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ON STRONGLY AFFINE EXTENSIONS OF COMMUTATIVE RINGS

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Abstract. A ring extension $R \subseteq S$ is said to be strongly affine if each R-subalgebra of S is a finite-type R-algebra. In this paper, several characterizations of strongly affine extensions are given. For instance, we establish that if R is a quasi-local ring of finite dimension, then $R \subseteq S$ is integrally closed and strongly affine if and only if $R \subseteq S$ is a Prüfer extension (i.e. (R, S) is a normal pair). As a consequence, the equivalence of strongly affine extensions, quasi-Prüfer extensions and INC-pairs is shown. Let G be a subgroup of the automorphism group of S such that R is invariant under action by G. If $R \subseteq S$ is strongly affine, then $R^G \subseteq S^G$ is strongly affine under some conditions.

Keywords: strongly affine; Prüfer extension; finitely many intermediate algebras property extension; finite chain propery extension; normal pair; integrally closed pair; ring of invariants

MSC 2010: 13B02, 13A15, 13A50, 13E05

1. Introduction and notation

All rings and algebras considered below are commutative with identity. All subrings, subalgebras, and inclusions of rings are (unital) ring extensions; all ring/algebra homomorphisms are unital. If R is a ring, by an overring of R we mean a subring of total quotient ring of R containing R. By a quasi-local ring we mean a ring with unique maximal ideal. The symbol $\subseteq$ is used for inclusion, while $\subset$ is used for proper inclusion. Any unexplained terminology is standard as in [13] and [19]. By affine extension of a ring R we mean an extension ring S of R that is finitely generated as a ring extension of R. Recall from [15] that a ring extension $R \subseteq S$ is said to be strongly affine if each R-subalgebra of S is a finite-type R-algebra. The notion of strongly affine extension of rings is related to another class of ring pairs, namely Noetherian pair. Noetherian pair was introduced by Wadsworth, see [26]. For Noetherian rings R and S with $R \subseteq S$, (R, S) is called a Noetherian

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pair if every intermediate ring T, $R \subseteq T \subseteq S$, is Noetherian. Attention is focussed on Noetherian rings. In that case, S strongly affine over R implies that each intermediate ring is Noetherian so that (R, S) is a Noetherian pair. The converse need not hold in general, see [15], Introduction.

The set of all R-subalgebras of S (that is, of rings T such that $R \subseteq T \subseteq S$) is denoted by $[R, S]$ and the integral closure of R in S by R^*. The extension $R \subseteq S$ is said to have (or to satisfy) FIP (finitely many intermediate algebras property) if $[R, S]$ is finite. The FIP property was introduced in [1] and, along with various related properties, has been treated in many other papers [8], [9], [10], [14], [17], [18]. A chain of R-subalgebras of S is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. We say that the extension $R \subseteq S$ has FCP (finite chain property) if each chain in $[R, S]$ is finite, see for example [2], [4], [9], [10], [14], [17], [18] for properties and characterizations of FCP extensions. It is clear that each extension that satisfies FIP must also satisfy FCP. The main tool that is used to study the FIP and FCP properties is the concept of a minimal (ring) extension, as introduced by Ferrand-Olivier, see [12]. Recall that an extension $R \subset S$ is called minimal if $[R, S] = \{R, S\}$.

In Section 2, we establish several characterizations of some strongly affine extensions. Knebusch and Zhang define Prüfer extensions in [20]. It is now well known that $R \subseteq S$ is Prüfer if and only if (R, S) is a normal pair, see [20], Theorem 5.2, case (4). (A pair (R, S) is a normal pair if S is integrally closed in T for all $T \in [R, S]$, see [5]). We refer the reader to [20] for the properties of Prüfer extensions, noting here only that a ring extension $R \subseteq S$ is Prüfer if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$. Motivated by this concept, Picavet and Picavet-L'Hermitte introduced the following definition, see [23]. An extension of rings $R \subseteq S$ is called quasi-Prüfer if $R^* \subseteq S$ is a Prüfer extension. An extension is clearly Prüfer if and only if it is quasi-Prüfer and integrally closed, see for example [24] for properties and characterizations of quasi-Prüfer extensions. An important result is that quasi-Prüfer extensions coincide with INC-pairs. Recall from [6] that (R, S) is said to be an INC-pair if $R \subseteq T$ satisfies INC for any intermediate ring $R \subseteq T \subseteq S$.

