathcal{M}$-supplemented in a finite group G if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for every maximal subgroup H_1 of H. This concept was introduced by Miao and Lempken in [5]. More recently, in [6] they generalized $\mathcal{M}$-supplemented subgroups to weakly $\mathcal{M}$-supplemented subgroups. A subgroup H of G is said to be weakly $\mathcal{M}$-supplemented in G if there exists a subgroup B of G such that (1) $G = HB$, and (2) if H_1/H_G is a maximal subgroup of H/H_G, then $H_1B = BH_1 < G$, where H_G is the largest normal subgroup of G contained in H. In this case, B is also called

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a weak $\mathcal{M}$-supplement of H in G. Clearly, every $\mathcal{M}$-supplemented subgroup of G is a weakly $\mathcal{M}$-supplemented subgroup of G, but the converse does not hold. The authors use weakly $\mathcal{M}$-supplemented subgroups when investigating the structure of G, as in [4] and [6]. For example, Miao in [4] proves the following result.

Theorem 1.1. *Let p be an odd prime divisor of $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and supposing that P has a subgroup D such that $1 < D < P$, and every subgroup E of P with order $|D|$ is weakly $\mathcal{M}$-supplemented in G, then G is p-nilpotent.*

A celebrated theorem of Frobenius (see [2], Satz IV.5.8) asserts that G is p-nilpotent if $N_G(H)$ is p-nilpotent for every p-subgroup H of G. In this article, we replace some of the conditions of the Frobenius theorem and Theorem 1.1, namely, H is restricted to be a p-subgroup of a fixed order, the condition of p-nilpotency of $N_G(P)$ is changed to the p-nilpotency of $N_G(H)$, and we assume that H is a weakly $\mathcal{M}$-supplemented subgroup of G. The results of this article can be viewed as extensions of the Frobenius theorem and Theorem 1.1 with weakly $\mathcal{M}$-supplemented subgroups. Our main theorem is the following result.

Theorem 1.2. *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime. If P has a subgroup D with $1 < |D| < |P|$ such that all subgroups H of P with order $|H| = |D|$ are weakly $\mathcal{M}$-supplemented in G and $N_G(H)$ is p-nilpotent, then G is p-nilpotent.*

2. Preliminary results

In this section, we collect some known results that are useful later.

Lemma 2.1 ([6]). *Let G be a group.*

- (i) *If H is weakly $\mathcal{M}$-supplemented in G, $H \leqslant M \leqslant G$, then H is weakly $\mathcal{M}$-supplemented in M.*
- (ii) *Let $N \trianglelefteq G$ and $N \leqslant H$. Then H is weakly $\mathcal{M}$-supplemented in G if and only if H/N is weakly $\mathcal{M}$-supplemented in G/N.*
- (iii) *Let π be a set of primes. Let K be a normal π'-subgroup and H a π-subgroup of G. If H is weakly $\mathcal{M}$-supplemented in G, then HK/K is weakly $\mathcal{M}$-supplemented in G/K.*
- (iv) *Let R be a solvable minimal normal subgroup of the group G and R_1 be a maximal subgroup of R. If R_1 is weakly $\mathcal{M}$-supplemented in G, then R is a cyclic group of prime order.*

(v) *Let P be a p-subgroup of G, where p is a prime divisor of $|G|$. If P is weakly $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $|G:TB| = p$ for every maximal subgroup T of P containing P_G.*

Lemma 2.2 ([2], Satz IV.5.4)**.** *Suppose that p is a prime and G is a minimal non-p-nilpotent group, i.e., G is not a p-nilpotent group but every proper subgroup of G is p-nilpotent. Then:*

(i) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(iii) *The exponent of P is p or 4.*

3. Main results

In this section, we prove our main results.

Theorem 3.1. *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime. If each maximal subgroup P_1 of P is weakly $\mathcal{M}$-supplemented in G and $N_G(P_1)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Assume that the theorem is not true and let G be a counterexample of minimal order. We derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Consider $G/O_{p'}(G)$. Let $M/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $M = M \cap PO_{p'}(G) = (M \cap P)O_{p'}(G)$. Let $P_1 = M \cap P$. It is easy to see that P_1 is maximal in P. Let $N_{G/O_{p'}(G)}(P_1O_{p'}(G)/O_{p'}(G)) = K/O_{p'}(G)$. Then $P_1O_{p'}(G) \lhd K$, and thus $K = N_K(P_1)P_1O_{p'}(G) = N_G(P_1)O_{p'}(G) \leqslant K$; that is,

$$N_{G/O_{p'}(G)}(P_1O_{p'}(G)/O_{p'}(G)) = N_G(P_1)O_{p'}(G)/O_{p'}(G).$$

By the hypothesis, $N_G(P_1)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. Then by Lemma 2.1, we have that $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

Step 2. Let T be a subgroup of G such that $P \leqslant T < G$, then T is p-nilpotent.

