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DERIVED EQUIVALENCES BETWEEN GENERALIZED MATRIX ALGEBRAS

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Abstract. We construct derived equivalences between generalized matrix algebras. We record several corollaries. In particular, we show that the n-replicated algebras of two derived equivalent, finite-dimensional algebras are also derived equivalent.

Keywords: derived equivalence; tilting complex; generalized matrix algebra

MSC 2010: 16G10, 16E35, 16S50

1. Introduction

Derived equivalence as Morita theory for derived categories provides a new method and tool for the classification of algebras. By a fundamental result of Rickard, for two algebras A and B, the derived categories of modules $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are equivalent as triangulated categories if and only if there exists a tilting complex $T^\bullet$ for A such that the endomorphism algebra $\mathrm{End}_{\mathcal{D}(A)}(T^\bullet) = B$. But in practice there are usually two obstacles when one constructs derived equivalences between algebras. One is finding the tilting complexes and the other is determining their endomorphism algebras. In order to construct new derived equivalence from the old one, Rickard in [17], [19] used tensor product and trivial extension to produce derived equivalences. These results were generalized by Ladkani, see [12] and Miyachi, see [16], respectively. For a list of more constructions of derived equivalences see [20] and the references therein.

In this short note we consider derived equivalences between generalized n-by-n matrix algebras. An n-by-n generalized matrix algebra Λ is defined by $\Lambda = (M_{ij})_{(\varphi_{ijk})}$,

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where $M_{ii} = A_i$ is an algebra, M_{ij} is an A_i-A_j-bimodule, and the multiplication in Λ is given by the bimodule map $\varphi_{ijk}\colon M_{ij} \otimes M_{jk} \to M_{ik}$ that satisfies the obvious associativity condition for any $1 \leqslant i, j, k \leqslant n$. A generalized matrix algebra is also called a formal matrix algebra. Some important classes of generalized matrix algebras consist of Morita context when $n = 2$ and triangular matrix algebras. These algebras are important and fundamental objects in the representation theory of algebras (e.g. see [5], [6], [7], [8]). Motivated by [8], we construct the functors T_{A_t} from the module category $\operatorname{mod} A_t$ to the module category $\operatorname{mod} \Lambda$ and its right adjoint functor U_{A_t} for any t. It is well known that the adjoint functor pair $(\mathrm{T}_{A_t}, \mathrm{U}_{A_t})$ of module categories can be lifted to the homotopy categories of complexes, which are still denoted by $(\mathrm{T}_{A_t}, \mathrm{U}_{A_t})$. Furthermore, for any tilting complex $P_t^\bullet$ for A_t we provide a sufficient and necessary condition such that $\bigoplus\limits_{t=1}^{n} \mathrm{T}_{A_t}(P_t^\bullet)$ is a tilting complex for Λ in this paper. Meanwhile, we determine its endomorphism algebra. Then we establish various important derived equivalences between generalized matrix algebras, unifying a few results described in [1], [12]. In particular, we point out that the n-replicated algebras of two derived equivalent, finite-dimensional but not necessary Gorenstein algebras are also derived equivalent, compared with Corollary 1.4 in [12].

We include some notation here. By an algebra we mean a finite-dimensional algebra over a field K. We denote by $\operatorname{mod} A$ the category of all finitely generated right A-modules. We denote by P_A the subcategory of $\operatorname{mod} A$ consisting of projective modules. We denote by $\mathcal{C}(A)$ (or $\mathcal{K}(A)$) the category of complexes (or the homotopy category) of finitely generated right A-modules, and denote by $\mathcal{K}^b(P_A)$ the subcategory of $\mathcal{K}(A)$ consisting of bounded complexes over P_A. For two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ we write gf for their composition. We denote by D the standard duality $\operatorname{Hom}_K(-, K)$.

2. The representation over the generalized matrix algebra

We recall the construction of the generalized matrix algebra in [4]. Let $(A_i)_{1 \leqslant i \leqslant n}$ be a family of K-algebras and $(M_{ij})_{1 \leqslant i,j \leqslant n}$ be a family of A_i-A_j-bimodules such that $M_{ii} = A_i$ for any $1 \leqslant i \leqslant n$. Moreover, assume that for any $1 \leqslant i, j, k \leqslant n$ such that $i \neq j$, $j \neq k$, there is an A_i-A_j-bimodule homomorphism

$$\varphi_{ijk}\colon M_{ij} \otimes_{A_j} M_{jk} \to M_{ik}.$$

For subscripts $i = j$ and $j = k$, the bimodule homomorphisms

$$\varphi_{iik}\colon A_i \otimes_{A_i} M_{ik} \to M_{ik}, \quad \varphi_{ijj}\colon M_{ij} \otimes_{A_j} A_j \to M_{ij}$$

are canonical isomorphisms. The bimodule homomorphisms mentioned above satisfy the associativity law

$$\varphi_{ijt}(1_{M_{ij}} \otimes \varphi_{jkt}) = \varphi_{ikt}(\varphi_{ijk} \otimes 1_{M_{kt}}) \tag{2.1}$$