The main result of Section 2 is Theorem 2.6, in which we characterize the integrally closed, strongly affine extensions. Precisely, if R is a quasi-local ring of finite krull dimension, then $R \subseteq S$ is an integrally closed, strongly affine extension of rings if and only if it is a Prüfer extension. An immediate consequence of our main result is that under certain conditions, the strongly affine ring extensions coincide with quasi-Prüfer extensions and INC-pairs (see Theorem 2.7). We conclude Section 2 with several results showing when the notions of strongly affine extension, FCP and FIP are equivalent.

In Section 3, we investigate the behavior of a subgroup G of the automorphism group $\mathrm{Aut}(S)$ on a strongly affine extension $R \subseteq S$. Indeed, let $S^G = \{s \in S:$

$\sigma(s) = s$ for all $\sigma \in G\}$ be the fixed subring under the action of G. We find conditions for which the integrally closed, strongly affine extension condition descends from $R \subseteq S$ down to $R^G \subseteq S^G$. It is worth noting that early studies in this area are due in large measure to Hilbert's fourteenth problem.

2. Properties of strongly affine extensions

We begin this section with some primary results which will be used in characterizing the integrally closed strongly affine extension of rings. Recall from [16] that an extension $R \subseteq S$ is called a P-extension if each $s \in S$ is the root of some polynomial $f(X) \in R[X]$ with unit content (that is, such that the coefficients of f generate the unit ideal of R). We will say that an extension $R \subseteq S$ satisfies ACC (or DCC) if each ascending (or descending) chain of members of $[R, S]$ terminates. Note that every strongly affine extension of rings is a P-extension as we have the first proposition.

Proposition 2.1. *Let $R \subseteq S$ be a strongly affine extension of rings. Then $R \subseteq S$ is a P-extension.*

Proof. In view of the comments in the end of Section 1 (after example 10) of [15], S is strongly affine over R if and only if the ACC condition holds for intermediate rings between R and S. The hypothesis gives an ascending finite maximal chain $R_0 \subset R_1 \subset \ldots \subset R_n = S$ of rings from R to S. Therefore, by applying [2], Theorem 2.3, we get that $R \subseteq S$ is a P-extension. $\square$

Corollary 2.2. *Let $R \subseteq S$ be a strongly affine extension of rings such that R is integrally closed in S, $u \in S$ and $P \in \operatorname{Spec}(R)$. Then u satisfies at least one of the following two conditions:*

(i) $u/1 \in R_P$;

(ii) *$u/1$ is a unit of S_P and* $(u/1)^{-1} \in R_P$.

Proof. Proposition 2.1 ensures that each element of S satisfies a polynomial with coefficients in R that has at least one coefficient in $R \setminus P$. Thus, by [9], Lemma 3.8, the result follows. $\square$

We next give two lemmas that will be used often in proving subsequent results.

Lemma 2.3. *If $R \subseteq S$ is an integrally closed strongly affine extension, then $R \subseteq S$ is a Prüfer extension.*

Proof. Let P be a prime ideal of R. According to Proposition 2.1, every $s/t \in S_P$ satisfies a polynomial $f(X) \in R_P[X]$ with at least one unit coefficient. It follows from [20], Theorem 5.2 that $R_P \subseteq S_P$ is a Prüfer extension. Since "integrally closed in" is a local property and P was an arbitrarily chosen prime ideal, $R \subseteq S$ is a Prüfer extension. $\square$

Lemma 2.4. *Let $R \subseteq S$ be a ring extension and I a common ideal of R and S. Then $R \subseteq S$ is strongly affine if and only if $R/I \subseteq S/I$ is strongly affine.*

Proof. Applying [8], Lemma II.3 to the pullback $R = S \times_{S/I} R/I$, we have an order-preserving bijection between the set of all R-subalgebras of S and the set of all R/I-subalgebras of S/I. The assertion now follows easily. $\square$

Remark 2.5. The "only if" assertion in Lemma 2.4 also follows from Gilmer and Heinzer, see [15], Proposition 1.1 case (1).