Let P_1 be a maximal subgroup of P. Obviously, $N_T(P_1) \leqslant N_G(P_1)$. By the hypothesis, we have $N_T(P_1)$ is p-nilpotent and by Lemma 2.1 P_1 is weakly $\mathcal{M}$-supplemented in T. Hence T satisfies the hypothesis of the theorem. The minimality of G forces that T is p-nilpotent.

Step 3. $O_p(G)$ is the unique minimal normal subgroup of G and $G/O_p(G)$ is p-nilpotent. Moreover, $\Phi(G) = 1$.

Since G is not p-nilpotent, by the Glauberman-Thompson Theorem we have that $N_G(Z(J(P)))$ is not p-nilpotent, where $J(P)$ is the Thompson subgroup of P. Noticing that $Z(J(P))$ is a characteristic subgroup of P, we get $N_G(P) \leqslant N_G(Z(J(P)))$. By Step 2, we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N = P$, then obviously G/N is p-nilpotent. If N is maximal in P, then by the hypothesis $G = N_G(N)$ is p-nilpotent, a contradiction. Hence we may assume that $|P : N| \geqslant p^2$. By Lemma 2.1, it is easy to see that G/N satisfies the hypothesis of the theorem, so the choice of G yields that G/N is p-nilpotent. Next we prove the uniqueness of N. If $O_p(G)$ contains a second minimal normal subgroup M of G then both G/N and G/M are p-nilpotent by the choice of G, and so $G \cong G/(M \cap N) \leqslant G/M \times G/N$ shows that G is p-nilpotent contrary to hypothesis. If $\Phi(G) \neq 1$, then by Lemma 2.1 and Step 1, it is easy to see that $G/\Phi(G)$ satisfies the hypothesis of the theorem, so G is p-nilpotent contrary to hypothesis. Thus $\Phi(G) = 1$. Now we show $N = O_p(G)$. Lemma 2.6 in [3] shows that if $K \neq 1$ is a normal subgroup of any finite group G and $K \cap \Phi(G) = 1$, then the Fitting subgroup $F(K)$ of K lies in the socle $\mathrm{Soc}(G)$ and therefore $F(K)$ is the direct product of minimal normal subgroups of G contained in $F(K)$. In our case, since N is the unique minimal normal p-subgroup of G, applying this lemma with $K = O_p(G) = F(K)$ shows that $O_p(G)$ is equal to N.

Step 4. G is p-solvable, $C_G(O_p(G)) \leqslant O_p(G)$.

By Step 3, the p-solvablity of G is obvious. So $C_G(O_p(G)) \leqslant O_p(G)$ follows from Step 1 and [7], Theorem 9.3.1.

Step 5. $G = PQ$, where Q is a Sylow q-subgroup of G with $q \neq p$.

For each prime $q \in \pi(G)$ and $q \neq p$, there exists a Sylow q-subgroup Q of G such that $G_1 = PQ$ is a subgroup of G by Step 4 and [1], Theorem 6.3.5. If $G_1 < G$, then Step 2 forces that G_1 is p-nilpotent and so $Q \trianglelefteq G_1$. Thus we have $NQ = N \times Q$. It follows that $Q \leqslant C_G(N) = C_G(O_p(G))$, which contradicts Step 4. Hence $G_1 = G$, that is, $G = PQ$.

Step 6. $|N| = p$ and $P \cap M$ is maximal in P.

By Step 3, $\Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Clearly, $P = N(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leqslant P_1$. First we may assume $P \cap M < P_1$. By hypotheses, P_1 is weakly $\mathcal{M}$-supplemented in G. There exists a subgroup B such that $G = P_1 B$ and $TB < G$ for every maximal subgroup $(P_1)_G \leqslant T$. If $(P_1)_G \neq 1$, then we have $N \leqslant (P_1)_G \leqslant P_1$, a contradiction. So we have $(P_1)_G = 1$. By Lemma 2.1 (v), $|G : TB| = p$ for every maximal subgroup T of P_1. Particularly, there exists at least a maximal subgroup T of P_1 such that $N \nleqslant TB$. We may choose a maximal subgroup T of P_1 such that $P \cap M \leqslant T$. Clearly, $N \nleqslant TB$. Otherwise, $N \leqslant TB$ and $TB = NTB = PB = G$, a contradiction. Thus $P \cap M = P_1$ is maximal in P and $|N| = p$.

Step 7. The final contradiction.