for any $1 \leqslant i, j, k, t \leqslant n$, that is, the the following diagram

$$\begin{tikzcd}
M_{ij} \otimes_{A_j} M_{jk} \otimes_{A_k} M_{kt} \arrow[r, "\varphi_{ijk}\otimes 1_{M_{kt}}"] \arrow[d, "1_{M_{ij}}\otimes\varphi_{jkt}"'] & M_{ik} \otimes_{A_k} M_{kt} \arrow[d, "\varphi_{ijt}"] \\
M_{ij} \otimes_{A_j} M_{jt} \arrow[r, "\varphi_{ikt}"] & M_{it}
\end{tikzcd}$$

commutes. For any $X_i, Y_i \in \operatorname{mod} A_i$ and $a_i\colon X_i \to Y_i$, the diagram

$$\begin{tikzcd}
X_i \otimes_{A_i} M_{ij} \otimes_{A_j} M_{jk} \arrow[r, "1_{X_i}\otimes\varphi_{ijk}"] \arrow[d, "a_i\otimes 1_{M_{ij}}\otimes 1_{M_{jk}}"'] & X_i \otimes_{A_i} M_{ik} \arrow[d, "a_i\otimes 1_{M_{ik}}"] \\
Y_i \otimes_{A_i} M_{ij} \otimes_{A_j} M_{jk} \arrow[r, "1_{Y_i}\otimes\varphi_{ijk}"] & Y_i \otimes_{A_i} M_{ik}
\end{tikzcd}$$

commutes for any $1 \leqslant i, j, k \leqslant n$. Hence, the bimodule map φ_{ijk} determines a natural transformation from the functor $- \otimes M_{ij} \otimes_{A_j} M_{jk}$ to the functor $- \otimes M_{ik}$, denoted by Φ_{ijk}. And we denote the morphism $1_{X_i} \otimes \varphi_{ijk}$ by $\Phi_{ijk}^{X_i}$ in the sequel. Then the n-by-n generalized matrix algebra is defined by

$$\Lambda = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{pmatrix}$$

with the bimodule map φ_{ijk}, and we denote it by $\Lambda = (M_{ij})_{(\varphi_{ijk})}$ for short. The addition of elements is componentwise and multiplication is given by $(AB)_{ij} = \sum\limits_{1\leqslant k\leqslant n} \varphi_{ikj}(a_{ik} \otimes b_{kj})$ for $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n} \in \Lambda$, where $(AB)_{ij}$ means the (i, j)-entry of AB.

When $n = 2$, the generalized matrix algebra is just the Morita ring in the sense of [7]. For any integer $n \geqslant 2$, we introduce the definition of the representation category of Λ, which is similar to the one when $n = 2$ in [7]. Denote the representation category of Λ by $\operatorname{rep}\Lambda$, whose objects are tuples $X = (X_1, \dots, X_n)$, where

$X_i \in \operatorname{mod} A_i$ and for any $1 \leqslant i,\ j \leqslant n$, X has the *structure map* of (i,j)-position $f_{ij} \in \operatorname{Hom}_{A_j}(X_i \otimes_{A_i} M_{ij}, X_j)$ $(f_{ii} = 1)$ such that the diagram

$$\begin{tikzcd} X_i \otimes_{A_i} M_{ij} \otimes_{A_j} M_{jk} \arrow[r, "\Phi_{ijk}^{X_i}"] \arrow[d, "f_{ij}\otimes 1_{M_{jk}}"'] & X_i \otimes_{A_i} M_{ik} \arrow[d, "f_{ik}"] \\ X_j \otimes_{A_j} M_{jk} \arrow[r, "f_{jk}"] & X_k \end{tikzcd}$$

commutes. We also denote X by $(X_1, \ldots, X_n)_{(f_{ij})}$. Let $X = (X_1, \ldots, X_n)_{(f_{ij})}$ and $Y = (Y_1, \ldots, Y_n)_{(g_{ij})}$ be objects of $\operatorname{rep} \Lambda$. Then a morphism $X \to Y$ in $\operatorname{rep} \Lambda$ is an n-tuples of homomorphisms $(a_1, \ldots, a_n)$, where $a_i \colon X_i \to Y_i$ is an A_i-morphism such that the diagram

$$\begin{tikzcd} X_i \otimes_{A_i} M_{ij} \arrow[r, "f_{ij}"] \arrow[d, "a_i\otimes 1_{M_{ij}}"'] & X_j \arrow[d, "a_j"] \\ Y_i \otimes_{A_i} M_{ij} \arrow[r, "g_{ij}"] & Y_j \end{tikzcd}$$

commutes for $1 \leqslant i,\ j \leqslant n$. The composition is in the natural way. Moreover, a sequence of $\operatorname{rep} \Lambda$

$$(X_1, \ldots, X_n)_{(f_{ij})} \to (Y_1, \ldots, Y_n)_{(g_{ij})} \to (Z_1, \ldots, Z_n)_{(h_{ij})}$$

is exact if and only if for any t the sequence $X_t \to Y_t \to Z_t$ of A_t-modules is exact.

