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GENERAL-AFFINE INVARIANTS OF PLANE CURVES AND SPACE CURVES

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Abstract. We present a fundamental theory of curves in the affine plane and the affine space, equipped with the general-affine groups $\mathrm{GA}(2) = \mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathrm{GA}(3) = \mathrm{GL}(3,\mathbb{R}) \ltimes \mathbb{R}^3$, respectively. We define general-affine length parameter and curvatures and show how such invariants determine the curve up to general-affine motions. We then study the extremal problem of the general-affine length functional and derive a variational formula. We give several examples of curves and also discuss some relations with equiaffine treatment and projective treatment of curves.

Keywords: plane curve; space curve; general-affine group; general-affine curvature; variational problem

MSC 2010: 53A15, 53A55, 53A20

1. Introduction

Let $\mathbb{A}^n$ be the affine n-space with coordinates $x = (x^1, \dots, x^n)$. It is called a unimodular affine space if it is equipped with a parallel volume element, namely a determinant function. The unimodular affine group $\mathrm{SA}(n) = \mathrm{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ acts as

$$x = (x^i) \mapsto gx + a = \left(\sum_j g^i_j x^j + a^i\right), \quad g = (g^i_j) \in \mathrm{SL}(n,\mathbb{R}),\ a = (a^i) \in \mathbb{R}^n,$$

which preserves the volume element. The study of geometric properties of submanifolds in $\mathbb{A}^n$ invariant under this group is called equiaffine differential geometry, while the study of properties invariant under the general affine group $\mathrm{GA}(n) =$

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$\mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ is called general-affine differential geometry. Furthermore, the study of geometric properties of submanifolds in the projective space $\mathbb{P}^n$ invariant under the projective linear group $\mathrm{PGL}(n) = \mathrm{GL}(n+1, \mathbb{R})/\text{center}$ is called projective differential geometry. Equiaffine differential geometry, as well as projective differential geometry, has long been studied and has yielded a plentiful amount of results, especially for curves and hypersurfaces; we refer to [3], [4], [15], [24], [30] and [33]. However, the study of general-affine differential geometry is little known even for curves. The purpose of this paper is to present a basic study of plane curves and space curves in general-affine differential geometry by recalling old results and by adding some new results. In addition, we relate them with the curve theory in equiaffine and projective differential geometry. Although the study of invariants of curves of higher-codimension could possibly be given by a similar formulation used in this paper, it probably requires a more complicated presentation and is not attempted here. For studies of submanifolds in affine spaces which correspond to other types of subgroups of $\mathrm{PGL}(n)$, we refer to, e.g. see [30].

Let us begin with equiaffine treatment of plane curves. Let $\mathbb{A}^2$ be the unimodular affine plane with the determinant function $|x\ y| = x^1y^2 - x^2y^1$ for two vectors $x = (x^1, x^2)$ and $y = (y^1, y^2)$, and let $x(t)$ be a curve with parameter t into $\mathbb{A}^2$, which is nondegenerate in the sense that $|x'\ x''| \neq 0$. When $|x'\ x''| = 1$, furthermore, the parameter t is called the equiaffine length parameter. In this case, it holds that $|x'\ x'''| = 0$, namely x''' is linearly dependent on x' and we can write this dependence as

$$x''' = -k_a x',$$

where k_a is a scalar-valued function called the equiaffine curvature. Conversely, given a differential equation of this form, the map defined by two linearly independent nonconstant solutions defines a curve whose equiaffine curvature is k_a.