We now present a characterization of integrally closed strongly affine extension of rings. We need the following concept from Papick, see [22]. An extension $R \subseteq S$ of domains is said to be open if $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is an open map. Following Gilmer and Heinzer, see [15], page 258, for a Prüfer domain R, openness is equivalent to the assertion that R has only finitely many maximal ideals and the prime ideals of R_M satisfy DCC for each maximal ideal M of R.

Theorem 2.6. *If R is quasi-local and $\dim(R)$ is finite, then $R \subseteq S$ is integrally closed and strongly affine if and only if $R \subseteq S$ is a Prüfer extension (i.e. (R, S) is a normal pair).*

Proof. The "only if part" follows immediately from Lemma 2.3.

Conversely, according to [9], Theorem 6.8, the hypothesis that (R, S) is a normal pair is equivalent to that there exists $Q \in \operatorname{Spec}(R)$ such that $S = R_Q$, $Q = SQ$ and R/Q is a valuation domain. Then R/Q is a Prüfer domain. Note that S/Q is the quotient field of R/Q. On the other hand, each overring of R/Q is a quasi-local integral domain and it is clear that R/Q is an open Prüfer domain. Hence, by [22], Theorem 2, $R/Q \subseteq S/Q$ is an integrally closed strongly affine extension of rings. Finally, since Q is a common ideal of R and S, Lemma 2.4 gives that $R \subset S$ is an integrally closed strongly affine extension of rings. $\square$

From [3], Definitions 1.1 and 2.1, an extension of integral domains $R \subseteq S$ is called residually algebraic if for any prime ideal Q of S and $P = Q \cap R$, the ring S/Q is algebraic over R/P. If the extension $R \subseteq T$ is residually algebraic for all $T \in [R, S]$, then (R, S) is called a residually algebraic pair. The concept of residually algebraic pairs is generalized by Dobbs to the context of arbitrary commutative rings, see [7], Section 2. The next result shows when the notions of strongly affine, quasi-Prüfer, INC, and residually algebraic are all equivalent.

Theorem 2.7. *If R is quasi-local,* $\dim(R)$ *is finite and* $R \subseteq S$ *is an integrally closed extension. Then the following are equivalent:*

(1) *$R \subseteq S$ is strongly affine,*

(2) *$R \subseteq S$ is quasi-Prüfer,*

(3) *(R, S) is an INC-pair,*

(4) *(R, S) is a residually algebraic pair.*

Proof. It suffices to notice that an extension is Prüfer if and only if it is quasi-Prüfer and integrally closed. Hence, the result follows immediately from Theorem 2.6 and [23], Theorem 2.3. □

By combining Theorem 2.7 and [3], Lemma 2.9, we immediately arrive at the following result.

Corollary 2.8. *If $R \subseteq S$ is a strongly affine extension of integral domains such that R is quasi-local and integrally closed in S, then* $\mathrm{Spec}(S) = \{PS\colon PS \neq S, P \in \mathrm{Spec}(R)\}$.

Remark 2.9. (1) By combining Theorem 2.6 and [9], Theorem 6.8, we can derive another characterization of strongly affine extension. If R is quasi-local and $\dim(R)$ is finite, then $R \subseteq S$ is integrally closed and strongly affine if and only if there exists $Q \in \mathrm{Spec}(R)$ such that $S = R_Q$, $Q = SQ$ and R/Q is a valuation domain. Under these conditions, S/Q is necessarily the quotient field of R/Q.

(2) According to [15], Section 6, the "strongly affine" property is not generally transitive. However, it is shown in [23], Corollary 3.3 that if $R \subseteq T \subseteq S$, then $R \subseteq S$ is a quasi-Prüfer extension if and only if $R \subseteq T$ and $T \subseteq S$ are quasi-Prüfer extensions. Hence, in view of Theorem 2.7 and [23], Corollary 3.3 we have the following result: Let R be a quasi-local ring such that $\dim(R)$ is finite and let $R \subseteq T \subseteq S$ be a tower of rings. Then $R \subseteq S$ is strongly affine if and only if $R \subseteq T$ and $T \subseteq S$ are strongly affine.