We call a category $\mathcal{C}$ a *K-linear category* if the class of the objects of $\mathcal{C}$ forms a set, the set of morphisms between two arbitrary objects of $\mathcal{C}$ is K-vector space and the composition of morphisms is K-bilinear. Then an n-by-n generalized matrix algebra $\Lambda = (M_{ij})_{(\varphi_{ijk})}$ defined as above can be viewed as a K-linear category $\mathcal{C}_\Lambda$ with n objects $x_1, x_2, \ldots, x_n$ such that the set of morphisms from x_j to x_i is M_{ij}. Thus, a representation of Λ is nothing but a right $\mathcal{C}_\Lambda$-module, i.e. a covariant K-linear functor from the category $\mathcal{C}_\Lambda$ to the category of K-modules. For example, one can get the explicit definition of a module over a K-linear category from [10]. And the category of right $\mathcal{C}_\Lambda$-modules is equivalent to the category of finitely generated right Λ-modules (see [13]). Hence we have the following proposition.

Proposition 2.1. *The module category* $\operatorname{mod} \Lambda$ *and the representation category* $\operatorname{rep} \Lambda$ *are equivalent.*

We will identify the modules in $\operatorname{mod}\Lambda$ with the objects of $\operatorname{rep}\Lambda$ from now on. For describing the derived equivalence among the generalized matrix algebras, motivated by [6] and [8], we define two kinds of functors as follows.

(1) For any $t = 1, 2, \ldots, n$, and $X_t \in \operatorname{mod} A_t$, the functor $\mathrm{T}_{A_t}\colon \operatorname{mod} A_t \to \operatorname{mod}\Lambda$ is defined by

$$\mathrm{T}_{A_t}(X_t) = (X_t \otimes M_{t1}, \ldots, X_t \otimes M_{tt} = X_t, \ldots, X_t \otimes M_{tn})_{(f_{ij})},$$

where the structure map of (i,j)-position $f_{ij} = \Phi_{tij}^{X_t}$ and for morphism $a_t \in \operatorname{Hom}_{A_t}(X_t, Y_t)$

$$\mathrm{T}_{A_t}(a_t) = (a_t \otimes 1_{M_{t1}}, \ldots, a_t, \ldots, a_t \otimes 1_{M_{tn}}) = (a_t \otimes 1_{M_{ti}})_{i=1}^{n}.$$

The associativity law of bimodule homomorphisms in (2.1) induce the following commutative diagram.

$$\begin{tikzcd}
X_t \otimes M_{ti} \otimes M_{ij} \otimes M_{jk} \arrow[r, "\Phi_{ijk}^{X_t \otimes M_{ti}}"] \arrow[d, "\Phi_{tij}^{X_t} \otimes 1_{M_{jk}}"'] & X_t \otimes M_{ti} \otimes M_{ik} \arrow[d, "\Phi_{tik}^{X_t}"] \\
X_t \otimes M_{tj} \otimes M_{jk} \arrow[r, "\Phi_{tjk}^{X_t}"] & X_t \otimes M_{tk}.
\end{tikzcd}$$

Hence, $\mathrm{T}_{A_t}(X_t) \in \operatorname{mod}(\Lambda)$ and the functor T_{A_t} is well defined.

(2) For any $t = 1, 2, \ldots, n$ and $(X_1, \ldots, X_n)_{(f_{ij})} \in \operatorname{mod}\Lambda$, the functor $\mathrm{U}_{A_t}\colon \operatorname{mod}\Lambda \to \operatorname{mod} A_t$ is defined by $\mathrm{U}_{A_t}(X_1, \ldots, X_n)_{(f_{ij})} = X_t$ and for Λ-morphism $(a_i)_{i=1}^{n}$, $\mathrm{U}_{A_t}((a_i)_{i=1}^{n}) = a_t$.

According to the construction of a representation over Λ, T_{A_t} is really the tensor functor $-\otimes_{A_t} e_t\Lambda\colon \operatorname{mod} A_t \to \operatorname{mod}\Lambda$, and U_{A_t} is the restriction functor from $\operatorname{mod}\Lambda$ to $\operatorname{mod} e_t\Lambda e_t = \operatorname{mod} A_t$, where e_t is the matrix with 1 in (t,t)-entry and 0 elsewhere. Naturally, we have the following analogs of [8], Proposition 2.4.

Lemma 2.2.

(1) *The above pair* $(\mathrm{T}_{A_t}, \mathrm{U}_{A_t})$ *is an adjoint pair of functors for any* $t = 1, 2, \ldots, n$.

(2) *The composed functor* $\mathrm{U}_{A_t}\mathrm{T}_{A_t}$ *is equal to* $1_{\operatorname{mod} A_t}$ *for any* $t = 1, 2, \ldots, n$.

P r o o f. (1) Let $X_t \in \operatorname{mod} A_t$ and $Y = (Y_1, \ldots, Y_n)_{(g_{ij})} \in \operatorname{mod}\Lambda$. Claim that $a = (a_i)_{i=1}^{n}\colon \mathrm{T}_{A_t}(X_t) \to Y$ is a morphism in $\operatorname{mod}\Lambda$ if and only if $a_i = g_{ti} \circ a_t \otimes 1_{M_{it}}$ for $1 \leqslant i \neq t \leqslant n$, that is, a_i is uniquely determined by a_t for any $i \neq t$. In fact, on one hand the morphism a in $\operatorname{mod}\Lambda$ gives the commutative diagram

$$\begin{tikzcd} X_t \otimes_{A_t} M_{ti} \arrow[r, "1"] \arrow[d, "a_t \otimes 1_{M_{ti}}"'] & X_t \otimes M_{ti} \arrow[d, "a_i"] \\ Y_t \otimes M_{ti} \arrow[r, "g_{ti}"] & Y_i, \end{tikzcd}$$

which implies $a_i = g_{ti} \circ a_t \otimes 1_{M_{it}}$. On the other hand, assume $g_{ti} \circ a_t \otimes 1_{M_{it}}$ for $1 \leqslant i \neq t \leqslant n$. Since Y is an object of $\operatorname{mod}\Lambda$, the structure maps of Y satisfy

$$g_{ij} \circ g_{ti} \otimes 1_{M_{ij}} = g_{tj} \circ \Phi_{tij}^{Y_t}. \tag{2.2}$$

