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ACYCLIC 4-CHOOSABILITY OF PLANAR GRAPHS WITHOUT 4-CYCLES

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Abstract. A proper vertex coloring of a graph G is acyclic if there is no bicolored cycle in G. In other words, each cycle of G must be colored with at least three colors. Given a list assignment $L = \{L(v) \colon v \in V\}$, if there exists an acyclic coloring π of G such that $\pi(v) \in L(v)$ for all $v \in V$, then we say that G is acyclically L-colorable. If G is acyclically L-colorable for any list assignment L with $|L(v)| \geqslant k$ for all $v \in V$, then G is acyclically k-choosable. In 2006, Montassier, Raspaud and Wang conjectured that every planar graph without 4-cycles is acyclically 4-choosable. However, this has been as yet verified only for some restricted classes of planar graphs. In this paper, we prove that every planar graph with neither 4-cycles nor intersecting i-cycles for each $i \in \{3, 5\}$ is acyclically 4-choosable.

Keywords: planar graph; acyclic coloring; choosability; intersecting cycle

MSC 2010: 05C10, 05C15

1. Introduction

Only simple graphs are considered in this article. A *plane* graph is a particular drawing of a planar graph in the Euclidean plane in such a way that any pair of edges intersect only at their endpoints. Formally, for a plane graph G we use $V(G)$, $E(G)$ and $F(G)$ to denote its vertex set, edge set and face set, respectively. Two cycles (or faces) are said to be *intersecting* if they share at least one boundary vertex.

A *proper vertex k-coloring* of a graph G is a mapping $\pi \colon V(G) \to \{1, 2, \ldots, k\}$ such that $\pi(u) \neq \pi(v)$ for adjacent vertices u and v. A proper vertex k-coloring of a graph G is called an *acyclic k-coloring* if G does not contain any bicolored cycle. The *acyclic chromatic number* $\chi_a(G)$ of G is the smallest integer k such that G has an acyclic k-coloring.

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The concept of acyclic coloring of graphs was introduced by Grünbaum, see [11], and was first studied by Mitchem, see [13], Albertson and Berman, see [1] and Kostochka, see [12]. In [11], Grünbaum conjectured that if G is a planar graph, then $\chi_a(G) \leqslant 5$. This challenging conjecture was positively confirmed by Borodin, see [2].

Given a list assignment $L = \{L(v)\colon v \in V(G)\}$ of a graph G, we say that G is *acyclically L-colorable* if there is an acyclic coloring π of the vertices such that $\pi(v) \in L(v)$ for each vertex v. This coloring π is said to be an *acyclic L-coloring* of G. If for any list assignment L with $|L(v)| \geqslant k$ for all $v \in V(G)$, G is always acyclically L-colorable, then G is called *acyclically k-choosable*. The *acyclic list chromatic number* of G, denoted by $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable.

In 2002, Borodin et al. in [3] first investigated the acyclic L-coloring of planar graphs. They proved that every planar graph is acyclically 7-choosable. Wang and Chen in [18] showed that if a planar graph does not contain 4-cycles, then it is acyclically 6-choosable. This result has been further slightly improved in [19] which states that if a planar graph G does not contain 4-cycles adjacent to 6-cycles, then G is acyclically 6-choosable.

The following conjecture was proposed in [3].

Conjecture 1. *Every planar graph is acyclically* 5*-choosable.*

Conjecture 1 has been verified only for some special planar graphs: those without 4-cycles and i-cycles for some fixed $i \in \{5, 6\}$, see [16]; with neither 4-cycles nor triangles at distance less than 3, see [10]; with neither 4-cycles nor intersecting triangles, see [7]; and with neither 4-cycles nor chordal 6-cycles, see [20]. Recently, Borodin and Ivanova in [4] proved that every planar graph without 4-cycles is acyclically 5-choosable. This nice result covers all previous consequences.

Now we turn our attention to the acyclic 4-choosability of planar graphs. Montassier, Raspaud and Wang [15] raised the following conjecture.

Conjecture 2. *Every planar graph without* 4*-cycles is acyclically* 4*-choosable.*

Note that if this conjecture were true, then it would strengthen a known result that every planar graph without 4-cycles is 4-choosable. However, it seems to be too difficult. Montassier, Raspaud and Wang in [15] proved that every planar graph without 4-, 5-, and 6-cycles, or without 4-, 5-, and 7-cycles, or without 4-, 5-cycles and intersecting 3-cycles is acyclically 4-choosable. Chen and Raspaud in [6] proved that every planar graph without 4-, 5-, and 8-cycles is acyclically 4-choosable. More recently, Chen and Raspaud in [8] improved all above-mentioned results by showing that every planar graph without 4- and 5-cycles is acyclically 4-choosable. Some other results regarding Conjecture 2 can be found in the references [5], [9], [14].

The purpose of this paper is to provide a new sufficient condition for planar graphs being acyclically 4-choosable. More precisely, we prove the following:

Theorem 1. *Every planar graph with neither* 4*-cycles nor intersecting* i*-cycles for each* $i \in \{3, 5\}$ *is acyclically* 4*-choosable.*

2. Notation

Before showing our main result, we need to introduce a few of concepts and notation. Let $G = (V, E, F)$ be a plane graph. A *k-vertex* (*k^+-vertex*, *k^--vertex*) is a vertex of degree k (at least k, at most k). Similar notation can be defined for faces. For $v \in V(G)$, let $N(v)$ denote the set of neighbors of v. Sometimes we use $n_k(v)$ to denote the number of k-vertices adjacent to v.

A vertex or edge is called *triangular* if it is incident with a 3-face. A triangular 3-vertex adjacent to a non-triangular vertex is said to be a *pendant triangular* 3-vertex. We usually denote by $p_3(v)$ the number of pendant triangular 3-vertices of vertex v. We call a 4-vertex v a *4^{mp}-vertex* if $p_3(v) = m$, and write *4^p-vertex* instead of 4^{1p}-vertex when $m = 1$. If $n_2(v) = 1$, then v is called a *4^*-vertex*. Moreover, a vertex v is said to be *weak* if either $d(v) = 3$ or v is a 4^*-vertex with $n_3(v) \geqslant 1$.

For $f \in F(G)$, we write $f = [u_1 u_2 \ldots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in clockwise order. For simpleness, we use $m_i(v)$ to denote the number of i-faces incident with a vertex v. A 3-face $f = [v_1 v_2 v_3]$ is called an *(a_1, a_2, a_3)-face* if it satisfies that $d(v_i) = a_i$ for $i = 1, 2, 3$. In addition, we write a_i^* (a_i^{mp}) instead of a_i if v_i is a 4^*-vertex (4^{mp}-vertex).

For all figures in this paper, a vertex is represented by a solid point when all of its incident edges are drawn; otherwise it is represented by a hollow point.

3. Proof of Theorem 1

Suppose that G is a counterexample to Theorem 1 with minimizing $|V(G)|$. Obviously, G is connected.

3.1. Structural properties of the minimum counterexample. The following Lemmas 1 to 4, whose proofs were given in [6], [14], [15], [17] are quite useful in the remaining argument.

Lemma 1 ([14], [15]). *Each vertex v in G satisfies the following:*

(A1) $d(v) \neq 1$;

(A2) *v cannot be triangular if $d(v) = 2$;*

(A3) *$n_2(v) = n_3(v) = 0$ if $d(v) = 2$;*

(A4) *v is adjacent to at most one weak vertex if $d(v) = 3$;*

(A5) *$n_2(v) \leqslant 1$ if $d(v) = 4$;*

(A6) *v is not adjacent to any triangular 3-vertex if $d(v) = 4$ and $n_2(v) = 1$;*

(A7) *$n_2(v) \leqslant 3$ if $d(v) = 5$;*

(A8) *$n_2(v) \leqslant 2$ if $d(v) = 5$ and $m_3(v) = 1$;*

(A9) *$p_3(v) = 0$ if $d(v) = 5$, $n_2(v) = 2$ and v is incident with a $(5, 3, 4^+)$-face;*

(A10) *$n_2(v) \leqslant 4$ if $d(v) = 6$.*

Lemma 2 ([15]). *No face f is a $(3, 3, 4)$-face or a $(3, 4, 4^{2p})$-face in G.*

Lemma 3 ([6]). *Each 3-vertex v satisfies that $p_3(v) = 0$.*

Lemma 4 ([17]). *There is no 5-vertex incident with a $(3, 3, 5)$-face and adjacent to two 2-vertices.*

In what follows, let L be a list assignment of G with $|L(v)| = 4$ for all $v \in V(G)$. Suppose that π is a partial acyclic L-coloring of G. Let a and b be any two colors under π. A *bicolored (a, b)-path* is a path $P = v_1v_2 \ldots v_m$ in G such that $\pi(v_i) = a$ if i is odd and $\pi(v_j) = b$ otherwise. A vertex v is said to be *properly colored under π* (or simply *properly colored*) if we may choose a color in $L(v)$ for v that is distinct from the colors of all its neighbors.