In light of Theorem 2.6 and [9], Theorem 6.10, we immediately get the following result showing that in the case of integrally closed extensions of rings the notions of FIP, FCP and strongly affine extensions are equivalent.

Theorem 2.10. *Let R be a quasi-local ring of finite dimension and $R \subseteq S$ an integrally closed extension. Then the following conditions are equivalent:*

(i) *$R \subseteq S$ has FCP;*

(ii) *$R \subseteq S$ has FIP;*

(iii) *$R \subseteq S$ is strongly affine.*

If the above equivalent conditions hold, then $|[R, S]| = l[R, S] + 1$ and $[R, S]$ is linearly ordered by inclusion.

Recall that if $R \subseteq S$ is an extension of rings, then $\operatorname{Supp}(S/R) = \{P \in \operatorname{Spec}(R)\colon R_P \subset S_P\}$. We say that $R \subseteq S$ is locally strongly affine if $R_M \subseteq S_M$ is an affine extension for each maximal ideal $M \in \operatorname{Max}(R)$. Combining Theorem 2.6, [9], Lemma 6.2 and [9], Theorem 6.9, we immediately arrive at the following result.

Proposition 2.11. *Suppose that $R \subseteq S$ is an integrally closed, locally strongly affine extension of rings. The following statements are equivalent:*
(i) *$R \subseteq S$ has FCP;*
(ii) *$R \subseteq S$ has FIP;*
(iii) *$\operatorname{Supp}(S/R)$ is finite.*

To facilitate the proof of Theorem 2.13, we isolate the following proposition, which is of some independent interest. This result is surely known, but we include a proof of it for the sake of completeness.

Proposition 2.12. *Let $R \subseteq S$ be a ring extension. If $R \subseteq S$ has FCP, then $R \subseteq S$ is strongly affine.*

Proof. Let T be an intermediate ring between R and S and let $t_1 \in T \setminus R$. If $T = R[t_1]$, we are done. Otherwise, we take another element $t_2 \in T \setminus R[t_1]$. If $T = R[t_1, t_2]$, there is nothing more to prove, otherwise we pick an element $t_3 \in T \setminus R[t_1, t_2]$ and we repeat the same argument. As $R \subset R[t_1] \subset R[t_1, t_2] \subset \ldots \subset T$ is an increasing chain of intermediate rings between R and T, then this process must terminate since $R \subseteq T$ inherits the "satisfies FCP" property from $R \subseteq S$. Hence, $T = R[t_1, t_2 \ldots, t_n]$ for some $t_1, t_2, \ldots, t_n \in T$. □

In the next theorem, we present a characterization for an integral strongly affine extension of rings to satisfy FCP. For convenience, we will use the symbol $\sqrt{I}$ to denote the radical of an ideal I of R. Recall that if $R \subseteq S$ is an extension of rings, the conductor of S in R is defined by $(R : S) = \{x \in R\colon xS \subseteq R\}$.

Theorem 2.13. *Let (R, M) be a quasi-local ring and $R \subseteq S$ an integral extension. Then $R \subseteq S$ has FCP if and only if $R \subseteq S$ is strongly affine and M is an ideal of S.*

Proof. First, suppose that $R \subseteq S$ has FCP. Then $R \subseteq S$ is strongly affine by Proposition 2.12. For the second assertion, it suffices to prove that $M = \sqrt{(R : S)}$. The hypothesis gives a finite maximal chain in $[R, S]$ from R to S. As S is integral over R, [2], Lemma 4.9 (b) ensures that $\sqrt{(R : S)}$ can be expressed as the intersection of finitely many maximal ideals of R. Hence $\sqrt{(R : S)} = M$. This proves that M is an ideal of S.