And applying the natural transformation $\Phi_{tij}\colon - \otimes M_{ti} \otimes M_{ij} \to - \otimes M_{tj}$ to the morphism $a_t\colon X_t \to Y_t$, we have

$$\Phi_{tij}^{Y_t} \circ a_t \otimes 1_{M_{ti}} \otimes 1_{M_{ij}} = a_t \otimes 1_{M_{tj}} \circ \Phi_{tij}^{X_t}. \tag{2.3}$$

Combining the two equations (2.2) and (2.3) above imply

$$\begin{aligned} g_{ij} \circ a_i \otimes 1_{M_{ij}} &= g_{ij} \circ (g_{ti} \circ a_t \otimes 1_{M_{it}}) \otimes 1_{M_{ij}} = g_{ij} \circ g_{ti} \otimes 1_{M_{ij}} \circ a_t \otimes 1_{M_{it}} \otimes 1_{M_{ij}} \\ &= g_{tj} \circ \Phi_{tij}^{Y_t} \circ a_t \otimes 1_{M_{it}} \otimes 1_{M_{ij}} = g_{tj} \circ a_t \otimes 1_{M_{tj}} \circ \Phi_{tij}^{X_t} = a_j \circ \Phi_{tij}^{X_t}. \end{aligned}$$

Hence $(a_i)_{i=1}^n$ is a morphism in $\operatorname{mod}\Lambda$ from $\mathrm{T}_{A_t}(X_t)$ to Y, and the claim follows. Consequently, there is an abelian group isomorphism $\mathcal{H}\colon \operatorname{Hom}_\Lambda(\mathrm{T}_{A_t}(X_t), Y) \to \operatorname{Hom}_{A_t}(X_t, Y_t)$ by $(a_i)_{i=1}^n \mapsto a_t$. And it is not difficult to check that $\mathcal{H}$ is natural.

(2) Simple verification. □

By taking advantage of the functors $\mathrm{T}_{A_1}, \ldots, \mathrm{T}_{A_n}$, we can get all the indecomposable projective Λ-modules by the following lemma.

Lemma 2.3. *Let P_t be an indecomposable projective A_t-module for any t. Then*

$$\mathrm{T}_{A_t}(P_t) = (P_t \otimes M_{t1}, \ldots, P_t \otimes M_{tn})$$

with the structure map of (i,j)-position $\Phi_{tij}^{P_t}$ is indecomposable projective Λ-module. Moreover, all the indecomposable projective Λ-modules are of this form.

Proof. See [8], Proposition 3.1. □

Let A be a finite-dimensional K-algebra. Following [18], an object $T^\bullet \in \mathcal{D}(A)$ is called a tilting complex provided the following conditions are satisfied:
(1) $T^\bullet \in \mathcal{K}^b(P_A)$,
(2) $\mathrm{Hom}_{\mathcal{K}(A)}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$, and
(3) $\mathrm{tria}(T^\bullet) = \mathcal{K}^b(P_A)$, where $\mathrm{tria}(T^\bullet)$ is the smallest triangulated category containing $T^\bullet$.

For later use, we recall here concepts of two kinds of total complexes. Given two complexes of right A-modules $X^\bullet = (X^m, d_X^m)$ and $Y^\bullet = (Y^m, d_Y^m)$, the total complex $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ is defined by

$$\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) := \dots \longrightarrow \prod_{p\in\mathbb{Z}} \mathrm{Hom}_A(X^p, Y^{m+p}) \xrightarrow{d^m_{\mathrm{Hom}}} \prod_{p\in\mathbb{Z}} \mathrm{Hom}_A(X^p, Y^{m+p+1}) \longrightarrow \dots,$$

where $d^m_{\mathrm{Hom}}\colon (\alpha^p)_{p\in\mathbb{Z}} \mapsto (d_Y^{m+p}\circ\alpha^p - (-1)^m\alpha^{p+1}\circ d_X^p)_{p\in\mathbb{Z}}$ for $\alpha^p \in \mathrm{Hom}_A(X^p, Y^{m+p})$. According to [11], we have the key formula $H^r(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{\mathcal{K}(A)}(X^\bullet, Y^\bullet[r])$ for any $r \in \mathbb{Z}$. If $P^\bullet = (P^m, d_P^m)$ and $Q^\bullet = (Q^m, d_Q^m)$ are complexes of K-modules, then the other total complex $P^\bullet \otimes Q^\bullet$ is defined by

$$P^\bullet \otimes Q^\bullet := \dots \longrightarrow \bigoplus_{p\in\mathbb{Z}} P^p \otimes Q^{m-p} \xrightarrow{d^m_\otimes} \bigoplus_{p\in\mathbb{Z}} P^p \otimes Q^{m-p+1} \longrightarrow \dots,$$