Lemma 5. *There is no $(3, 4^p, 4^p)$-face in G.*

P r o o f. Suppose that $f = [uvw]$ is a 3-face with $d(u) = 3$ and $d(v) = d(w) = 4$. Denote by u_1 another neighbor of u different from v and w. Let $N(v) = \{v_1, v_2, u, w\}$ and $N(w) = \{w_1, w_2, u, v\}$. By the absence of 4-cycles, $v_i \neq w_j$ for each $i, j \in \{1, 2\}$. Suppose to the contrary that both v and w are 4^p-vertices. Namely, $p_3(v) = 1$ and $p_3(w) = 1$. Without loss of generality (w.l.o.g.), assume that v_1 and w_1 are pendant triangular 3-vertices. Let v_1', v_1'' and w_1', w_1'' denote the other two neighbors of v_1 and w_1, respectively. Notice that $v_1v_1'v_1''v_1$ and $w_1w_1'w_1''w_1$ are both 3-cycles.

Clearly, $G - u$ admits an acyclic L-coloring π by the minimality of G. If u_1, v and w have pairwise distinct colors, then u can be properly colored, which leads to an acyclic L-coloring of G. So, in what follows, by symmetry, assume that $\pi(u_1) = \pi(v)$ due to $\pi(v) \neq \pi(w)$. Assume that $L(u) = \{1, 2, 3, 4\}$. If we are still not able to find a possible color for u, then w.l.o.g., suppose that $\pi(u_1) = \pi(v) = 1$, $\pi(w) = 2$, $\pi(v_1) = 3$ and $\pi(v_2) = 4$. Moreover, one of v_1' and v_1'' is colored with 1, say $\pi(v_1') = 1$. Note that $L(v) = \{1, 2, 3, 4\}$, for otherwise we may recolor v by a color belonging to

$L(v) \setminus \{1,2,3,4\}$ and then go back to the previous case. At this moment, it suffices to recolor v with 3, properly recolor v_1 and then color u with 4. Thus, we always get an acyclic L-coloring of G, a contradiction. □

Lemma 6. *There is no $(m, 4^*, 4^{2p})$-face in G.*

P r o o f. Suppose to the contrary that $f = [v_1v_2v_3]$ is an $(m, 4^*, 4^{2p})$-face such that $d(v_1) = m$, v_2 is a 4^*-vertex and v_3 is a 4^{2p}-vertex. For $i = 2, 3$, let x_i, y_i be other two neighbors of v_i. By definition, both x_3, y_3 are pendant triangular 3-vertices and w.l.o.g., let x_2 be a 2-vertex. Let $N(x_2) = \{v_2, x_2'\}$, $N(x_3) = \{v_3, x_3', x_3''\}$, and $N(y_3) = \{v_3, y_3', y_3''\}$.

Clearly, $G - x_2$ admits an acyclic L-coloring π by the minimality of G. If $\pi(v_2) \neq \pi(x_2')$, then we are easily done by properly coloring x_2. So next assume that $\pi(v_2) = \pi(x_2')$. Let $L(x_2) = \{1,2,3,4\}$. If x_2 cannot be given a color, we get an acyclic coloring, then w.l.o.g., suppose that $\pi(v_2) = \pi(x_2') = 1$, $\pi(y_2) = 2$, $\pi(v_3) = 3$ and $\pi(v_1) = 4$. Moreover, there is a bicolored $(1,3)$-path joining x_2' and v_3 in $G - \{x_2\}$. By symmetry, let $\pi(x_3) = 1$ and $\pi(x_3') = 3$. If $L(v_2) \neq \{1,2,3,4\}$, then we may recolor v_2 by a color in $L(v_2) \setminus \{1,2,3,4\}$ and then further color x_2 with 2. The resultant coloring of G is obviously an acyclic L-coloring, a contradiction. So, in the following, assume that $L(v_2) = \{1,2,3,4\}$.

First consider the case when $\pi(y_3) \notin \{1,4\}$. If there is a color a in $L(v_3) \setminus \{1,3,4,\pi(y_3)\}$, then recolor v_3 with a, recolor v_2 with 3, and then color x_2 with 2. Now, assume that $L(v_3) = \{1,3,4,\pi(y_3)\}$. In order to extend π to G, we do like this: recolor v_3 with 1, properly recolor x_3, and then color v_2, x_2 with 3 and 2, respectively.

Now consider the case when $\pi(y_3) = 4$. If there is a color $b \in L(v_3) \setminus \{3,4,\pi(y_3'),\pi(y_3'')\}$, then we recolor v_3 with b, v_2 with 3, and then color x_2 with 2. If the obtained coloring is not acyclic, then it must be the case when $b = 1$. At this moment, we only need to further properly recolor x_3. Now, assume that $L(v_3) = \{3,4,\pi(y_3'),\pi(y_3'')\}$. We are sure that there is at least one color belonging to $\{\pi(y_3'),\pi(y_3'')\}$ that is different from $\pi(x_3'')$, say $\pi(y_3')$. Then recolor v_3 with $\pi(y_3')$ and recolor y_3 with α in $L(y_3) \setminus \{4,\pi(y_3'),\pi(y_3'')\}$. If $\pi(y_3') \notin \{1,2\}$, then it suffices to color x_2 with 3. If $\pi(y_3') = 1$, we may first properly recolor x_3, then recolor v_2 with 3, and afterwards color x_2 with 2. Otherwise, $\pi(y_3') = 2$. If $\alpha \neq 1$, we may similarly color x_2 with 3. Or else, $\alpha = 1$. In this case, one can reassign color 3 to v_2 and then assign color 2 to x_2.

Finally consider the case when $\pi(y_3) = 1$. Recolor v_3 with $c \in L(v_3) \setminus \{3,4,\pi(x_3'')\}$. If $c \notin \{1,2\}$, it suffices to color x_2 with 3. If $c = 2$, then recolor v_2 with 3 and further color x_2 with 2. Otherwise, $c = 1$. It remains us to properly recolor x_3 and y_3, and then recolor v_2 with 3. Afterwards, assign color 2 to x_2 successfully. It is not difficult to inspect that there is no bicolored cycle produced in recoloring process, and therefore G is acyclically L-colorable, a contradiction. $\square$

Lemma 7. *There is no* $(4^*,4^*,4)$*-face in* G.

Proof. Suppose that $f = [v_1v_2v_3]$ is a $(4,4,4)$-face with $d(v_1) = d(v_2) = d(v_3) = 4$. For each $i \in \{1,2,3\}$, denote by x_i and y_i the other two neighbors of v_i that are not on the boundary of f. Suppose to the contrary that v_1 and v_2 are both 4^*-vertices, that is, each v_i is adjacent to exactly one 2-vertex. W.l.o.g., assume that $d(x_1) = d(x_2) = 2$. Let $N(x_1) = \{v_1, x_1'\}$ and $N(x_2) = \{v_2, x_2'\}$. Obviously, $x_1 \neq x_2$ due to the absence of 4-cycles in G.

Let $G' = G - x_1$. By the minimality of G, G' admits an acyclic L-coloring π. If $\pi(v_1) \neq \pi(x_1')$, then it is easy to extend π to the whole graph G by properly coloring x_1. Otherwise, $\pi(v_1) = \pi(x_1')$. Let $L(x_1) = \{1,2,3,4\}$. If we are not able to find a way to acyclically color x_1, then assume w.l.o.g. that $\pi(v_1) = \pi(x_1') = 1$, $\pi(y_1) = 2$, $\pi(v_2) = 3$ and $\pi(v_3) = 4$. Furthermore, in G' there exist one bicolored $(1,3)$-path, denoted by P_1, joining x_1' and v_2 and one bicolored $(1,4)$-path, denoted by P_2, joining x_1' and v_3. It follows that $1 \in \{\pi(x_2), \pi(y_2)\}$ and $1 \in \{\pi(x_3), \pi(y_3)\}$. By symmetry, assume that $\pi(x_3) = 1$. If $L(v_1) \neq \{1,2,3,4\}$, then it suffices to recolor v_1 by a color in $L(v_1) \setminus \{1,2,3,4\}$ and then color x_1 with 2. So, in what follows, assume that $L(v_1) = \{1,2,3,4\}$. To extend π from G' to G, we consider the following cases according to the colors of x_2 and y_2.

Case 1: $\pi(y_2) = 1$. If there exists a color $a \in L(v_2) \setminus \{1,3,4,\pi(x_2')\}$, then we first recolor v_2 with a and v_1 with 3. Then properly recolor x_2, and finally color x_1 with 2. Otherwise, assume that $L(v_2) = \{1,3,4,\pi(x_2')\}$. It follows that $\pi(x_2') \neq 3$. If $\pi(y_3) = 1$, we can recolor v_2 with $\pi(x_2')$, x_2 with a color $c \in L(x_2) \setminus \{1,3,\pi(x_2')\}$, v_1 with 3, and afterwards color x_1 with 2.