With reference to this presentation of equiaffine notions, we first define the general-affine length parameter and the general-affine curvature relative to the full affine group $\mathrm{GA}(2)$ for a nondegenerate curve in Section 2. In contrast to the differential equation above, we have

$$x''' = -\frac{3}{2}kx'' - \left(\varepsilon + \frac{1}{2}k' + \frac{1}{2}k^2\right)x',$$

where k is the general-affine curvature and $\varepsilon = \pm 1$ denotes additional information of the curve; see Section 2.1. Conversely, for a given function k and a sign $\varepsilon = \pm 1$ there exists a nondegenerate curve x for which x satisfies the above equation and the curvature of x is k, uniquely up to a general-affine motion (see Theorem 2.5). We then give a procedure of how to compute the curvature and a list of plane curves with constant general-affine curvature. Furthermore, we show that the total curvature of any nondegenerate closed curve vanishes (see Corollary 2.9) and give remarks on the number of general-affine vertices. Some discussions on the relation with equiaffine treatment and projective treatment of plane curves will be given in Appendices A.1 and B.1.

We next consider an extremal problem of the general-affine length functional and derive a variational formula, i.e., a nonlinear ordinary differential equation that characterizes an extremal plane curve, as

$$k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' + \varepsilon k' = 0$$

(see Proposition 3.1). Here we give remarks on the preceding studies: the formulation used to define general-affine curvature of plane curves in this paper is very similar to that in [17]. For example, the ordinary differential equation above for x and Theorem 2.5 were already given in [17]. The formula of the general-affine plane curvature was given also in [25], [30] in a different context. The variational formula above for k was first given in [18], equation (33) though some modifications are necessary. The same formula for $\varepsilon = 1$ was then rediscovered by Verpoort in the equiaffine setting (see [32], page 432). Furthermore, the author of [32] gave a relation of solutions of this nonlinear equation with the coordinate functions of the immersion; we reprove this relation in Corollary 3.2.

We then remark that the differential equation above is very similar to some of the nonlinear differential equations of Chazy type. Furthermore, we give a remark to the variational formula for a certain generalized curvature functional.

When the curve is given as a graph immersion, the general-affine curvature is written as a nonlinear form of a certain intermediate function. It is interesting to obtain a graph immersion from a given curvature function, which is treated in Section 4: We see that its integration can be reduced to solving the Abel equations of the first kind and the second kind (see Theorem 4.2).

The second aim of this paper is to study general-affine invariants of space curves. The procedure is similar to that for plane curves. We give the definition of general-affine space curvatures of two kinds k and M, and the ordinary differential equation of rank four

$$x'''' = -3kx''' - \Big(2k' + \frac{11}{4}k^2 + \varepsilon\Big)x'' - \Big(M + \frac{1}{2}\varepsilon k + \frac{1}{2}k'' + \frac{7}{4}kk' + \frac{3}{4}k^3\Big)x'$$

(see Lemma 5.3), which defines the immersion. Then we show how to compute curvatures. Discussions on relations with the equiaffine and projective treatment of space curves will be given in Appendices A.2 and B.2.

We next solve the extremal problem of the general-affine length functional: A nondegenerate space curve without affine inflection point is general-affine extremal if and only if the pair of ordinary differential equations

$$k'''+\frac{3}{2}kk''+\frac{1}{2}k^2k'+\frac{1}{2}k'^2-\frac{1}{5}\varepsilon k'+\frac{6}{5}M'=0,\quad k''+\frac{2}{3}kk'+\frac{5}{6}\varepsilon k'M-\frac{3}{2}\varepsilon kM'-\varepsilon M''=0$$

are satisfied (see Theorem 6.1). Then we discuss a similarity amongst the nonlinear differential equations for the curvature functions, one for plane curves, and the other for space curves belonging to a linear complex, i.e., $M=\varepsilon k$ (see Corollaries 6.3 and 6.4).

In Appendix A, we give a sketch of how to treat curves in equiaffine differential geometry and discuss general-affine invariants in terms of equiaffine invariants. In Appendix B, we discuss projective invariants in terms of general-affine invariants. Furthermore, in Appendix C, we shortly recall how to define projective invariants and summarize the variational formula of the projective length functional due to [6] and [14].