where $d^m_\otimes\colon (x\otimes y)_{p\in\mathbb{Z}} \mapsto (d_X^p(x)\otimes y + (-1)^p x \otimes d_Y^{m-p}(y))_{p\in\mathbb{Z}}$ for $x \in P^p$, $y \in Q^{m-p}$. If $P^\bullet$ is a complex of right A-modules and $Q^\bullet$ is a complex of A-B-bimodules, then $P^\bullet \otimes Q^\bullet$ is a complex of right B-modules. Note that we view modules as stalk complexes concentrated in degree zero. Then for a complex $X^\bullet = (X^p) \in \mathcal{C}(A)$ and an A-B-bimodule M we have the total complex $X^\bullet \otimes M$ with the form

$$X^\bullet \otimes M := \dots \longrightarrow X^p \otimes M \xrightarrow{d_X^p \otimes 1_M} X^{p+1} \otimes M \longrightarrow \dots$$

Let $\Lambda = (M_{ij})_{\varphi_{ijk}}$ be a generalized matrix algebra, where $M_{ii} = A_i$ is an algebra, M_{ij} is an A_i-A_j-bimodule and the bimodule map φ_{ijk} corresponds to the natural transformation Φ_{ijk}. For any $t = 1, 2, \dots, n$ and complex $X_t^\bullet = (X_t^m, d_{X_t}^m)$ in $\mathcal{C}(A_t)$, the morphism $\Phi_{tij}^{X_t^m}\colon X_t^m \otimes M_{ti} \otimes M_{ij} \to X_t^m \otimes M_{tj}$ induces the morphism of complex $\Phi_{tij}^{X_t^\bullet} = (\Phi_{tij}^{X_t^m})\colon X_t^\bullet \otimes M_{ti} \otimes M_{ij} \to X_t^\bullet \otimes M_{tj}$. Let $(\mathrm{T}_{A_t}, \mathrm{U}_{A_t})$ be the adjoint pair defined in Section 2. Denote the induced adjoint pair of complex categories by $(C(\mathrm{T}_{A_t}), C(\mathrm{U}_{A_t}))$. Then we have

$$C(\mathrm{T}_{A_t})(X_t^\bullet) = (X_t^\bullet \otimes M_{t1}, \dots, X_t^\bullet \otimes M_{tn})_{(\Phi_{tij}^{X_t^\bullet})},$$

where the homogeneous component of degree m is $(X_t^m \otimes M_{t1}, \ldots, X_t^m \otimes M_{tn})$ with the structure map $\Phi_{tij}^{X_t^m}$, and the mth differential is $(d_{X_t}^m \otimes M_{ti})_{i=1}^n$. And for Λ complex $(Y_1^\bullet, \ldots, Y_n^\bullet)_{(g_{ij}^\bullet)}$ with the homogeneous component of degree m being $(Y_1^m, Y_2^m, \ldots, Y_n^m)_{(g_{ij}^m)}$, we have

$$C(\mathrm{U}_{A_t})(Y_1^\bullet, \ldots, Y_n^\bullet)_{(g_{ij}^\bullet)} = Y_t^\bullet.$$

Moreover, if $(a_1^\bullet, \ldots, a_n^\bullet)\colon \mathrm{T}_{A_t}(X_t^\bullet) \to (Y_1^\bullet, \ldots, Y_n^\bullet)$ is a chain map of complex, then by the proof of Lemma 2.2, $a_i^\bullet$ is determined by $a_t^\bullet$ and $a_i^\bullet = g_{ti}^\bullet \circ a_t^\bullet \otimes 1_{M_{it}}$ for any $i \neq t$.

We can continue to lift the adjoint pair $(C(\mathrm{T}_{A_t}), C(\mathrm{U}_{A_t}))$ to homotopy categories with a standard proof, see for instance [15], Chapter 5, Proposition 1.1.3. We still denote the induced adjoint pair between the homotopy categories by $(\mathrm{T}_{A_t}, \mathrm{U}_{A_t})$. Under the assumption above, we will state and prove our main result as follows.

Theorem 3.1. *Let Λ be a generalized matrix algebra as above and $P_t^\bullet \in \mathcal{D}(A_t)$ a tilting complex for any $t = 1, 2, \ldots, n$. Then $\bigoplus_{t=1}^{n} \mathrm{T}_{A_t}(P_t^\bullet) \in \mathcal{D}(\Lambda)$ is a tilting complex if and only if $\mathrm{Hom}_{\mathcal{D}(A_i)}(P_i^\bullet, (P_j^\bullet \otimes M_{ji})[r]) = 0$ for any $1 \leqslant i \neq j \leqslant n$ and $r \neq 0$. In this case, Λ is derived equivalent to the following generalized matrix algebra*

$$\Gamma = \begin{pmatrix} N_{11} & N_{12} & \ldots & N_{1n} \\ N_{21} & N_{22} & \ldots & N_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ N_{n1} & N_{n2} & \ldots & N_{nn} \end{pmatrix},$$

where $N_{ij} = \mathrm{Hom}_{\mathcal{D}(A_j)}(P_j^\bullet, P_i^\bullet \otimes M_{ij})$ and the multiplication map $N_{ij} \otimes N_{jk} \to N_{ik}$ is given by

$$a_{ij}^\bullet * a_{jk}^\bullet = \Phi_{ijk}^{P_i^\bullet} \circ a_{ij}^\bullet \otimes 1_{M_{jk}} \circ a_{jk}^\bullet$$

for any $1 \leqslant i, j, k \leqslant n$, and $a_{ij}^\bullet \in N_{ij}$.