Next assume that $\pi(y_3) \neq 1$. Then we first recolor v_3 with $d \in L(v_3) \setminus \{1,4,\pi(y_3)\}$. If $d \notin \{2,3\}$, then we only need to color x_1 with 4 directly and continue to recolor x_2 with a color different from 1, 3, d in the case when $\pi(x_2) = d$ and $\pi(x_2') = 3$. If $d = 2$, then recolor v_1 with 4 and further color x_1 with 2, and similarly further recolor x_2 with a color different from 1, 2, 3 when $\pi(x_2) = 2$ and $\pi(x_2') = 3$. Now consider the case that $d = 3$. At this moment, we need to recolor v_2 with $\pi(x_2')$, x_2 with $c_1 \in L(x_2) \setminus \{1,3,\pi(x_2')\}$, and then color x_1 with 4. Noting that $c_1 \neq 1$, so the obtained coloring is not acyclic, then it must be the case when $\pi(x_2') = 2$. If $c_1 \neq 4$, it suffices to recolor v_1 with 4 and then color x_1 with 2. Or else, $c_1 = 4$, implying that $L(x_2) = \{1,2,3,4\}$. Recall that there is one bicolored $(1,3)$-path P_1 joining x_1' and y_2 and one bicolored $(1,4)$-path P_2 joining x_1' and x_3. Then we recolor v_1 with 4 and then color x_1 with 2. If the resultant coloring is not acyclic, then there exists one bicolored $(2,4)$-path in G', say P_3, joining y_1 and x_2'. Since $\{1,3\} \cap \{2,4\} = \emptyset$, together with the planarity of G, one may obtain that only two possible cases may occur, as depicted in Fig 1.

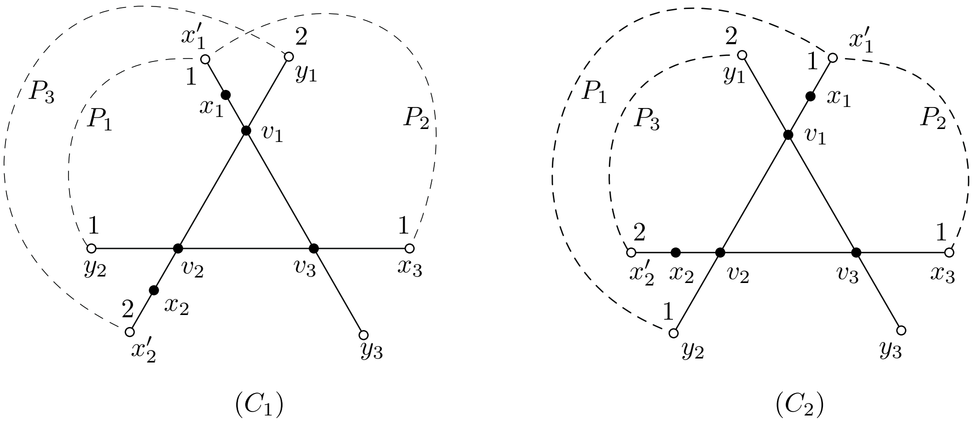

Figure 1. Two possible cases occurred in Lemma 7.

Clearly, for configuration (C1), we first recolor x_2 with 3. Since P_2 is a bicolored $(1,4)$-path, we deduce that there is no bicolored $(2,3)$-path connecting y_1 and x_2 outside of f. Thus, we can obtain an acyclic L-coloring of G by recoloring v_1 with 3, v_2 with 2, v_3 with 4 and later coloring x_1 with 2. Similarly, for configuration (C2), we can recolor x_2 and v_3 with 4, v_2 with 2, v_1 with 3 and then color x_1 with 2. Since P_1 is a bicolored $(1,3)$-path, we declare that there is no bicolored $(2,4)$-path connecting x_2' and v_3 outside of f. Hence, the obtained coloring is an acyclic L-coloring, a contradiction.

Case 2: $\pi(x_2) = 1$. Since $\pi(y_2) \neq 1$, we deduce that $\pi(x_2') = 3$ by the existence of bicolored $(1,3)$-path P_1.

- ▷ $\pi(y_2) = 4$. If there is a color $a \in L(v_2) \setminus \{1,3,4,\pi(y_3)\}$, then we recolor v_2 with a and then color x_1 with 3. Otherwise, assume that $L(v_2) = \{1,3,4,\pi(y_3)\}$. It follows that $\pi(y_3) \neq 1$. If $L(v_3) \neq \{1,2,4,\pi(y_3)\}$, then we may recolor v_3 with $b \in L(v_3) \setminus \{1,2,4,\pi(y_3)\}$, v_2 with 1, v_1 with 4 and then color x_1 with 2. Afterwards, x_2 can be properly recolored. Now suppose that $L(v_3) = \{1,2,4,\pi(y_3)\}$. It is easy to recolor v_3 with 2, v_1 with 4 and then color x_1 with 2. In each case, we always reach an acyclic L-coloring of G, a contradiction.
- ▷ $\pi(y_2) \neq 4$. We first recolor x_2 with $a \in L(x_2) \setminus \{1,3,\pi(y_2)\}$. If $a \neq 4$, then we immediately color x_1 with 3. Or else, $a = 4$. This guarantees us that $L(x_2) = \{1,3,4,\pi(y_2)\}$. We can do as follows: recolor x_2 with $\pi(y_2)$, v_2 with a color in $L(v_2) \setminus \{3,4,\pi(y_2)\}$, v_1 with 3 and finally color x_1 with 2. □

Lemma 8. *Let v be a 5-vertex. Then*

(F1) *if v is incident to a $(3,3,5)$-face and $n_2(v)=1$, then $p_3(v)=0$;*

(F2) *if v is incident to a $(3,4^{2p},5)$-face, then $n_2(v)\leqslant 1$;*

(F3) *if v is incident to a $(4^*,4^*,5)$-face and $n_2(v)=2$, then $p_3(v)=0$.*

Proof. Let $v_1,v_2,\ldots,v_5$ denote all the neighbors of v in a cyclic order. In what follows, in each case, we always denote by v_i' the other neighbor of v_i (different from v) if $d(v_i)=2$, and x_i, y_i the other two neighbors of v_i (different from v) if v_i is a pendant triangular 3-vertex of v. Here, $v_ix_iy_iv_i$ forms a 3-cycle. We will make use of contradictions to show (F1) to (F3).

(F1) Suppose to the contrary that $f_1=[v_1v_2v]$ is a $(3,3,5)$-face, v_3 is a 2-vertex and v_4 is a pendant triangular 3-vertex of v. By definition, both v_1 and v_2 are 3-vertices. We let u_1 or u_2 denote the neighbor of v_1 or v_2 that is not on the boundary of f_1.

Let $G'=G-\{v,v_1,v_2,v_3,v_4\}$. It is obvious that G' has an acyclic L-coloring π by the minimality of G. Let $S=\{u_1,u_2,v_3',x_4,y_4\}$. Since $|L(v)\setminus\{\pi(v_5)\}|\geqslant 3$ and $|S|=5$, we conclude that there exists a color α belonging to $L(v)\setminus\{\pi(v_5)\}$ appearing at most once on the set S. We first color v with α. Clearly, if no vertex of S is colored with α, then we may firstly assign a color distinct from that of v, u_1, u_2 to v_1, and then properly color remaining vertices v_2, v_3, v_4, in succession. So in the following, assume that such color α appears exactly once on S.

By symmetry, we have three cases to handle. If $\pi(u_1)=\alpha$, then it suffices to color v_1 with a color in $L(v_1)\setminus\{\alpha,\pi(v_5),\pi(u_2)\}$ and then properly color v_2, v_3, v_4 in the given order. If $\pi(v_3')=\alpha$, then we color v_3 with a color in $L(v_3)\setminus\{\alpha,\pi(v_5)\}$, v_1 with a color in $L(v_1)\setminus\{\alpha,\pi(u_1),\pi(u_2)\}$, and finally properly color v_2 and v_4 in the given order. Now consider the case when $\pi(x_4)=\alpha$. We can first color v_4 with a color in $L(v_4)\setminus\{\alpha,\pi(v_5),\pi(y_4)\}$. Then color v_1 with a color belonging to $L(v_1)\setminus\{\alpha,\pi(u_1),\pi(u_2)\}$. Finally, v_2 and v_3 can be further properly colored without any trouble. One may verify that in each case we always obtain an acyclic L-coloring of G, a contradiction.

(F2) Suppose to the contrary that $f_1=[v_1v_2v]$ is a $(3,4^{2p},5)$-face and v_3, v_4 are both 2-vertices. Then $d(v_1)=3$ and v_2 is a 4^{2p}-vertex, namely, $p_3(v_2)=2$. Let $N(v_1)=\{v,u_1,v_2\}$ and $N(v_2)=\{v,v_1,w_1,w_2\}$, where w_1 and w_2 are pendant triangular 3-vertices of v_2. For each $i\in\{1,2\}$, denote by x_i, y_i the other two neighbors of w_i such that $w_ix_iy_iw_i$ forms a 3-cycle.

Let $G'=G-\{v,v_1,v_2,v_3,v_4\}$. By the minimality of G, G' admits an acyclic L-coloring π. Let $S=\{u_1,w_1,w_2,v_3',v_4'\}$. Similarly, since $|L(v)\setminus\{\pi(v_5)\}|\geqslant 3$ and $|S|=5$, we assert that there exists a color $\alpha\in L(v)\setminus\{\pi(v_5)\}$ appearing at most once on the set S. Firstly, assign such color α to v. The following discussion is split into two cases:

Case 1: Assume that exactly one of w_1 and w_2 is colored with α, say $\pi(w_1) = \alpha$.

