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ON TI-SUBGROUPS AND QTI-SUBGROUPS OF FINITE GROUPS

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Abstract. Let G be a group. A subgroup H of G is called a TI-subgroup if $H \cap H^g = 1$ or H for every $g \in G$ and H is called a QTI-subgroup if $C_G(x) \leqslant N_G(H)$ for any $1 \neq x \in H$. In this paper, a finite group in which every nonabelian maximal is a TI-subgroup (QTI-subgroup) is characterized.

Keywords: TI-subgroup; QTI-subgroup; maximal subgroup; Frobenius group; solvable group

MSC 2010: 20D10

1. Introduction

All groups considered in this paper are finite and G always denotes a group.

A subgroup H of G is called a TI-subgroup if $H \cap H^g = 1$ or H for every $g \in G$. It is obvious that a minimal subgroup or a normal subgroup is a TI-subgroup. Recall that a group G is a Frobenius group if G has a nontrivial subgroup H such that $H \cap H^g = 1$ for every $g \in G - H$, and H is said to be a Frobenius complement of G. It is easy to see that a Frobenius complement of a Frobenius group is a TI-subgroup. In [13], Walls determined the finite groups in which every subgroup is a TI-subgroup. As generalizations, Li in [6] described the finite non-nilpotent groups in which every second maximal subgroup is a TI-subgroup. Moreover, Guo, Li and Flavell in [3] gave a complete classification of finite groups in which every abelian subgroup is a TI-subgroup. Most recently, in [8] and [12], Lu et al. and Shi et al. investigated finite groups in which every nonabelian subgroup is a TI-subgroup, respectively.

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On the other hand, Qian and Tang in [10] introduced the definition of QTI-subgroups. A subgroup H of G is called a QTI-subgroup if $C_G(x) \leqslant N_G(H)$ for any $1 \neq x \in H$. Clearly a TI-subgroup is a QTI-subgroup. However, the converse is not true, see [10], Example 1.2.

Furthermore, for every nonnormal maximal subgroup H of G, if H is a QTI-subgroup of G, then H is a CC-subgroup. In [10], Qian and Tang classify the groups all of whose abelian subgroups are QTI-subgroups. And in [7], Lu and Guo studied finite groups with all second maximal subgroups being QTI-subgroups.

Recall that a group G is p-closed if and only if it contains a normal Sylow p-subgroup (see e.g. [5], page 132).

In [9], Lu, Pang and Zhong characterized the structure of a finite G satisfying that all non-nilpotent maximal subgroups of G are QTI-subgroups, and they obtained the following theorem.

Theorem 1.1. *Assume that every non-nilpotent maximal subgroup of G is a QTI-subgroup. Then*

(1) *G is solvable;*

(2) *G is p-closed and q-closed for some primes p and q.*

In this paper, we consider an analogous problem as in [9] but putting stronger assumptions on the maximal subgroups. We investigate a finite group in which every nonabelian maximal subgroup is a QTI-subgroup, and for convenience we call such a group an NMQTI-group. Clearly, when G is an NMQTI-group, it satisfies the premises of this theorem. In fact, our main result, the classification of NMQTI-groups, reads as follows:

Theorem 1.2. *Let every nonabelian maximal subgroup of a group G be a QTI-subgroup. Then G is one of the following types:*

(a) *All maximal subgroups of G are normal. Then G is nilpotent.*

(b) *There is a nonnormal maximal subgroup of G and all nonnormal maximal subgroups are abelian. Then G is solvable and the maximal nonnormal subgroups are all conjugate to a Carter subgroup of G.*

(c) *There is a nonnormal nonabelian maximal subgroup H of G. Then $G = N \rtimes H$ is a Frobenius group with Frobenius kernel N a minimal normal subgroup of G and a nilpotent Frobenius complement H. The Sylow 2-subgroup H_2 of H is a generalized quaternion group.*

Conversely, if a group G satisfies either of conditions (a)–(c)*, then every maximal nonabelian subgroup is a QTI-subgroup.*

2. Proof of Theorem 1.2

Lemma 2.1. *The following statements about a nonnormal maximal subgroup H of a group G are equivalent:*
(a) *H is a QTI-subgroup of G.*
(b) *H is a CC-subgroup of G.*

Proof. Since H is a nonnormal maximal subgroup of G, we deduce that $H = N_G(H)$, and the conclusion comes easily. □

Lemma 2.2 ([1], Lemma 2.8)**.** *Let G contain a nilpotent CC-subgroup H with $N_G(H) = H$. Then G is a Frobenius group and H a Frobenius complement.*

Lemma 2.3 ([1], Lemma 2.9)**.** *Let a solvable group G contain a CC-subgroup H satisfying $N_G(H) = H$. Then G is a Frobenius group and H a complement.*

Lemma 2.4 ([7], Theorem 2.3)**.** *Let G be a finite group in which every maximal subgroup is a QTI-subgroup. Then G is of one of the following types:*
(a) *G is nilpotent; or*
(b) *$G = N \rtimes H$ is a Frobenius group with a kernel N and a complement H and N is the unique minimal normal subgroup of G.*

Now we will give the proof of Theorem 1.2.