Finally, we give a few remarks on general-affine extremal curves. We have derived the differential equations for the curves to be extremal; however, except a few examples, the authors do not know how to integrate these equations. To get any method to describe extremal (especially closed) curves is an important problem; we refer to the book [11] and also the paper [22] for further study. From a general point of view, the study of motion of curves is also a related problem; we refer to the papers [16], [21].

In this paper, we use the classical moving frame method; we refer to [7], [29]. For the equiaffine differential geometry and its terminologies, we refer to the books [3], [24], [30], and for the projective treatment of curves to Wilczynski in [33] and Lane in [15].

2. General-affine curvature of plane curves

Let $x\colon M\to\mathbb{A}^2$ be a curve into the 2-dimensional affine space, where M is a 1-dimensional parameter space. Let $e=\{e_1,e_2\}$ be a frame along x; at each point of $x(M)$ it is a set of independent vectors of $\mathbb{A}^2$ that depend smoothly on the parameter. The vector-valued 1-form $\mathrm{d}x$ is written as

$$\mathrm{d}x=\omega^1e_1+\omega^2e_2, \tag{2.1}$$

and the dependence of e_i on the parameter is described by the equation

$$\mathrm{d}e_i=\sum_{j=1,2}\omega_i^je_j, \tag{2.2}$$

where ω^j and ω_i^j are 1-forms, and the matrix of 1-forms

$$\Omega = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix}$$

is called the coframe.

2.1. Choice of frames for plane curves and definition of general-affine curvature. We now reduce the choice of frames in order to define certain invariants. First, we assume $\omega^2 = 0$, which means that e_1 is tangent to the curve, and we set $\omega^1 = \omega$ for simplicity. The vector e_2 is arbitrary at present, as long as it is independent of e_1. Let $\widetilde{e} = \{\widetilde{e}_1, \widetilde{e}_2\}$ be another choice of such a frame. Then it is written as

$$\widetilde{e}_1 = \lambda e_1, \quad \widetilde{e}_2 = \mu e_1 + \nu e_2,$$

where $\lambda\nu \neq 0$. The coframe is written as $\widetilde{\omega}$ and $\widetilde{\omega}_i^j$, which satisfy

$$\mathrm{d}x = \widetilde{\omega}\widetilde{e}_1, \quad \mathrm{d}\widetilde{e}_i = \sum_{j=1,2} \widetilde{\omega}_i^j \widetilde{e}_j.$$

Then we certainly have

$$\widetilde{\omega} = \lambda^{-1}\omega. \tag{2.3}$$

Since $\mathrm{d}\widetilde{e}_1$ is represented in two ways, one being

$$\mathrm{d}\widetilde{e}_1 = (\mathrm{d}\lambda)e_1 + \lambda(\omega_1^1 e_1 + \omega_1^2 e_2)$$

and the other being

$$\mathrm{d}\widetilde{e}_1 = \widetilde{\omega}_1^1(\lambda e_1) + \widetilde{\omega}_1^2(\mu e_1 + \nu e_2),$$

by comparing the coefficients of e_1 and e_2 in these expressions, we get

$$\lambda\omega_1^2 = \nu\widetilde{\omega}_1^2, \quad \mathrm{d}\lambda + \lambda\omega_1^1 = \lambda\widetilde{\omega}_1^1 + \mu\widetilde{\omega}_1^2. \tag{2.4}$$

Similarly, considering $\mathrm{d}\widetilde{e}_2$, we have

$$\mu\omega_1^2 + \mathrm{d}\nu + \nu\omega_2^2 = \nu\widetilde{\omega}_2^2, \quad \mathrm{d}\mu + \mu\omega_1^1 + \nu\omega_2^1 = \lambda\widetilde{\omega}_2^1 + \mu\widetilde{\omega}_2^2. \tag{2.5}$$