It follows that none of u_1, w_2, v_3' and v_4' has color α. Because v_3 and v_4 are 2-vertices, we may first properly color each of them. Then, select a color $c \in L(v_2) \setminus \{\alpha, \pi(w_2), \pi(v_5)\}$ for v_2. If $c \neq \pi(u_1)$, then we further properly color v_1 and thus we are done. Otherwise, we can choose a color in $L(v_1) \setminus \{c, \alpha, \pi(w_2)\}$ for v_1. In each case, one may verify that the resulting coloring of G is an acyclic L-coloring, a contradiction.

Case 2: Assume that $\pi(w_1) \neq \alpha$ and $\pi(w_2) \neq \alpha$.

Subcase 2.1: $\pi(w_1) \neq \pi(w_2)$. Noting that w_1, w_2 and v have pairwise distinct colors, we can choose a color c belonging to $L(v_2) \setminus \{\alpha, \pi(w_1), \pi(w_2)\}$ for v_2. Suppose that $c \neq \pi(u_1)$. If $\pi(v_3') \neq \alpha$ and $\pi(v_4') \neq \alpha$, then we may first properly color each of v_3 and v_4. Then if $\pi(u_1) \neq \alpha$, it suffices to further properly color v_1; otherwise, we only need to choose a color for v_1 that is distinct from α, c and $\pi(v_5)$. In what follows, w.l.o.g., suppose that $\pi(v_3') = \alpha$. It grantees us that $\pi(u_1) \neq \alpha$. Thus, we first properly color v_4, then color v_3 with a color different from α and $\pi(v_5)$, and afterwards properly color v_1.

Next, suppose that $c = \pi(u_1)$. It implies that $\pi(u_1) \neq \alpha$ due to $c \neq \alpha$. First, we can find a possible way for coloring v_3 and v_4 as follows: if $\pi(v_i) \neq \alpha$ for each $i = 3, 4$, then properly color each of v_3 and v_4; otherwise, say $\pi(v_3') = \alpha$, and thus we can color v_3 with a color different from α and $\pi(v_5)$, and then properly color v_4. In what follows, in order to obtain an acyclic L-coloring of G, it remains us to show how to color v_1. If $L(v_2) \neq \{c, \alpha, \pi(w_1), \pi(w_2)\}$, then recolor v_2 with a color in $L(v_2) \setminus \{c, \alpha, \pi(w_1), \pi(w_2)\}$ and then further properly color v_1. Or else, suppose that $L(v_2) = \{c, \alpha, \pi(w_1), \pi(w_2)\}$. At this moment, if v_1 cannot be given a proper color, then one may easily deduce that $L(v_1) = \{c, \alpha, \pi(w_1), \pi(w_2)\}$ and $c \in \{\pi(x_i), \pi(y_i)\}$ for each $i \in \{1, 2\}$. By symmetry, let $\pi(x_1) = \pi(x_2) = c$. In this case, we may do as follows: recolor v_2 with $\pi(w_1)$, then color v_1 with $\pi(w_2)$, and finally properly recolor w_1.

Subcase 2.2: $\pi(w_1) = \pi(w_2)$. Recolor w_1 with a color $\beta \in L(w_1) \setminus \{\pi(w_1), \pi(x_1), \pi(y_1)\}$. If $\beta \neq \alpha$, then we go back to the previous Subcase 2.1. Otherwise, assume that $\beta = \alpha$. If none of u_1, v_3' and v_4' has been colored with α, then we reduce the following proof to Case 1. Or else, exactly one of u_1, v_3', v_4' is colored with α. It implies that there exists a color $\gamma \in L(v) \setminus \{\alpha, \pi(v_5)\}$ such that γ appears at most once on the set $S' = \{u_1, w_2, v_3', v_4'\}$. Now we recolor v with γ. If $\gamma = \pi(w_2)$, then the following argument is reduced to Case 1. Otherwise, we go back to Subcase 2.1.

(F3) Suppose to the contrary that $f_1 = [v_1 v_2 v]$ is a $(4^*, 4^*, 5)$-face, v_3 and v_4 are 2-vertices, and v_5 is a pendant triangular 3-vertex. By definition, both v_1 and v_2

are special 4-vertices. For each $i = 1, 2$, let u_i, w_i denote the other two neighbors of v_i not on the boundary of f_1 such that $d(u_i) = 2$. Denote by u'_1 and u'_2 another neighbor of u_1 and u_2, respectively.

By the minimality of G, $G - \{v, v_3, v_4, v_5\}$ has an acyclic L-coloring π. Let $S = \{v'_3, v'_4, x_5, y_5\}$. Obviously, there exists a color $\alpha \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ that appears at most twice on the set S. We assign such color α to v firstly. If there is no vertex of S colored with α, then it is easy to extend π to G by properly coloring v_3, v_4, v_5 in succession. Next, let us discuss the following two cases depending on the number of occurrences of α.

Case 1: There is exactly one vertex of S colored with α. By symmetry, we have two possibilities as follows:

- ▷ $\pi(v'_3) = \alpha$. We can color v_3 with a color belonging to $L(v_3) \setminus \{\alpha, \pi(v_1), \pi(v_2)\}$ and then properly color each of v_4 and v_5.
- ▷ $\pi(x_5) = \alpha$. First properly color each of v_3 and v_4. Assume, w.l.o.g., that $\pi(v_1) = 1$ and $\pi(v_2) = 2$. If $L(v_5) \neq \{1, 2, \alpha, \pi(y_5)\}$, then it suffices to further color v_5 with a color in $L(v_5) \setminus \{1, 2, \alpha, \pi(y_5)\}$. Next, assume that $L(v_5) = \{1, 2, \alpha, \pi(y_5)\}$. If we are still not able to successfully color v_5, then it must be the case when in G' there exists a bicolored $(1, \alpha)$-path P_1 connecting v_1 and x_5 and a bicolored $(2, \alpha)$-path P_2 connecting v_2 and x_5. That is, $\alpha \in \{\pi(u_1), \pi(w_1)\}$ and $\alpha \in \{\pi(u_2), \pi(w_2)\}$.

  Consider the case when $\pi(w_1) = \alpha$. If there is a color β belonging to $L(v_1) \setminus \{1, 2, \alpha, \pi(u'_1)\}$, then we recolor v_1 with β, color v_5 with 1, and then properly color u_1 (if needed). Otherwise, let $L(v_1) = \{1, 2, \alpha, \pi(u'_1)\}$. We recolor v_1 with $\pi(u'_1)$ and then color v_5 with 1. Finally, it remains us to recolor u_1 with a color in $L(u_1) \setminus \{2, \alpha, \pi(u'_1)\}$.

  Since the discussion for the case when $\pi(w_2) = \alpha$ is the same as the above case, in what follows, assume that $\pi(w_1) \neq \alpha$ and $\pi(w_2) \neq \alpha$. By the existences of P_1 and P_2, we deduce that $\pi(u_1) = \pi(u_2) = \alpha$. Moreover, $\pi(u'_1) = 1$ and $\pi(u'_2) = 2$. If $L(v_1) \neq \{1, 2, \alpha, \pi(w_1)\}$, then recolor v_1 with a color c in $L(v_1) \setminus \{1, 2, \alpha, \pi(w_1)\}$ and further color v_5 with 1. If the resulting coloring is not acyclic, then we deduce that $\pi(w_2) = c$ and $\pi(w_1) = 2$. In this case, we further recolor v_2 with a color distinct from 2, c, α, and finally recolor v_5 with 2. Otherwise, $L(v_1) = \{1, 2, \alpha, \pi(w_1)\}$. Similarly, we deduce that $L(v_2) = \{1, 2, \alpha, \pi(w_2)\}$. At this moment, we may destroy P_1 and P_2 by switching the colors of v_1 and v_2 and then color v_5 with 1 successfully.

Case 2: There are exactly two vertices of S colored with α. Since $\pi(x_5) \neq \pi(y_5)$, w.l.o.g., assume that $\pi(v'_3) = \pi(x_5) = \alpha$. In this case, we may further suppose that $L(v) = \{1, 2, \alpha, \beta\}$ such that $\pi(v'_4) = \pi(y_5) = \beta$. First consider the case when

$L(v_5) \neq \{1, 2, \alpha, \beta\}$. Choose a color $c \in L(v_5) \setminus \{1, 2, \alpha, \beta\}$ for v_5. If $L(v_3) \neq \{1, 2, \alpha, c\}$, then it is easy to further color v_3 with a color in $L(v_3) \setminus \{1, 2, \alpha, c\}$ and afterwards properly color v_4. Similarly, if $L(v_4) \neq \{1, 2, \beta, c\}$, then we can continue to color v_4 with a color in $L(v_4) \setminus \{1, 2, \beta, c\}$, recolor v with β and lastly properly color v_3. Now assume that $L(v_3) = \{1, 2, \alpha, c\}$ and $L(v_4) = \{1, 2, \beta, c\}$. Moreover, one may easily inspect that $\{\pi(u_1), \pi(w_1)\} = \{\pi(u_2), \pi(w_2)\} = \{\alpha, \beta\}$. W.l.o.g., assume that $\pi(u_1) = \alpha$ and $\pi(w_1) = \beta$. At present, we color v_3 with 1 and properly color v_4. If the resulting coloring is not acyclic, then it should be the case when $\pi(u_1') = 1$. We only need to further recolor u_1 by a color distinct from 1, α, β.