Proof of Theorem 1.2. In (a) and (b) the condition of G being an NMQTI-group is satisfied in a trivial manner. A case by case discussion will follow according to premises (a)–(c). We also establish, in each case separately, the converse direction.

(a) Certainly G must be a nilpotent group.

Conversely, if G is nilpotent, every maximal subgroup is normal in G and hence G is an NMQTI-group.

(b) Exercise 7 on page 309 in [11] implies that G is solvable. Let H be any nonnormal abelian maximal subgroup. Then H is self-normalizing and nilpotent and thus it is a Carter subgroup of G. It follows that all nonnormal maximal subgroups of G are conjugate.

Conversely, let a nonabelian solvable group G have abelian Carter subgroups which are at the same time maximal subgroups. Suppose H is any maximal subgroup which is abelian and not normal. Then H is self-normalizing and hence agrees with one of the Carter subgroups.

(c) We first claim that G is a Frobenius group with a nonabelian Frobenius complement. We discuss the cases:

Case 1: Let G not contain nonnormal abelian maximal subgroups. Then, as every normal subgroup of G is QTI, every nonnormal maximal subgroup of G is nonabelian and hence is a QTI-subgroup; therefore, by Theorem 5 of [7], G is a solvable Frobenius group and H is a complement.

Case 2: Group G contains a nonnormal abelian maximal subgroup, say H. Then Exercise 7 on page 309 in [11] implies that G must be solvable. As H contains the centralizer of each of its nontrivial elements, it is a CC-subgroup of G. It follows from Lemma 3 of [1] that G is a solvable Frobenius group and H is a complement.

For proving that the Frobenius kernel N is a minimal normal subgroup of G, suppose, by way of contradiction, that there is a normal subgroup M of G, properly contained in N, and let it be maximal with respect to this property. Then $MH = M \rtimes H$ is a maximal nonnormal subgroup of G. As G is a Frobenius group and MH is neither contained nor contains the Frobenius kernel, it turns out that MH is a nonnormal nonabelian maximal subgroup of G and hence it is a CC-subgroup of G. Since M is nilpotent, it contains a nontrivial element centralized by an element $x \in N \setminus M$. Since $x \notin MH$, it follows that MH is not a QTI-subgroup, contrary to the assumptions. Hence, N is a minimal normal subgroup of G and all maximal nonnormal subgroups of G are conjugates of H.

Next let us prove that the Frobenius complement H is nilpotent and it will suffice to show that every maximal subgroup of H is normal in H (see e.g. [11], page 130). Suppose, by way of contradiction, that there is a nonnormal maximal subgroup S of H. Then NS cannot be normal in G and is nonabelian. Hence, NS must be a QTI-subgroup of G. Therefore, for every $1 \neq s \in S$ the centralizer $C_G(s)$ must be contained in $N_G(NS) = NS$. Since $s \in H$ and H is a CC-subgroup of G, we deduce $C_G(s) \leqslant NS \cap H = S$. Thus, any nonnormal maximal subgroup S of H must be a QTI-subgroup of H. In particular, such S is a CC-subgroup of H and hence a Hall subgroup. Lemma 4 of [1] implies that H is a Frobenius group and S is the complement. Then, however, $Z(H) = 1$, contradicting the fact that H is a Frobenius complement, since any Frobenius complement has a nontrivial center (see [4], page 506, Satz (c)).

Finally, we observe that for $q \in \pi(H)$ the Sylow q-subgroup H_q is either cyclic or, when $q = 2$, it can be a generalized quaternion as well. Thus, $H_{\pi\setminus\{2\}}$ is finite cyclic and, as we assumed G to contain a maximal nonabelian QTI-subgroup, it follows that H and therefore H_2 is nonabelian.

Conversely, let $G = N \rtimes H$ be a solvable Frobenius group with Frobenius complement H a maximal subgroup and Frobenius kernel N minimal normal. Then N is an elementary abelian p-group for some prime p. Consider any maximal nonnormal subgroup M of G, set $\pi = \pi(H)$ and note that N is an elementary abelian Sylow p-subgroup of G. If M has a nontrivial π-subgroup, by conjugating, we can arrange that $M_\pi \leqslant H$. If $M_p = 1$, then $M = H$ is thus a CC-subgroup and hence even a TI-subgroup of G.