Since the immersion is 1-dimensional, we can set $\omega_1^2 = h\omega$ and $\widetilde{\omega}_1^2 = \widetilde{h}\widetilde{\omega}$, and then the first identity of (2.4) implies that

$$\widetilde{h} = \nu^{-1}\lambda^2 h.$$

Hence, the non-vanishing property for h is independent of the frame. Geometrically it means that the curve is locally strictly convex at each point. We assume this property in the following. Such a curve is said to be *nondegenerate*. Then, by the identity above, provided that h is nonzero, we can choose a frame $\widetilde{e}$ so that $\widetilde{h} = 1$ and we treat such frames with $h = 1$ in the following. Then $\nu = \lambda^2$ immediately follows. Next, we see from (2.4) and (2.5) that

$$\widetilde{\omega}_1^1 = \omega_1^1 + \lambda^{-1}\,\mathrm{d}\lambda - \mu\lambda^{-2}\omega, \quad \widetilde{\omega}_2^2 = \omega_2^2 + \nu^{-1}\,\mathrm{d}\nu + \mu\nu^{-1}\omega.$$

Hence,

$$2\widetilde{\omega}_1^1 - \widetilde{\omega}_2^2 = 2\omega_1^1 - \omega_2^2 - 3\mu\lambda^{-2}\omega,$$

which enables us to choose μ so that $2\widetilde{\omega}_1^1 - \widetilde{\omega}_2^2 = 0$, and by considering only such frames in the following, we must have $\mu = 0$. Thus, we have determined the frame e up to a change of the form

$$\widetilde{e}_1 = \lambda e_1, \quad \widetilde{e}_2 = \lambda^2 e_2.$$

We call the direction determined by e_2 the *general-affine normal direction*. Furthermore, we have

$$\widetilde{\omega}_1^1 = \mathrm{d}\log\lambda + \omega_1^1, \quad \widetilde{\omega}_2^1 = \lambda\omega_2^1.$$

From the first identity, we can choose λ so that $\widetilde{\omega}_1^1 = 0$. Hence, we consider the frame with $\omega_1^1 = 0$ and λ is assumed to be constant. From the second identity, by setting

$$\omega_2^1 = -l\omega,$$

and similarly $\widetilde{\omega}_2^1 = -\widetilde{l}\widetilde{\omega}$, we get

$$\widetilde{l} = \lambda^2 l. \tag{2.6}$$

We call this scalar function l the *equiaffine curvature*, see Appendix A.1, though it still depends on the chosen frame. A point where $l = 0$ is called an *affine inflection point*; we refer to [13] for its geometrical meaning. Thus, we have seen that given a nondegenerate curve x, there exists a frame e with coframe of the form

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \\ -l\omega & 0 \end{pmatrix} \tag{2.7}$$

and that such frames are related by $\widetilde{e}_1 = \lambda e_1$ and $\widetilde{e}_2 = \lambda^2 e_2$ for a nonzero constant λ. In the following, given a curve $x = x(t)$ with parameter t, we assume that the vector e_1 is a positive multiple of the tangent vector $\mathrm{d}x/\mathrm{d}t$. Then the choice of λ is limited to be positive and the form ω is a positive multiple of $\mathrm{d}t$.

We now assume $l \neq 0$ and let ε denote the sign of l:

$$\varepsilon = \mathrm{sign}(l).$$

It is a locally defined invariant of the curve called the sign of the curve. Then we define a form

$$\omega_s = \sqrt{\varepsilon l}\omega, \tag{2.8}$$

which is unique, independent of the frame, in view of (2.3) and (2.6). We call this form the *general-affine length element* and call the parameter s such that $\mathrm{d}s = \omega_s$ the *general-affine length parameter*, determined up to an additional constant.