Conversely, suppose that M is an ideal of S. Since $R \subseteq S$ is strongly affine and integral, it follows from Lemma 2.4 that $R/M \subseteq S/M$ is also strongly affine and integral. It follows that $R/M \subseteq S/M$ is module finite. This proves that S/M is finite dimensional R/M-vector space, and hence $R/M \subseteq S/M$ satisfies FCP. Finally, $R \subseteq S$ has FCP by [9], Proposition 3.7, case (c). $\square$

3. Group action on strongly affine extension of rings

Throughout this section, we assume that $R \subseteq S$ and G is a subgroup of the automorphism group of S such that $\sigma(R) \subseteq (R)$ for all $\sigma \in G$. In the first theorem, we prove that strongly affine extension of rings is a G-invariant property under the stated conditions on G. We need the following two lemmas.

Lemma 3.1. *Assume $|G|$ is finite and a unit in R, and $R \subseteq S$ is integral. If $R \subseteq S$ has FCP, then so does $R^G \subseteq S^G$.*

Proof. Let $R^G = T_0 \subset T_1 \subset T_2 \subset \ldots$ be an ascending chain of intermediate rings in $[R^G, S^G]$. According to [25], Lemma 3.32, $T_i = T_iR \cap S^G$, hence this chain can be written as $R^G = T_0R \cap S^G \subset T_1R \cap S^G \subset T_2 \cap S^G \subset \ldots$ Note that $R \subseteq T_iR \subseteq S$, then $R = T_0R \subset T_1R \subset T_2R \subset \ldots$ is an ascending chain of intermediate rings in $[R, S]$. Henceforth, $T_nR = T_{n+1}R = \ldots$ for some n since $R \subseteq S$ has FCP. Also, it is clear that if $A \neq B \in [R^G, S^G]$, then $AR \neq BR$, it follows that $T_n = T_{n+1} = \ldots$ Whence, ACC holds in $[R^G, S^G]$.

A similar argument shows that DCC also holds in $[R^G, S^G]$, and hence $R^G \subset S^G$ has FCP, as required. $\square$

Lemma 3.2 ([9], Theorem 3.13)**.** *The extension $R \subseteq S$ has FCP (or FIP) if and only if $R \subseteq R^*$ and $R^* \subseteq S$ have FCP (or FIP).*

Theorem 3.3. *Assume $|G|$ is finite and a unit in R. If $R \subseteq S$ is strongly affine and satisfies DCC, then $R^G \subseteq S^G$ is strongly affine.*

Proof. Since $R \subseteq S$ is strongly affine, then $R \subseteq S$ satisfies ACC. As in addition $R \subseteq S$ satisfies DCC, it follows that the extension $R \subseteq S$ has FCP. According to Lemma 3.2, each of the extensions $R \subseteq R^*$ and $R^* \subseteq S$ has FCP. Hence, by virtue of Lemma 3.1, $R^G \subseteq (R^*)^G$ has FCP and by [25], Theorem 3.28, $(R^*)^G \subseteq S^G$ has FCP. Now, let $(R^G)^*$ denote the integral closure of R^G in S^G. We wish to show that $(R^G)^* = (R^*)^G$. According to [11], Lemma 2.2, the extensions $R^G \subseteq R$ and $R \subseteq R^*$ are both integral. Thus, we conclude by the transitivity of integrality that $R^G \subseteq R^*$ is an integral extension. Moreover, since $(R^*)^G \in [R^G, R^*]$, it follows that $(R^*)^G$ is integral over R^G. On the other hand, as R^* is integrally closed in S, hence $(R^*)^G$ is integrally closed in S^G by [25], Proposition 3.2, case (a). This proves that $(R^G)^* = (R^*)^G$. Henceforth, we may again apply Lemma 3.2 to obtain that the extension $R^G \subseteq S^G$ has FCP, and then it is strongly affine by Proposition 2.12. $\square$

Interestingly, for an integrally closed extension of a quasi-local ring, we can remove the condition that $|G|$ is a unit. We are also able to relax the finiteness condition to local finiteness on G. In fact, if G is locally finite and R is quasi-local, then "integrally closed, strongly affine extension" is a G-invariant property.