Proof. By Lemma 2.3, the functor T_{A_t} gives the bijection between the isomorphism classes of indecomposable summands of A_t and the isomorphism classes of indecomposable summands of $e_{tt}\Lambda$, where e_{tt} is the unit in A_t. Therefore T_{A_t} provides a triangulated equivalence between $\mathcal{K}^b({}_{A_t}P)$ and $\mathrm{tria}(e_{tt}\Lambda)$, which implies $\mathrm{tria}(\mathrm{T}_{A_t}(P_t^\bullet)) = \mathrm{tria}(e_{tt}\Lambda)$. In particular, $e_{tt}\Lambda \in \mathrm{tria}(\mathrm{T}_{A_t}(P_t^\bullet))$. Consequently, $\Lambda = \bigoplus_{t=1}^{n} e_{tt}\Lambda \in \mathrm{tria}\Big(\bigoplus_{t=1}^{n} (\mathrm{T}_{A_t}(P_t^\bullet))\Big)$. Hence $\bigoplus_{i=t}^{n} \mathrm{T}_{A_t}(P_t^\bullet)$ generates $\mathcal{K}^b(P_\Lambda)$ as a triangulated category.

By the definitions of T_{A_t} and U_{A_t} and their adjointness, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{K}(\Lambda)}\left(\bigoplus_{i=t}^{n}\mathrm{T}_{A_t}(P_t^\bullet),\bigoplus_{i=t}^{n}\mathrm{T}_{A_t}(P_t^\bullet)[r]\right)\cong H^r\mathrm{Hom}_{\Lambda}\left(\bigoplus_{i=1}^{n}\mathrm{T}_{A_i}(P_i^\bullet),\bigoplus_{j=1}^{n}\mathrm{T}_{A_j}(P_j^\bullet)\right)\\
&\quad\cong\bigoplus_{i,j}H^r\mathrm{Hom}_{\Lambda}(\mathrm{T}_{A_i}(P_i^\bullet),\mathrm{T}_{A_j}(P_j^\bullet))\cong\bigoplus_{i,j}H^r\mathrm{Hom}_{A_i}(P_i^\bullet,\mathrm{U}_{A_i}\mathrm{T}_{A_j}(P_j^\bullet))\\
&\quad\cong\bigoplus_{i,j}H^r\mathrm{Hom}_{A_i}(P_i^\bullet,P_j^\bullet\otimes M_{ji})\cong\bigoplus_{i,j}\mathrm{Hom}_{\mathcal{K}(A_i)}(P_i^\bullet,(P_j^\bullet\otimes M_{ji})[r])\\
&\quad\cong\bigoplus_{i,j}\mathrm{Hom}_{\mathcal{D}(A_i)}(P_i^\bullet,(P_j^\bullet\otimes M_{ji})[r])
\end{aligned}$$

as vector space. Therefore $\bigoplus\limits_{t=1}^{n}\mathrm{T}_{A_t}(P_t^\bullet)$ is a tilting complex if and only if

$$\mathrm{Hom}_{\mathcal{D}(A_i)}(P_i^\bullet,(P_j^\bullet\otimes M_{ji})[r])=0$$

for any $1\leqslant i,\, j\leqslant n$ and $r\neq 0$. This observation establishes the first statement.

For any $1\leqslant i,j\leqslant n$, the vector space $N_{ij}=\mathrm{Hom}_{\mathcal{D}(A_j)}(P_j^\bullet,P_i^\bullet\otimes M_{ij})$ has a natural structure of $\mathrm{End}(P_i^\bullet)$-$\mathrm{End}(P_j^\bullet)$-bimodule under the action

$$\begin{aligned}
&\mathrm{End}_{A_i}(P_i^\bullet)\times\mathrm{Hom}_{\mathcal{D}(A_j)}(P_j^\bullet,P_i^\bullet\otimes M_{ij})\\
&\qquad\times\mathrm{End}_{A_j}(P_j^\bullet)\longrightarrow\mathrm{Hom}_{\mathcal{D}(A_j)}(P_j^\bullet,P_i^\bullet\otimes M_{ij})(a_{ii}^\bullet,a_{ij}^\bullet,a_{jj}^\bullet)\\
&\qquad\qquad\longmapsto a_{ii}^\bullet\otimes 1_{M_{ij}}\circ a_{ij}^\bullet\circ a_{jj}^\bullet,
\end{aligned}$$