Next, consider the case when $L(v_5) = \{1, 2, \alpha, \beta\}$. If $\{\pi(u_1), \pi(w_1)\} \neq \{\alpha, \beta\}$, say $\alpha \notin \{\pi(u_1), \pi(w_1)\}$, then color v_5 with 1, and v_3 with a color in $L(v_3) \setminus \{\alpha, 1, 2\}$, and then properly color v_4. So next, assume that $\{\pi(u_1), \pi(w_1)\} = \{\alpha, \beta\}$, say $\pi(u_1) = \alpha$ and $\pi(w_1) = \beta$. Similarly, we derive that $\pi(u_1') = 1$. Now we can recolor u_1 with a color in $L(u_1) \setminus \{1, \alpha, \beta\}$, then color v_5 with 1, v_3 with a color in $L(v_3) \setminus \{1, 2, \alpha\}$, and finally properly color v_4. □

Lemma 9. *Let v be a 6-vertex incident to a $(6, 3, 3^+)$-face f. Then*

(Q1) *$n_2(v) \leqslant 3$;*

(Q2) *if $n_2(v) = 3$, then f cannot be a $(6, 3, 3)$-face.*

Proof. Let $v_1, v_2, \ldots, v_6$ denote all the neighbors of v in cyclic order. Suppose that $f = [vv_1v_2]$ is a $(6, 3, 3^+)$-face such that $d(v_1) = 3$ and $d(v_2) \geqslant 3$. Let $N(v_1) = \{u_1, v_2, v\}$. In each case of the following discussion, we denote by v_i' the other neighbor of v_i whenever $d(v_i) = 2$. Next, we shall make use of contradictions to show (Q1) and (Q2).

(Q1) Suppose to the contrary that $n_2(v) = 4$ so that v_3, v_4, v_5, v_6 are all 2-vertices. Let $G' = G - \{v, v_3, v_4, v_5, v_6\}$. Then, by the minimality of G, G' admits an acyclic L-coloring π. Let $S = \{v_3', v_4', v_5', v_6'\}$. Since $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geqslant 2$ and $|S| = 4$, we declare that there exists a color, say α, belonging to $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ such that α appears at most twice on the set S. Firstly, we assign α to v. If there is no vertex of S colored with α, then it is easy to properly color each v_i, where $i \in \{3, 4, 5, 6\}$. If exactly one vertex of S, say v_3', is colored with α, then we can color v_3 with $a \in L(v_3) \setminus \{\alpha, \pi(v_1), \pi(v_2)\}$ and then properly color each remaining 2-vertex v_i for each $i \in \{4, 5, 6\}$. So, in what follows, assume that there are exactly two vertices of S colored with α. At this moment, one may immediately deduce that $L(v) = \{\alpha, \beta, \pi(v_1), \pi(v_2)\}$ such that $\pi(v_3) = \pi(v_4) = \alpha$ and $\pi(v_5) = \pi(v_6) = \beta$. Further, w.l.o.g., we may assume that $\pi(u_1) \neq \alpha$ since otherwise we may choose β. Then it suffices to color v_3 with $a \in L(v_3) \setminus \{\alpha, \pi(v_2)\}$, color v_4 with $b \in L(v_4) \setminus \{a, \alpha, \pi(v_2)\}$, and finally properly color each of v_5 and v_6.

(Q2) Suppose to the contrary that f is a $(6,3,3)$-face such that $d(v_1) = d(v_2) = 3$. Let v_3, v_4, v_5 be all 2-vertices. Denote by u_2 another neighbor of v_2 that is different from v and v_1. By the minimality of G, $G - \{v, v_1, v_2, v_3, v_4, v_5\}$ admits an acyclic L-coloring π. Let $S = \{u_1, u_2, v_3', v_4', v_5'\}$. Similarly, there exists a color $\alpha \in L(v) \setminus \{\pi(v_6)\}$ that appears at most once on the set S. We first color v with α. If no vertex of S has been colored with α, then it is easy to first color v_1 with $a \in L(v_1) \setminus \{\alpha, \pi(u_1), \pi(u_2)\}$, and then properly color each of v_2, v_3, v_4, v_5 in order. Next, by symmetry, we have to deal with two cases below in light of the location of the vertex whose color is α.

- ▷ $\pi(u_1) = \alpha$. Then color v_1 with $a \in L(v_1) \setminus \{\alpha, \pi(v_6), \pi(u_2)\}$. Afterwards, each of v_2, v_3, v_4, v_5 can be further properly colored.
- ▷ $\pi(v_3') = \alpha$. Then color v_1 with $c \in L(v_1) \setminus \{\alpha, \pi(u_1), \pi(u_2)\}$, v_3 with $d \in L(v_3) \setminus \{\alpha, \pi(v_6)\}$, and finally properly color each of v_2, v_4, v_5 in order.

In each case, one may easily verify that the obtained coloring of G is an acyclic L-coloring. This contradicts the choice of G. □

3.2. Discharging process. Next, we are going to apply a discharging procedure to reach a contradiction. We define a weight function ω on $V(G) \cup F(G)$ by letting $\omega(v) = 2d(v) - 6$ if $v \in V(G)$ and $\omega(f) = d(f) - 6$ if $f \in F(G)$. By Euler's formula, we have that $\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12$. By transferring weights from one element to another, we shall obtain a new non-negative weight function $\omega^*(x)$ for all $x \in V(G) \cup F(G)$. Since the total sum of weights is kept fixed when the discharging is in process, this leads to obvious contradiction

$$-12 = \sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} \omega^*(x) \geqslant 0,$$

and hence we complete the proof of Theorem 1.

Let $v \in V(G)$. If v is a 4^{mp}-vertex with $m \in \{1,2\}$, then we denote $v \in V_{4^{mp}}$. Similarly, if v is a 4^*-vertex, then we say that $v \in V_{4^*}$.

In what follows, for $x, y \in V(G) \cup F(G)$ we use $\sigma(x \to y)$ to denote the amount of weights transferred from x to y. Our discharging rules are defined as follows:

(R1) Every 4^+-vertex sends 1 to each adjacent 2-vertex and $\frac{1}{4}$ to each adjacent pendant triangular 3-vertex.

(R2) Every 5^+-vertex sends $\frac{1}{2}$ to each incident 5-face.

(R3) Let x be a 4-vertex incident to a 5-face $f = [xyzuv]$. Then

(R3.1) $\sigma(x \to f) = \frac{1}{2}$ if f is either a $(4,3,4,2,4^+)$-face or a $(4,3,3^+,3^+,3)$-face;

(R3.2) $\sigma(x \to f) = \frac{3}{8}$ if f is a $(4,3,3,4^+,4^+)$-face;

(R3.3) $\sigma(x \to f) = \frac{1}{4}$ otherwise.

(R4) Let $f = [xyz]$ be a 3-face such that $d(x) \leqslant d(y) \leqslant d(z)$. We set

(R4.1) $(3,3,5^+) \to (\frac{1}{4}, \frac{1}{4}, \frac{5}{2})$;

(R4.2) $(3,4,4) \to \begin{cases} (\frac{1}{4}, \frac{5}{4}, \frac{3}{2}) & \text{if } y \in V_{4^p}; \\ (\frac{1}{4}, \frac{11}{8}, \frac{11}{8}) & \text{otherwise.} \end{cases}$

(R4.3) $(3,4^+,5^+) \to \begin{cases} (\frac{1}{4}, 1, \frac{7}{4}) & \text{if } y \in V_{4^{2p}}; \\ (\frac{1}{4}, \frac{5}{4}, \frac{3}{2}) & \text{if } y \in V_{4^p}; \\ (\frac{1}{4}, \frac{11}{8}, \frac{11}{8}) & \text{otherwise.} \end{cases}$

(R4.4) $(4,4,4) \to \begin{cases} (\frac{3}{4}, \frac{9}{8}, \frac{9}{8}) & \text{if } x \in V_{4^*} \text{ and } y, z \notin V_{4^*}; \\ (1,1,1) & \text{otherwise.} \end{cases}$

(R4.5) $(4,4,5^+) \to \begin{cases} (\frac{3}{4}, \frac{3}{4}, \frac{3}{2}) & \text{if } x, y \in V_{4^*}; \\ (\frac{3}{4}, \frac{9}{8}, \frac{9}{8}) & \text{if } x \in V_{4^*} \text{ and } y \notin V_{4^*}; \\ (1,1,1) & \text{otherwise.} \end{cases}$

(R4.6) $(4^+,5^+,5^+) \to \begin{cases} (\frac{3}{4}, \frac{9}{8}, \frac{9}{8}) & \text{if } x \in V_{4^*}; \\ (1,1,1) & \text{otherwise.} \end{cases}$

Fact 1. *By* (R3), *every* 4*-vertex sends weight at least* $\frac{1}{4}$ *to each of incident* 5*-face.*

Let us first check that $\omega^*(f) \geqslant 0$ for each k-face f. Obviously, $k \geqslant 3$. Moreover, $k \neq 4$ due to the absence of 4-cycles. If $k \geqslant 6$, then it is trivial that $\omega^*(f) = \omega(f) \geqslant 0$ since f does not participate in the discharging by (R1)–(R4). Next, we consider the case when $k \in \{3, 5\}$.