Assume next that $M_p \neq 1$. Then $M_\pi < H$, because otherwise $M_\pi = H$ and hence $M = \langle M_p, M_\pi \rangle = G$, a contradiction. Thus, indeed $M_\pi < H$ is a maximal subgroup of H and hence we need to have $M_p = N$ since M is maximal in G. Since H is nilpotent and M_π is a maximal subgroup of H, we note that $M_\pi \trianglelefteq H$ and hence $M \trianglelefteq G$. Therefore M is a QTI-subgroup of G and hence G is an NMQTI-group. □

3. Consequences

A concrete type of examples for situation (b) follows:

Example 3.1. Let p be a prime and R an arbitrary nontrivial abelian p'-group. Consider an irreducible nontrivial representation of R over the field with p-elements and let N denote the resulting R-module. Thus, R acts on N and we may form the semidirect product $N \rtimes R$. Fix an arbitrary abelian p-group L and let G be the direct product $G = (N \rtimes R) \times L$.

The following facts can be seen immediately:

(1) N is a minimal normal subgroup of G.

(2) The Sylow p-subgroup of G agrees with $P = N \times L$.

(3) $H := R \times L$ is an abelian maximal subgroup and hence both, a Carter subgroup and a system normalizer for the Sylow system consisting of P and all the Sylow r-subgroups of R for $r \in \pi(R)$.

(4) $Z(G) = L \times C_R(N)$.

In particular, G is an NMQTI-group, having a nonabelian nonnormal maximal subgroup and all nonnormal maximal subgroups are abelian.

We can give a rather explicit description of the solvable groups in situation (b). Recall that a group with only abelian Sylow subgroups is called an A-group ([4], page 751).

Corollary 3.1. *Let G be as in* (b) *of the preceding theorem and H a Carter subgroup. Let N be a minimal normal subgroup of G with $G = NH$ and p be the prime for which N is a p-group. Fix a Sylow system $\mathcal{S} = \{G_r \colon r \in \pi(G)\}$ so that $N \leqslant G_p$ and $H_r = G_r \cap H$ for $r \in \pi(H)$. Set $R = H_{p'}$ and $P = G_p$.*

Then all of the following holds:

(i) *$G' = N$ and $G = P \rtimes H$.*

(ii) *$H_p \leqslant Z(G)$ and P is abelian.*

(iii) *G is an A-group.*

(iv) *$Z(G) = C_H(N) = H_p C_R(N)$ intersects N trivially.*

In particular, letting $L := H_p$, the group G is of the type given in Example 3.1.

P r o o f. (i) Since S is a Sylow system, we have $G = PR$. Moreover, $G = HN$ by the maximality of H and as N is not a subgroup of H. Therefore $G/N = HN/N \cong H/H \cap N$ is abelian and hence $N \geqslant G'$. By the minimality of N we deduce that $N = G'$. Since $N \leqslant P$, it follows that P is normal and has complement R, i.e. $G = P \rtimes R$.

(ii) Since the conjugation with elements $R = H_{p'}$ induces the action of a p'-automorphism group on P, application of Theorem 3.6 in [2] implies $P = [P, R]C_P(R)$ and $[[P, R], R] = [P, R]$. Since $G = NH$ and $N \leqslant P$ and $H \cap P = H \cap G_p = H_p$, we have $P = NH_p$. Then, since N and P are both normal in G, one has that $[N, P] = [N, NH_p] = [N, H_p]$ is normal in G. Since N is a minimal normal subgroup of G, we can only have $[N, P] = N$ or $[N, P] = 1$. If $[N, P] = N$, then $[P, P] = [NH_p, NH_p] = N$, and $[[P, P], P] = [N, P] = N$, contradicting P being nilpotent. Hence $[N, P] = 1$, and therefore $[N, H_p] = 1$. Since $G = NH$ and $[H, H_p] = 1$, we deduce $H_p \leqslant Z(G)$.

Taking into account that H_p is abelian, we find that $P = NH_p$ is abelian. Since $R = G_{p'}$ is abelian, G is an A-group.

(iii) Since $G_p = P$ is abelian and H is an abelian Hall p'-subgroup of G, it follows that G is an A-group.

Since G is an A-group, 14.3 Satz on page 751 in [4] in conjunction with (i) implies $1 = Z(G) \cap G' = Z(G) \cap N$. By (ii) $H_p = P \cap Z(G)$ and certainly $C_R(N) \leqslant Z(G)$. Hence $H_pC_R(N) \leqslant Z(G)$.