where $a_{ij}^\bullet\in N_{ij}$. Moreover, by Lemma 2.2, there exist isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}(A_j)}(P_j^\bullet,P_i^\bullet\otimes M_{ij})\longrightarrow\mathrm{Hom}_{\mathcal{D}(\Lambda)}(\mathrm{T}_{A_j}(P_j^\bullet),\mathrm{T}_{A_i}(P_i^\bullet))a_{ij}^\bullet\longmapsto\alpha^\bullet\\
&\qquad=(\Phi_{ij1}^{P_i^\bullet}\circ a_{ij}^\bullet\otimes 1_{M_{j1}},\ldots,\underset{(j)}{a_{ij}^\bullet},\ldots,\Phi_{ijn}^{P_i^\bullet}\circ a_{ij}^\bullet\otimes 1_{M_{jn}})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}(A_k)}(P_k^\bullet,P_j^\bullet\otimes M_{jk})\longrightarrow\mathrm{Hom}_{\mathcal{D}(\Lambda)}(\mathrm{T}_{A_k}(P_k^\bullet),\mathrm{T}_{A_j}(P_j^\bullet))a_{jk}^\bullet\longmapsto\beta^\bullet\\
&\qquad=(\Phi_{jk1}^{P_j^\bullet}\circ a_{jk}^\bullet\otimes 1_{M_{k1}},\ldots,\underset{(k)}{a_{jk}^\bullet},\ldots,\Phi_{jkn}^{P_i^\bullet}\circ a_{jk}^\bullet\otimes 1_{M_{kn}}).
\end{aligned}$$

Hence, the kth term of the morphism composition $\alpha^\bullet\beta^\bullet$ is $\Phi_{ijk}^{P_i^\bullet}\circ a_{ij}^\bullet\otimes 1_{M_{jk}}\circ a_{jk}^\bullet$, which determines the following bimodule map in the generalized matrix algebra Γ:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}(A_j)}(P_j^\bullet,P_i^\bullet\otimes M_{ij})\otimes\mathrm{Hom}_{\mathcal{D}(A_k)}(P_k^\bullet,P_j^\bullet\otimes M_{jk})\\
&\qquad\longrightarrow\mathrm{Hom}_{\mathcal{D}(A_k)}(P_k^\bullet,P_i^\bullet\otimes M_{ik})a_{ij}^\bullet\otimes a_{jk}^\bullet\longmapsto\Phi_{ijk}^{P_i^\bullet}\circ a_{ij}^\bullet\otimes 1_{M_{jk}}\circ a_{jk}^\bullet.
\end{aligned}$$

This finishes the proof. $\square$

Let A be an algebra. We denote by $\mathcal{T}_n(A)$ the n-by-n lower triangular matrix algebra with all entries in A. In order to get the derived equivalences among line-likely algebras (algebra with a linear quiver), rectangle-likely algebras and triangle-likely algebras, Ladkani in [12] constructed interesting derived equivalences between tensor algebras and generalized matrix algebras using componentwise tensor product. We give a new method to prove the main results in [12] by applying Theorem 3.1.

Corollary 3.2 ([12], Theorem B)**.** *Let A be an algebra and let $T_1^\bullet, \ldots, T_n^\bullet$ be tilting complexes in $\mathcal{D}(A)$ satisfying* $\operatorname{Hom}_{\mathcal{D}(A)}(T_i^\bullet, T_{i+1}^\bullet[r]) = 0$ *for all $1 \leqslant i < n$ and $r \neq 0$. Then the matrix algebra*

$$\begin{pmatrix} \operatorname{End} T_1^\bullet & 0 & 0 & \ldots & 0 \\ \operatorname{Hom}(T_1^\bullet, T_2^\bullet) & \operatorname{End} T_2^\bullet & 0 & \ldots & \vdots \\ 0 & \operatorname{Hom}(T_2^\bullet, T_3^\bullet) & \operatorname{End} T_3^\bullet & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \operatorname{Hom}(T_{n-1}^\bullet, T_n^\bullet) & \operatorname{End} T_n^\bullet \end{pmatrix}$$

is derived equivalent to $\mathcal{T}_n(A)$.

P r o o f. Consider the n-by-n matrix

$$\Gamma = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \ldots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \ldots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{pmatrix},$$

where A_{ij} satisfies $A_{21} = A_{32} = \ldots = A_{n,n-1} = A$, $A_{ii} = A$ for all i and 0 elsewhere. View $T_i^\bullet$ as a tilting complex for A_{ii}. One has

$$\operatorname{Hom}_{\mathcal{D}(A_{ii})}(T_i^\bullet, (T_j^\bullet \otimes A_{ji})[r]) = 0$$

for all $1 \leqslant i \neq j \leqslant n$ and $r \neq 0$. Then by Theorem 3.1, Γ is derived equivalent to the given one in the assertion. Therefore, the conclusion follows by observing that Γ and $\mathcal{T}_n(A)$ are derived equivalent for a well-known result, e.g. see [9]. $\square$

Corollary 3.3 ([12], Theorem C)**.** *Let A be an algebra and let $T_1^\bullet, \ldots, T_n^\bullet$ be tilting complexes in $\mathcal{D}(A)$ satisfying* $\operatorname{Hom}_{\mathcal{D}(A)}(T_i^\bullet, T_j^\bullet[r]) = 0$ *for all $1 \leqslant i < j \leqslant n$*

and $r \neq 0$. Then the matrix algebra

$$\begin{pmatrix} \operatorname{End} T_1^\bullet & 0 & 0 & \dots & 0 \\ \operatorname{Hom}(T_1^\bullet, T_2^\bullet) & \operatorname{End} T_2^\bullet & 0 & \dots & 0 \\ \operatorname{Hom}(T_1^\bullet, T_3^\bullet) & \operatorname{Hom}(T_2^\bullet, T_3^\bullet) & \operatorname{End} T_3^\bullet & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}(T_1^\bullet, T_n^\bullet) & \operatorname{Hom}(T_2^\bullet, T_n^\bullet) & \operatorname{Hom}(T_3^\bullet, T_n^\bullet) & \dots & \operatorname{End} T_n^\bullet \end{pmatrix}$$

is derived equivalent to $\mathcal{T}_n(A)$.