First suppose that f is a 3-face. Then $\omega(f) = -3$. Denote $f = [xyz]$ such that $d(x) \leqslant d(y) \leqslant d(z)$. By (A2), none of x, y, z can be a 2-vertex. We have the following three cases:

- ▷ $d(x) = 3$. If $d(y) = 3$, then $d(z) \geqslant 5$ by (A4) and Lemma 2. So $\omega^*(f) \geqslant -3 + \frac{1}{4} + \frac{1}{4} + \frac{5}{2} = 0$ by (R4.1). If $d(y) \geqslant 5$, implying that $d(z) \geqslant 5$, then $\omega^*(f) \geqslant -3 + \frac{1}{4} + \frac{11}{8} \times 2 = 0$ by (R4.3). In what follows, assume that $d(y) = 4$. If $d(z) \geqslant 5$, namely f is a $(3,4,5^+)$-face, then by (R4.3) we see that f receives weight at least W from all incident vertices, where $W = \min\{\frac{1}{4} + 1 + \frac{7}{4}, \frac{1}{4} + \frac{5}{4} + \frac{3}{2}, \frac{1}{4} + \frac{11}{8} \times 2\} = 3$. Hence, $\omega^*(f) \geqslant -3 + 3 = 0$. Now we suppose that $d(z) = 4$. It follows that f is a $(3,4,4)$-face. By (R4.2), one may deduce that either $\omega^*(f) \geqslant -3 + \frac{1}{4} + \frac{5}{4} + \frac{3}{2} = 0$ or $\omega^*(f) \geqslant -3 + \frac{1}{4} + \frac{11}{8} \times 2 = 0$.
- ▷ $d(x) = 4$. If $d(y) \geqslant 5$, that is, f is a $(4,5^+,5^+)$-face, then by (R4.6), f gets weight at least W from all its incident vertices, where $W = \min\{\frac{3}{4} + \frac{9}{8} \times 2, 1 \times 3\} = 3$.

Hence, $\omega^*(f) \geqslant -3+3=0$. Now assume that $d(y)=4$. If $d(z) \geqslant 5$, namely f is a $(4,4,5^+)$-face, then by (R4.5), one may easily obtain that f gets weight at least 3 and therefore $\omega^*(f) \geqslant -3+3=0$. Next, it remains us to consider the case when $d(z)=4$. In other words, f is a $(4,4,4)$-face at this moment. By (R4.4), it is easy to calculate that either $\omega^*(f) \geqslant -3+\frac{3}{4}+\frac{9}{8}\times 2=0$ or $\omega^*(f) \geqslant -3+1\times 3=0$.

- ▷ $d(x) \geqslant 5$. It follows that f is a $(5^+,5^+,5^+)$-face. By (R4.6), we have that either $\omega^*(f) \geqslant -3+\frac{3}{4}+\frac{9}{8}\times 2=0$ or $\omega^*(f) \geqslant -3+1\times 3=0$.

Now suppose that f is a 5-face. Obviously, $\omega(f)=-1$. By (A1), there is no 1-vertex in G, and thus we know that the boundary of f is a cycle. Let $f=[v_1v_2v_3v_4v_5]$. Denote by $n_k(f)$ the number of k-vertices incident to f. By (A3), $n_2(f) \leqslant 2$. If $n_2(f)=2$, w.l.o.g., assume that $d(v_1)=d(v_3)=2$, then $d(v_i) \geqslant 4$ for all $i \in \{2,4,5\}$ by applying (A3) again. Moreover, $d(v_2) \geqslant 5$ by (A5). So by (R2), $\sigma(v_2 \to f)=\frac{1}{2}$. Thus, $\omega^*(f) \geqslant -1+\frac{1}{2}+\frac{1}{4}\times 2=0$ by Fact 1. Next consider the case when $n_2(f)=1$. W.l.o.g., let $d(v_1)=2$. Then $d(v_i) \geqslant 4$ for each $i \in \{2,5\}$ by (A3). If one of v_3 and v_4 is a 5^+-vertex, then we may similarly deduce that $\omega^*(f) \geqslant -1+\frac{1}{2}+\frac{1}{4}\times 2=0$ by (R2) and Fact 1. If v_3 and v_4 are both 4-vertices, then by Fact 1, we have that $\omega^*(f) \geqslant -1+\frac{1}{4}\times 4=0$. So, in what follows, assume that for each $i \in \{3,4\}$, $d(v_i) \leqslant 4$, and at most one of them can be a 4-vertex. We have two possibilities:

- ▷ $d(v_3)=d(v_4)=3$. Then $d(v_2) \neq 4$ since otherwise v_3 is adjacent to two weak vertices v_2 and v_4, which contradicts (A4). By symmetry, $d(v_5) \neq 4$, and hence both v_2 and v_5 are of degree at least 5. It is easy to obtain that $\omega^*(f) \geqslant -1+\frac{1}{2}\times 2=0$ by (R2).
- ▷ $d(v_3)=3$ and $d(v_4)=4$. In this case, we may further suppose that $d(v_2)=d(v_5)=4$ by similar discussion as above. Notice that f is a $(4,3,4,2,4)$-face. By (R3.1), $\sigma(v_4 \to f)=\frac{1}{2}$, and therefore $\omega^*(f) \geqslant -1+\frac{1}{2}+\frac{1}{4}\times 2=0$ by Fact 1.

Finally, suppose that $n_2(f)=0$, meaning that $d(v_i) \geqslant 3$ for all $i \in \{1,2,\ldots,5\}$. Since each 3-vertex is so called weak, by (A4) we confirm that $n_3(f) \leqslant 3$. If $n_3(f) \leqslant 1$, then it is obvious that $\omega^*(f) \geqslant -1+\frac{1}{4}\times 4=0$ by Fact 1. If $n_3(f)=3$, say $d(v_1)=d(v_3)=d(v_4)=3$, then $d(v_i) \geqslant 4$ for each $i \in \{2,5\}$. Notice that if $d(v_i)=4$, then f is a $(4,3,3,4^+,3)$-face, and thus by (R3.1) we have that $\sigma(v_i \to f)=\frac{1}{2}$. If $d(v_i) \geqslant 5$, then by (R2) we know that v_i also sends weight $\frac{1}{2}$ to f. Hence, $\omega^*(f) \geqslant -1+\frac{1}{2}\times 2=0$. So, next assume that $n_3(f)=2$. If $d(v_1)=d(v_2)=3$, then each v_i is a 4^+-vertex for the remaining index $i \in \{3,4,5\}$. If $n_{5^+}(f) \geqslant 1$, then $\omega^*(f) \geqslant -1+\frac{1}{2}+\frac{1}{4}\times 2=0$ by (R2) and Fact 1. Or else, $d(v_3)=d(v_4)=d(v_5)=4$. By (R3.2), each of v_3 and v_5 sends weight $\frac{3}{8}$ to f, and thus $\omega^*(f) \geqslant -1+\frac{1}{4}+\frac{3}{8}\times 2=0$ by Fact 1. Otherwise, assume that $d(v_1)=d(v_3)=3$. Then $d(v_i) \geqslant 4$ for all $i \in \{2,4,5\}$. If $d(v_2)=4$, then f is a $(4,3,4^+,4^+,3)$-face. By (R3.1), $\sigma(v_2 \to f)=\frac{1}{2}$.

If $d(v_2) \geqslant 5$, then again we deduce that $\sigma(v_2 \to f) = \frac{1}{2}$ by applying (R2). Consequently, $\omega^*(f) \geqslant -1 + \frac{1}{2} + \frac{1}{4} \times 2 = 0$ by Fact 1.

In what follows, it remains us to verify that $\omega^*(v) \geqslant 0$ for each k-vertex v. By (A1), $k \geqslant 2$. For our convenience, we let $v_1, v_2, \ldots, v_k$ denote the neighbors of v in clockwise order. Let f_i be the face with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \ldots, k$, where indices are taken modulo k. If $k = 2$, then $\omega(v) = -2$. It follows from (A3) that v is adjacent to two 4^+-vertices, implying that $\omega^*(v) \geqslant -2 + 1 \times 2 = 0$ by (R1). If $k = 3$, then $\omega(v) = 0$. Notice that v only sends weight $\frac{1}{4}$ to its incident 3-face by (R4). If it works, then v is a pendant triangular 3-vertex of a neighbor which is of degree at least 4 by (A3) and Lemma 3. Thus, by (R1), v receives the same weight $\frac{1}{4}$ from it and hence $\omega^*(v) \geqslant -\frac{1}{4} + \frac{1}{4} = 0$. So, in what follows, we are going to show that $\omega^*(v) \geqslant 0$ for each k-vertex, where $k \geqslant 4$.

Case 1: $k = 4$. Then $\omega(v) = 2$. By (A5), $n_2(v) \leqslant 1$. Moreover, $m_3(v) \leqslant 1$ and $m_5(v) \leqslant 1$ by the assumption of G.

First suppose that $m_3(v) = 0$. Then $d(f_i) \geqslant 5$ for all $i \in \{1, \ldots, 4\}$ due to $m_4(v) = 0$. Since $m_5(v) \leqslant 1$, v sends weight at most $\frac{1}{2}$ in total to all its incident faces by (R3). So if $n_2(v) = 0$, then it is easy to obtain that $\omega^*(v) \geqslant 2 - \frac{1}{2} - \frac{1}{4} \times 4 = \frac{1}{2}$ by (R1). Otherwise $n_2(v) = 1$. At present, by (A6) we know that $p_3(v) = 0$, and therefore $\omega^*(v) \geqslant 2 - 1 - \frac{1}{2} = \frac{1}{2}$ by (R1).