For proving the converse pick $1 \neq z \in Z(G)$. Then $z = nh_pr$ for suitable $n \in N$, $h_p \in H_p$, and $r \in R$. Since $h_p \in Z(G)$, we must have $n_r \in Z(G)$. Since $n_r \in C_G(N)$, we have $r \in C_G(N)$ and hence $r \in C_R(N)$. Therefore $n \in Z(G)$. And since $N \cap Z(G) = 1$, we conclude that $n = 1$. Thus $Z(G) \leqslant H_pC_R(N)$, as claimed. The last statement is clear from (i)–(iv) and the discussion in Example 3.1. $\square$

Corollary 3.2. *The following statements for a finite group G are equivalent:*

(A) *G is an NMTI-group.*

(B) *G is exactly one of the following types:*

  (B.1) *G is nilpotent.*

  (B.2) *$G = N \rtimes H$ is a Frobenius group for N a minimal normal subgroup of G and H is either abelian or nonabelian nilpotent of even order with $H_{2'}$ cyclic.*

P r o o f. Since every NMTI-group is NMQTI, we may use Theorem 1.2 and inspect conditions (a)–(c).

(a) Certainly every nilpotent group is NMTI since in fact all maximal subgroups are TI-subgroups.

(b) Since $H \cap H^x \geqslant Z(G)$ holds for all $x \in G$, we conclude from the fact that H

is not normal in G that $Z(G)=1$. In particular, $H_p=1$, $H=R$ and $P=N$ is a minimal normal subgroup of $G=N\rtimes H$.

Next suppose there is $1\neq x\in R$ with $C_N(x)\neq 1$. Then selecting any $1\neq n\in C_N(x)$, one finds $1\neq x\in R\cap R^n\leqslant H\cap H^n$. Since H is supposed to be a TI-subgroup, we conclude $H=H^n$. Therefore $[H,\langle n\rangle]\leqslant H\cap N=1$ implies the contradiction that $G=\langle n,\, H\rangle=\langle n\rangle H$ is abelian. Thus, $G=N\rtimes R$ is a Frobenius group with kernel N and abelian complement $H=R$. This yields the first type of Frobenius groups in (B.2).

(c) Now $G=N\rtimes H$ is a Frobenius group and the only nonnormal maximal subgroups of G are the conjugates of H, and all of them are TI-subgroups. The second situation in (B.2) arises. $\square$

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References

[1] *Z. Arad, W. Herfort*: Classification of finite groups with a CC-subgroup. Commun. Algebra *32* (2004), 2087–2098. zbl MR doi

[2] *D. Gorenstein*: Finite Groups. Harper's Series in Modern Mathematics, Harper & Row, New York, 1968. zbl MR

[3] *X. Guo, S. Li, P. Flavell*: Finite groups whose abelian subgroups are TI-subgroups. J. Algebra *307* (2007), 565–569. zbl MR doi

[4] *B. Huppert*: Endliche Gruppen I. Die Grundlehren der Mathematischen Wissenschaften 134, Springer, Berlin, 1967. (In German.) zbl MR doi

[5] *H. Kurzweil, B. Stellmacher*: The Theory of Finite Groups. An Introduction. Universitext, Springer, New York, 2004. zbl MR doi

[6] *S. Li*: Finite non-nilpotent groups all of whose second maximal subgroups are TI-groups. Math. Proc. R. Ir. Acad. *100A* (2000), 65–71. zbl MR

[7] *J. Lu, X. Guo*: Finite groups all of whose second maximal subgroups are QTI-subgroups. Commun. Algebra *40* (2012), 3726–3732. zbl MR doi

[8] *J. Lu, L. Pang*: A note on TI-subgroups of finite groups. Proc. Indian Acad. Sci., Math. Sci. *122* (2012), 75–77. zbl MR doi

[9] *J. Lu, L. Pang, X. Zhong*: Finite groups with non-nilpotent maximal subgroups. Monatsh. Math. *171* (2013), 425–431. zbl MR doi

[10] *G. Qian, F. Tang*: Finite groups all of whose abelian subgroups are QTI-subgroups. J. Algebra *320* (2008), 3605–3611. zbl MR doi

[11] *D. J. S. Robinson*: A Course in the Theory of Groups. Graduate Texts in Mathematics 80, Springer, New York, 1996. zbl MR doi

[12] *J. Shi, C. Zhang*: Finite groups in which all nonabelian subgroups are TI-subgroups. J. Algebra Appl. *13* (2014), Article ID 1350074, 3 pages. zbl MR doi

[13] *G. Walls*: Trivial intersection groups. Arch. Math. *32* (1979), 1–4. zbl MR doi

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