Let Q_1 be a Sylow q-subgroup of M such that $M = (P \cap M)Q_1$. If $p < q$, then by [8], Lemma 2.8, $O_p(G)Q_1$ is p-nilpotent, and so $Q_1 \leqslant C_G(O_p(G))$, which contradicts Step 4. So $q < p$. By Step 5, G is solvable. Then by Step 3, we have $F(G) = N = C_G(N)$. It follows that $M \cong G/N = N_G(N)/C_G(N)$, which is isomorphic to a subgroup of $\mathrm{Aut}(N)$. Because $|N| = p$ by Step 6, $\mathrm{Aut}(N)$ is a cyclic group of order $p - 1$. It follows that M is cyclic, and so $Q_1 \leqslant N_G(P \cap M)$. Since $P \cap M$ is maximal in P, we have $P \cap M \trianglelefteq P$ and $G = PM = PQ_1 \leqslant N_G(P \cap M)$. Now by the hypothesis $G = N_G(P \cap M)$ is p-nilpotent, the final contradiction. □

P r o o f of Theorem 1.2. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 2.1 guarantees that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. Let T be a subgroup of G such that $P \leqslant T < G$, then T is p-nilpotent.

This is proved by the same arguments as those shown in Step 2 of the proof of Theorem 3.1.

Step 3. $|P : D| > p$.

By Theorem 3.1.

Step 4. $|D| > p$.

Suppose that $|D| = p$. Clearly, the hypothesis is inherited by all proper subgroups of G by Lemma 2.1. Thus, G is a minimal non-p-nilpotent group. Then by Lemma 2.2, G has a normal Sylow p-subgroup P and $G = [P]Q$, where Q is a non-normal cyclic Sylow q-subgroup of G, and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Since p is an odd prime, by Lemma 2.2, the exponent of P is p. Let L be a minimal subgroup of P. By hypotheses, L is weakly $\mathcal{M}$-supplemented in G. If L is non-normal in G, then L has a complement B in G. By [8], Lemma 2.8, $B \trianglelefteq G$ and hence G is nilpotent, a contradiction. Since every minimal subgroup of P is normal in G, we also get a contradiction.

Step 5. $O_p(G) \neq 1$ and $G = PQ$, where $Q \in \mathrm{Syl}_q(G)$ and $q \neq p$.

Since G is not p-nilpotent, by the Glauberman-Thompson theorem we have that $N_G(Z(J(P)))$ is not p-nilpotent, where $J(P)$ is the Thompson subgroup of P. Noticing that $Z(J(P))$ is a characteristic subgroup of P, we get $N_G(P) \leqslant N_G(Z(J(P)))$. By Step 2, we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$. Consider $\overline{G} = G/O_p(G)$

and let G_1 be the inverse image of $N_{\overline{G}}(Z(J(\overline{P})))$ in G. Since $O_p(G)$ is the largest normal subgroup of G contained in P, we have $N_G(P) \leqslant G_1 < G$. By Step 2, G_1 is p-nilpotent and by [1], Theorem 8.3.1 again, G is p-nilpotent. Then there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G for any $q \in \pi(G)$ with $q \neq p$ by [1], Theorem 6.3.5. If $PQ < G$, then PQ is p-nilpotent by Step 2. Hence $Q \leqslant C_G(O_p(G)) \leqslant O_p(G)$ by [1], Theorem 6.3.2, a contradiction. Thus $PQ = G$.

Step 6. Let N be a minimal normal subgroup of G, then $|N| < |D|$.

If $|N| = |D|$, then by the hypothesis, $G = N_G(N)$ is p-nilpotent, a contradiction. Suppose that $|N| > |D|$. By hypotheses we may choose a subgroup E of P with order $|D|$ such that $E < N$. Since E is weakly $\mathcal{M}$-supplemented in G, there exists a subgroup B of G such that $G = EB$ and $TB < G$ for every maximal subgroup T of E. Since N is a minimal normal subgroup of G, we have $N \cap B = 1$ or N. If $N \cap B = 1$, then $N = E$, a contradiction. If $N \cap B = N$, then $B = G$, which is also a contradiction.

Step 7. G/N is p-nilpotent, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

By Step 6 and Lemma 2.1, it is easy to see that G/N satisfies the hypothesis of the theorem, so the choice of G yields that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious.

Step 8. The final contradiction.

Since G is solvable by Step 5, there is a maximal subgroup M of G such that $|G : M|$ is a prime. If $|G : M| \neq p$, then M is p-nilpotent by Step 2 and therefore $P = M \trianglelefteq G$ by Step 1, a contradiction. Thus we may assume that $|G : M| = p$. Then it follows that $P \cap M$ is a maximal subgroup of P and also a Sylow p-subgroup of M. If $N_G(P \cap M) < G$, then $N_G(P \cap M)$ is p-nilpotent by Step 2 and so is $N_M(P \cap M)$. Since $|P : D| > p$ by Step 3, every subgroup of $P \cap M$ of order $|D|$ is weakly $\mathcal{M}$-supplemented in M by Lemma 2.1. Consequently, M satisfies the hypotheses of our theorem and therefore the choice of G implies that M is p-nilpotent, a contradiction. Hence $P \cap M \trianglelefteq G$ and $N = O_p(G) = P \cap M$ is a maximal subgroup of P by Step 7. This leads to $|D| < |N|$ by Theorem 3.1, in contradiction to Step 6, the final contradiction. The proof of the theorem is complete. □

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