P r o o f. $\mathcal{T}_n(A)$ is equal to

$$\Gamma = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \dots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{pmatrix},$$

where A_{ij} satisfies $A_{ij} = A$ for all $1 \leqslant j \leqslant i \leqslant n$ and 0 elsewhere. View $T_i^\bullet$ as a tilting complex for A_{ii}. One has

$$\operatorname{Hom}_{\mathcal{D}(A_{ii})}(T_i^\bullet, (T_j^\bullet \otimes A_{ji})[r]) = 0$$

for all $1 \leqslant i \neq j \leqslant n$ and $r \neq 0$. Then by Theorem 3.1, Γ is derived equivalent to the given one in the assertion. $\square$

The n-replicated algebra of A is defined as the following $(n+1)$-by-$(n+1)$ matrix algebra

$$A^{(n)} = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ DA & A & 0 & \dots & \vdots \\ 0 & DA & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & DA & A \end{pmatrix},$$

where the multiplication is induced from the canonical isomorphisms $A \otimes DA \cong DA \cong DA \otimes A$ and the zero map $DA \otimes DA \to 0$. If $n = 1$, then $A^{(1)}$ is the duplicated algebra of A (see [2]). When A is hereditary, n-replicated algebra $A^{(n)}$ of A has a close relation to n-cluster categories (see [3]), and Zhang in [21] studied the partial tilting modules over $A^{(n)}$. If A is a finite-dimensional algebra over K, then DA is an injective co-generator. When DA_A has finite projective dimension and A_A has finite injective dimension, the algebra A is called Gorenstein. Ladkani proved in [12] that derived equivalent, Gorenstein algebras are derived equivalent. Here, we have the following corollary.

Corollary 3.4. *Let A and B be algebras. If A and B are derived equivalent, then $A^{(n)}$ and $B^{(n)}$ are also derived equivalent for any $n \geqslant 1$.*

P r o o f. Assume that A and B are derived equivalent. Then there exists a tilting complex $T^\bullet$ in $\mathcal{D}(A)$ such that $\operatorname{End}(T^\bullet) = B$. Since the Nakayama functor $-\otimes_A DA$: $\mathcal{D}(A) \to \mathcal{D}(A)$ gives a Serre functor, there exists an isomorphism

$$\operatorname{Hom}_{\mathcal{D}(A)}(T^\bullet, T^\bullet \otimes_A DA) \cong D \operatorname{Hom}_{\mathcal{D}(A)}(T^\bullet, T^\bullet) = DB.$$

Then the claim follows from Theorem 3.1. $\square$

The n-fold trivial extension algebra of A is defined as $\mathbb{T}_n(A) = \widehat{A}/\nu^n$, where ν is the Nakayama functor of the repetitive algebra $\widehat{A}$ and $n \geqslant 1$. In particular, $\mathbb{T}_1(A)$ is the usual trivial extension algebra $A \ltimes DA$. For $n \geqslant 2$, $\mathbb{T}_n(A)$ is isomorphic to the n-by-n matrix algebra

$$\mathbb{T}_n(A) = \begin{pmatrix} A & & & & DA \\ DA & A & & & \\ & DA & A & & \\ & & \ddots & \ddots & \\ & & & DA & A \end{pmatrix},$$

where the multiplication is induced from the canonical isomorphisms $A \otimes DA \cong DA \cong DA \otimes A$ and the zero map $DA \otimes DA \to 0$. Then $\mathbb{T}_n(A)$ is a finite-dimensional, self-injective K-algebra.

Corollary 3.5 ([1], Theorem 4.1)**.** *Let A and B be algebras. If A and B are derived equivalent, then $\mathbb{T}_n(A)$ and $\mathbb{T}_n(B)$ are also derived equivalent for any $n \geqslant 1$.*

P r o o f. In [17], Rickard showed that trivial extension algebras of two derived equivalent finite-dimensional algebras are also derived equivalent. And the proof is similar to the proof of Corollary 3.4 when $n \geqslant 2$. $\square$

The following corollary is the special case of [14], Theorem 4.3.

Corollary 3.6. *Let R, S be algebras and T_S a tilting right S-module. Let ${}_RM_S$ be a R-S-bimodule such that as R-module, $\operatorname{Ext}_S^n(T_S, M_S) = 0$ for all $n > 0$. Then the 2-by-2 triangular matrix algebras*

$$\Gamma_1 = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \Gamma_2 = \begin{pmatrix} R & \operatorname{Hom}_S(T, M) \\ 0 & \operatorname{End}_S(T) \end{pmatrix}$$

are derived equivalent.

P r o o f. The relation

$$\mathrm{Hom}_{\mathcal{D}(S)}(T, S \otimes_S M[n]) \cong \mathrm{Hom}_{\mathcal{D}(S)}(T, M[n]) \cong \mathrm{Ext}^n_S(T, M)$$

infers the assertion by Theorem 3.1. □

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