Theorem 3.4. *Let (R, M) be a quasi-local ring and let G be locally finite subgroup of $\mathrm{Aut}(S)$. If $R \subseteq S$ is an integrally closed, strongly affine extension, then $R^G \subseteq S^G$ is also an integrally closed, strongly affine extension.*

P r o o f. By Remark 2.9, case (1), there exists $Q \in \mathrm{Spec}(R)$ such that $S = R_Q$, $Q = QS$ and R/Q is a valuation domain with quotient field S/Q. First, we need to prove that S is quasi-local. If $R = S$, then $Q = M$ will work. Now, assume that $R \subset S$. Let $s \in S$. By Corollary 2.2, either (i) $s \in R$ or (ii) s is a unit of S and $s^{-1} \in R$. It follows that if M_1 and M_2 are maximal ideals of S, then M_1 and M_2 are contained in R. Thus, $S = M_1 + M_2 \subseteq R$ and hence $R = S$, which is a contradiction. Hence, S must be quasi-local, say with maximal ideal Q. Since (R, M) is quasi-local, (R^G, M^G) is quasi-local by [11], Lemma 2.1, case (b). By the same result, (S^G, Q^G) is quasi-local. We may again apply Remark 2.9, case (1), it is enough to show that $Q^G \in \mathrm{Spec}(R^G)$ such that $S^G = R^G_{Q^G}$, $Q^G = Q^G S^G$ and R^G/Q^G is a valuation domain with quotient field S^G/Q^G.

Now, for the rest of the proof we will argue as in the proof of [21], Theorem 3.2. For any $\sigma \in G$, $r \in R$ and $s \in R \setminus Q$, we have $r/s \in R_Q = S$ and $\sigma(r) = \sigma(r/s)\sigma(s)$. Then $\sigma(r/s) = \sigma(r)/\sigma(s)$. First, we assert that $S^G = (R_Q)^G = (R^G)_{Q^G}$. Since $Q^G = Q \cap R^G$, $(R^G)_{Q^G} \subseteq (R_Q)^G$. For the reverse inclusion, let $r/s \in (R_Q)^G$. Let $\widetilde{s} := \prod\limits_{\sigma \in G} \sigma(s) \in R^G \cap (R \setminus Q)$. If I is the identity map in G, then

$$r\Bigl(\prod_{\sigma \in G, \sigma \neq I} \sigma(s)\Bigr)/\widetilde{s} = \frac{r}{s} \in (R_Q)^G.$$

It follows that

$$r\Bigl(\prod_{\sigma \in G, \sigma \neq I} \sigma(s)\Bigr) = \widetilde{s}\Bigl(\frac{r}{s}\Bigr) \in (R_Q)^G \cap R = R^G.$$

On the other hand, $\widetilde{s} \notin Q$ implies $\widetilde{s} \notin Q^G$, and so $r/s \in (R^G)_{Q^G}$. It follows that $S^G = (R_Q)^G = (R^G)_{Q^G}$ and $Q^G = Q^G S^G$. Now, we can extend this G-action to S/Q via $\sigma(s+q) = \sigma(s)+q$ for each $s \in S$. Clearly this G-action on S/Q is well defined as $\sigma(Q) = Q$ for all $\sigma \in G$. Next, we define $\varphi\colon S^G/Q^G \to (S/G)^G$ by $\varphi(s+Q^G) = s+Q$. Clearly, φ is well-defined. Let $s + Q^G \in \ker(\varphi)$, that is, $s \in Q$. Then $s \in Q^G$ since

$Q \cap S^G = Q^G$. This implies that φ is injective, and hence $S^G/Q^G \subseteq (S/G)^G$, up to isomorphism. Similarly $R^G/Q^G \subseteq (R/Q)^G$, up to isomorphism. Now, it remains to show that R^G/Q^G is a valuation domain with quotient field S^G/Q^G. It is clear that the quotient field of R^G/Q^G is $(R^G)_{Q^G}/Q_G((R^G)_{Q^G}) = S^G/Q^G$. Take $s + Q = \varphi(s + Q^G) \in \varphi(S^G/Q^G) \subseteq (S/G)^G$. Since R/Q is a valuation domain with quotient field S/Q, $(R/Q)^G$ is a valuation domain with quotient field $(S/Q)^G$ by [12], Proposition 2.7. $\square$

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