Next suppose that $m_3(v) = 1$, w.l.o.g., say $d(f_1) = 3$. Then for each $i \in \{2, 3, 4\}$, $d(f_i) \geqslant 5$. Note that $p_3(v) \leqslant 2$. Let us first consider the case when f_1 is incident to a 3-vertex, i.e., $d(v_1) = 3$. By (A6), $n_2(v) = 0$.

- ▷ $p_3(v) = 0$. Then $\omega^*(v) \geqslant 2 - \frac{3}{2} - \frac{1}{2} = 0$ by (R3) and (R4).
- ▷ $p_3(v) = 1$. Namely, $v \in V_{4^p}$. If $d(v_2) \geqslant 5$, say f_1 is a $(3, 4^p, 5^+)$-face, then by (R4.3), $\sigma(v \to f_1) = \frac{5}{4}$. Otherwise, $d(v_2) = 4$. Namely, f_1 is a $(3, 4^p, 4)$-face. By Lemma 5, v_2 cannot be any 4^p-vertex. So by (R4.2), we know that v sends weight $\frac{5}{4}$ to f_1 rather than $\frac{3}{2}$. In both cases, one may always obtain that $\omega^*(v) \geqslant 2 - \frac{5}{4} - \frac{1}{4} - \frac{1}{2} = 0$ by applying (R1) and (R3).
- ▷ $p_3(v) = 2$. That is, $v \in V_{4^{2p}}$. By Lemma 2, we confirm that $d(v_2) \geqslant 5$. In other words, f_1 is a $(3, 4^{2p}, 5^+)$-face. By (R4.3), $\sigma(v \to f_1) = 1$ and thus $\omega^*(v) \geqslant 2 - 1 - \frac{1}{4} \times 2 - \frac{1}{2} = 0$ by (R1) and (R3).

Now let us consider the remaining case when f_1 is a $(4, 4^+, 4^+)$-face such that $d(v_1) \geqslant 4$ and $d(v_2) \geqslant 4$. We have two cases to discuss depending on the value of $n_2(v)$.

- ▷ $n_2(v) = 0$. By (R4.4) to (R4.6), we know that v sends weight at most $\frac{9}{8}$ to f_1. So if $p_3(v) \leqslant 1$, then $\omega^*(v) \geqslant 2 - \frac{9}{8} - \frac{1}{4} - \frac{1}{2} = \frac{1}{8}$ by (R1) and (R3). Otherwise, $p_3(v) = 2$, meaning that $v \in V_{4^{2p}}$. If $d(v_1) \geqslant 5$ and $d(v_2) \geqslant 5$, then by (R4.6) we know that $\sigma(v \to f_1) = 1$ basing on the fact that $v \notin V_4^*$. Hence, $\omega^*(v) \geqslant 2 - 1 - \frac{1}{4} \times 2 - \frac{1}{2} = 0$

by (R1) and (R3). Or else, by symmetry, assume that $d(v_1) = 4$. Namely, f is a $(4, 4^{2p}, 4^+)$-face. By Lemma 6, $v_1 \notin V_4^*$ and $v_2 \notin V_4^*$, and thus $\sigma(v \to f_1) = 1$ by (R4.4) and (R4.5). Therefore $\omega^*(v) \geqslant 2 - 1 - \frac{1}{4} \times 2 - \frac{1}{2} = 0$ by (R1) and (R3).

- ▷ $n_2(v) = 1$. Namely, $v \in V_4^*$. W.l.o.g., assume that $d(v_3) = 2$. Moreover, by (A6), $p_3(v) = 0$. It tells us that $v \notin V_{4^p} \cup V_{4^{2p}}$. If f_1 is a $(4, 5^+, 5^+)$-face, then $\sigma(v \to f_1) = \frac{3}{4}$ by (R4.6). If f_1 is a $(4, 4, 5^+)$-face, then by (R4.5) v sends weight exactly $\frac{3}{4}$ to f_1. If f_1 is a $(4, 4, 4)$-face, then by Lemma 7 we see that neither v_1 nor v_2 can be a 4^*-vertex, and thus $\sigma(v \to f_1) = \frac{3}{4}$ by (R4.4). These facts enable us to confirm that $\sigma(v \to f_1) = \frac{3}{4}$ regardless the situation of f_1. So if $m_5(v) = 0$ or $m_5(v) = 1$ such that the unique incident 5-face only gets weight at most $\frac{1}{4}$ from v, then we are done by showing that $\omega^*(v) \geqslant 2 - \frac{3}{4} - 1 - \frac{1}{4} = 0$ by (R1). In what follows, assume that f_j is a 5-face for some fixed $j \in \{2, 3, 4\}$. If $j = 2$, then f_2 is a $(4, 2, 4^+, 2^+, 4^+)$-face and thus by (R3.3) v sends weight $\frac{1}{4}$ to f_2 and then we go back to the previous case. If $j = 3$, then f_3 is a $(4, 2, 4^+, 2^+, 3^+)$-face. By (R3.3), $\sigma(v \to f_3) = \frac{1}{4}$ and then we also go back to the former case. Otherwise, $j = 4$. Let $f_4 = [vv_4w_1w_2v_1]$. If $d(v_4) \geqslant 4$, then by (R3.3), v sends at most $\frac{1}{4}$ to f_4 and thus we are done. Or else, $d(v_4) = 3$. Obviously, $d(w_1) \neq 2$ by (A3). Moreover, by (A4), w_1 cannot be a weak vertex. It follows that $d(w_1) \neq 3$ and if $d(w_1) = 4$, then $d(w_2) \neq 2$. So f_4 can be either a $(4, 3, 5^+, 2^+, 4^+)$-face or a $(4, 3, 4, 3^+, 4^+)$-face. By (R3.3), $\sigma(v \to f_4) = \frac{1}{4}$ and then we are done by the former case argument.

Case 2: $k = 5$. Then $\omega(v) = 4$. By (A7), $n_2(v) \leqslant 3$. Moreover, $m_i(v) \leqslant 1$ for each $i \in \{3, 5\}$ by the assumption on G.

First suppose that $m_3(v) = 0$. Then by (R1) and (R2) we deduce that $\omega^*(v) \geqslant 4 - n_2(v) - \frac{1}{4} \times p_3(v) - \frac{1}{2} = 4 - n_2(v) - \frac{1}{4} \times (5 - n_2(v)) - \frac{1}{2} = \frac{9}{4} - \frac{3}{4} n_2(v) \geqslant 0$. Next, suppose that $m_3(v) = 1$. W.l.o.g., assume that $f_1 = [v_1 v_2 v]$ is a 3-face. Here, $n_2(v) \leqslant 2$ by (A8). The following discussion is divided into several cases according to the condition on f_1.

- ▷ $d(v_1) = d(v_2) = 3$. That is, f_1 is a $(3, 3, 5)$-face. By (R4.1), $\sigma(v \to f_1) = \frac{5}{2}$. By Lemma 4, we see that $n_2(v) \leqslant 1$. If $n_2(v) = 1$, then $p_3(v) = 0$ by (F1), and so $\omega^*(v) \geqslant 4 - \frac{5}{2} - 1 - \frac{1}{2} = 0$ by (R1) and (R2). Otherwise, $n_2(v) = 0$. Then $p_3(v) \leqslant 3$ and therefore $\omega^*(v) \geqslant 4 - \frac{5}{2} - \frac{1}{4} \times 3 - \frac{1}{2} = \frac{1}{4}$ by (R1) and (R2).
- ▷ $d(v_1) = 3$ and $d(v_2) = 4$. If $v_2 \in V_{4^{2p}}$, namely f_1 is a $(3, 4^{2p}, 5)$-face. then $\sigma(v \to f_1) = \frac{7}{4}$ by (R4.3). Moreover, by (F2) we are sure that $n_2(v) \leqslant 1$. Thus, $\omega^*(v) \geqslant 4 - \frac{7}{4} - n_2(v) - \frac{1}{4} \times (3 - n_2(v)) - \frac{1}{2} = 1 - \frac{3}{4} n_2(v) \geqslant \frac{1}{4}$ by (R1) and (R2). Otherwise, $v_2 \notin V_{4^{2p}}$. Clearly, by (R4.3), $\sigma(v \to f_1) \leqslant \frac{3}{2}$. If $n_2(v) = 2$, then $p_3(v) = 0$ by (A9), implying that $\omega^*(v) \geqslant 4 - \frac{3}{2} - 1 \times 2 - \frac{1}{2} = 0$ by (R1) and (R2). Or else, $n_2(v) \leqslant 1$. It is easy to deduce that $\omega^*(v) \geqslant 4 - \frac{3}{2} - 1 - \frac{1}{4} \times 2 - \frac{1}{2} = \frac{1}{2}$.

▷ $d(v_1) = 3$ and $d(v_2) \geqslant 5$. It follows that f_1 is a $(3, 5^+, 5)$-face. Then by (R4.3), $\sigma(v \to f_1) = \frac{11}{8}$. If $n_2(v) \leqslant 1$, then $\omega^*(v) \geqslant 4 - \frac{11}{8} - 1 - \frac{1}{4} \times 2 - \frac{1}{2} = \frac{5}{8}$ by (R1) and (R2). Otherwise $n_2(v) = 2$. Again, by (A9), $p_3(v) = 0$. Hence, $\omega^*(v) \geqslant 4 - \frac{11}{8} - 1 \times 2 - \frac{1}{2} = \frac{1}{8}$ by (R1) and (R2).

▷ $d(v_1) = d(v_2) = 4$. Then f_1 is a $(4, 4, 5)$-face. By (R4.5), the weight sent from v to f_1 is either $\frac{3}{2}$, $\frac{9}{8}$ or 1. If $\sigma(v \to f_1) \leqslant \frac{9}{8}$, then $\omega^*(v) \geqslant 4 - \frac{9}{8} - 1 \times 2 - \frac{1}{4} - \frac{1}{2} = \frac{1}{8}$ by (R1) and (R2). Otherwise assume that $\sigma(v \to f_1) = \frac{3}{2}$. It follows from (R4.5) that v_1 and v_2 are both 4^*-vertices. In other words, f_1 is a $(4^*, 4^*, 5)$-face. If $n_2(v) \leqslant 1$, then $\omega^*(v) \geqslant 4 - \frac{3}{2} - 1 - \frac{1}{4} \times 2 - \frac{1}{2} = \frac{1}{2}$ by (R1) and (R2). Otherwise, $n_2(v) = 2$. By (F3), $p_3(v) = 0$. Hence, it is easy to deduce that $\omega^*(v) \geqslant 4 - \frac{3}{2} - 1 \times 2 - \frac{1}{2} = 0$.

▷ $d(v_1) \geqslant 4$ and $d(v_2) \geqslant 5$. Then f_1 is a $(4^+, 5^+, 5)$-face. By (R4.6), $\sigma(v \to f_1) \leqslant \frac{9}{8}$, and therefore $\omega^*(v) \geqslant 4 - \frac{9}{8} - 1 \times 2 - \frac{1}{4} - \frac{1}{2} = \frac{1}{8}$ by (R1) and (R2).

Case 3: $k = 6$. Clearly $\omega(v) = 6$. By (A10), $n_2(v) \leqslant 4$. If $d(f_i) \geqslant 5$ for all $i = 1, 2, \ldots, 6$, then $\omega^*(v) \geqslant 6 - 1 \times 4 - \frac{1}{4} \times 2 - \frac{1}{2} = 1$ by (R1) and (R2). Next, suppose that there exists a face f_i, say $f_1 = [vv_1v_2]$, such that $d(f_1) = 3$. If f_1 is a $(6, 4^+, 4^+)$-face, then v sends at most $\frac{3}{2}$ to f_1 by (R4.5) and (R4.6), and thus $\omega^*(v) \geqslant 6 - \frac{3}{2} - 1 \times 4 - \frac{1}{2} = 0$ by (R1) and (R2). Now suppose that f_1 is a $(6, 3, 3^+)$-face. By (Q1), we are sure that $n_2(v) \leqslant 3$. Moreover, v sends at most $\frac{5}{2}$ to f_1 by (R4). If $n_2(v) \leqslant 2$, then it is obvious that $\omega^*(v) \geqslant 6 - \frac{5}{2} - 1 \times 2 - \frac{1}{4} \times 2 - \frac{1}{2} = \frac{1}{2}$ by (R1) and (R2). Otherwise, $n_2(v) = 3$. At this moment, it is guaranteed by (Q2) that f_1 cannot be a $(6, 3, 3)$-face, which implies that $\sigma(v \to f_1) \leqslant \frac{7}{4}$ by (R4). Hence, $\omega^*(v) \geqslant 6 - \frac{7}{4} - 1 \times 3 - \frac{1}{4} - \frac{1}{2} = \frac{1}{2}$ by (R1) and (R2).

Case 4: $k \geqslant 7$. If $m_3(v) = 0$, then it is obvious that $\omega^*(v) \geqslant 2d(v) - 6 - n_2(v) - \frac{1}{4}p_3(v) - \frac{1}{2} \geqslant 2d(v) - 6 - n_2(v) - \frac{1}{4}(d(v) - n_2(v)) - \frac{1}{2} \geqslant \frac{7}{4}d(v) - \frac{3}{4}n_2(v) - \frac{13}{2} \geqslant \frac{7}{4}d(v) - \frac{3}{4}d(v) - \frac{13}{2} \geqslant d(v) - \frac{13}{2} \geqslant \frac{1}{2}$. Or else, assume that f_1 is the unique 3-face that is incident to v. By (R1), (R2) and (R4), one can easily derive that $\omega^*(v) \geqslant 2d(v) - 6 - \frac{5}{2} - n_2(v) - \frac{1}{4}p_3(v) - \frac{1}{2} \geqslant 2d(v) - \frac{17}{2} - n_2(v) - \frac{1}{4}(d(v) - 2 - n_2(v)) - \frac{1}{2} \geqslant \frac{7}{4}d(v) - \frac{17}{2} - \frac{3}{4}n_2(v) \geqslant \frac{7}{4}d(v) - \frac{17}{2} - \frac{3}{4}(d(v) - 2) \geqslant d(v) - 7 \geqslant 0$.

Therefore, we complete the proof of Theorem 1. □

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References

[1] *M. O. Albertson, D. M. Berman*: Every planar graph has an acyclic 7-coloring. Isr. J. Math. *28* (1977), 169–174. zbl MR doi

[2] *O. V. Borodin*: On acyclic colorings of planar graphs. Discrete Math. *306* (2006), 953–972. zbl MR doi

[3] *O. V. Borodin, D. G. Fon-Der Flaass, A. V. Kostochka, A. Raspaud, E. Sopena*: Acyclic list 7-coloring of planar graphs. J. Graph Theory *40* (2002), 83–90. zbl MR doi
[4] *O. V. Borodin, A. O. Ivanova*: Acyclic 5-choosability of planar graphs without 4-cycles. Sib. Math. J. *52* (2011), 411–425. zbl MR doi
[5] *O. V. Borodin, A. O. Ivanova, A. Raspaud*: Acyclic 4-choosability of planar graphs with neither 4-cycles nor triangular 6-cycles. Discrete Math. *310* (2010), 2946–2950. zbl MR doi
[6] *M. Chen, A. Raspaud*: On acyclic 4-choosability of planar graphs without short cycles. Discrete Math. *310* (2010), 2113–2118. zbl MR doi
[7] *M. Chen, A. Raspaud*: A sufficient condition for planar graphs to be acyclically 5-choosable. J. Graph Theory *70* (2012), 135–151. zbl MR doi
[8] *M. Chen, A. Raspaud*: Planar graphs without 4- and 5-cycles are acyclically 4-choosable. Discrete Appl. Math. *161* (2013), 921–931. zbl MR doi
[9] *M. Chen, A. Raspaud, N. Roussel, X. Zhu*: Acyclic 4-choosability of planar graphs. Discrete Math. *311* (2011), 92–101. zbl MR doi
[10] *M. Chen, W. Wang*: Acyclic 5-choosability of planar graphs without 4-cycles. Discrete Math. *308* (2008), 6216–6225. zbl MR doi
[11] *B. Grünbaum*: Acyclic colorings of planar graphs. Isr. J. Math. *14* (1973), 390–408. zbl MR doi
[12] *A. V. Kostočka*: Acyclic 6-coloring of planar graphs. Metody Diskretn. Anal. *28* (1976), 40–54. (In Russian.) zbl MR
[13] *J. Mitchem*: Every planar graph has an acyclic 8-coloring. Duke Math. J. *41* (1974), 177–181. zbl MR doi
[14] *M. Montassier*: Acyclic 4-choosability of planar graphs with girth at least 5. Graph Theory in Paris (A. Bondy, et al., eds.). Trends in Mathematics, Birkhäuser, Basel, 2007, pp. 299–310. zbl MR doi
[15] *M. Montassier, A. Raspaud, W. Wang*: Acyclic 4-choosability of planar graphs without cycles of specific lengths. Topics in Discrete Mathematics (M. Klazar, et al., eds.). Algorithms and Combinatorics 26, Springer, Berlin, 2006, pp. 473–491. zbl MR doi
[16] *M. Montassier, A. Raspaud, W. Wang*: Acyclic 5-choosability of planar graphs without small cycles. J. Graph Theory *54* (2007), 245–260. zbl MR doi
[17] *Y. Sun, M. Chen, D. Chen*: Acyclic 4-choosability of planar graphs without intersecting short cycles. Discrete Math. Algorithms Appl. *10* (2018), Article ID 1850014, 11 pages. zbl MR doi
[18] *W. Wang, M. Chen*: Planar graphs without 4-cycles are acyclically 6-choosable. J. Graph Theory *61* (2009), 307–323. zbl MR doi
[19] *W. Wang, G. Zhang, M. Chen*: Acyclic 6-choosability of planar graphs without adjacent short cycles. Sci. China, Math. *57* (2014), 197–209. zbl MR doi
[20] *H. Zhang, B. Xu*: Acyclic 5-choosability of planar graphs with neither 4-cycles nor chordal 6-cycles. Discrete Math. *309* (2009), 6087–6091. zbl MR doi

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