Definition 2.1. We call the scalar function k defined as

$$\frac{\mathrm{d}l}{l} = k\omega_s$$

the *general-affine curvature*. In other words,

$$k = \frac{\mathrm{d}\log l}{\mathrm{d}s}. \tag{2.9}$$

We define a new frame $\{E_1, E_2\}$ by setting

$$E_1 = \frac{1}{\sqrt{\varepsilon l}}e_1, \quad E_2 = \frac{1}{\varepsilon l}e_2. \tag{2.10}$$

Then

$$\mathrm{d}x = \omega_s E_1.$$

For another frame $\{\widetilde{e}_1, \widetilde{e}_2\}$, where $\widetilde{e}_1 = \lambda e_1$ and $\widetilde{e}_2 = \lambda^2 e_2$, we similarly define $\widetilde{E}_1$ and $\widetilde{E}_2$. Then we can see that

$$\widetilde{E}_1 = \frac{1}{\sqrt{\varepsilon\widetilde{l}}}\widetilde{e}_1 = \frac{1}{\sqrt{\varepsilon\lambda^2 l}}\lambda e_1 = E_1 \quad \text{and} \quad \widetilde{E}_2 = \frac{1}{\varepsilon\widetilde{l}}\widetilde{e}_2 = \frac{1}{\varepsilon\lambda^2 l}\lambda^2 e_2 = E_2.$$

Thus, we have proved the following.

Proposition 2.2. *Assume $l \neq 0$. Then the frame $\{E_1, E_2\}$ is uniquely defined from the immersion and it satisfies a Pfaffian equation*

$$\mathrm{d}\begin{pmatrix} x \\ E_1 \\ E_2 \end{pmatrix} = \Omega\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega_s & 0 \\ -\frac{1}{2}k\omega_s & \omega_s \\ -\varepsilon\omega_s & -k\omega_s \end{pmatrix}, \tag{2.11}$$

where ω_s is the general-affine length form, k is the general-affine curvature and ε is sign(l).

By use of this choice of frame, we have the following lemma.

Lemma 2.3. *The immersion x satisfies the ordinary differential equation*

$$x''' + \frac{3}{2}kx'' + \Bigl(\varepsilon + \frac{1}{2}k' + \frac{1}{2}k^2\Bigr)x' = 0, \tag{2.12}$$

relative to a general-affine length parameter.

P r o o f. Equation (2.11) shows that $x' = E_1$, $E_1' = -\frac{1}{2}kE_1 + E_2$, and $E_2' = -\varepsilon E_1 - kE_2$, where the derivation $\{'\}$ is taken relative to the length parameter. Then, combining these derivations, we easily obtain the differential equation above. $\square$

Remark 2.4. The definition of the curvature depends on the orientation of the parameter t. If we let the parameter be $u = -t$ and denote by an overhead dot the derivation relative to u, then we have

$$\dddot{x} - \frac{3}{2}k\ddot{x} + \Bigl(\varepsilon - \frac{1}{2}\dot{k} + \frac{1}{2}k^2\Bigr)\dot{x} = 0.$$

Namely, the curvature changes sign and its absolute value is a true invariant independent of the orientation of the parameter.

With this remark in mind, we have the following theorem.

Theorem 2.5 ([17])**.** *Given a function $k(t)$ of a parameter t and $\varepsilon = \pm 1$, there exists a nondegenerate curve $x(t)$ for which t is a length-parameter, k the curvature function and ε the* sign *of l, uniquely up to a general-affine transformation.*

P r o o f. Given k and ε, we solve the ordinary differential equation in (2.12) to get the vector $x'(t)$, which is determined up to a general linear transformation. Then we get $x(t)$ up to an additional translation by a constant vector; that is, the curve $x(t)$ is determined up to a transformation in GA(2). $\square$

Theorem 2.5 and the ordinary differential equation (2.12) were first given by Mihăilescu in [17], to the authors' knowledge; refer also to [5], [18], and [19].

Example 2.6. Ellipse and hyperbola. Let x denote an ellipse $x(\theta) = (a\cos\theta, b\sin\theta)$ or a hyperbola $x(\theta) = (a\cosh\theta, b\sinh\theta)$. Then $x''' = -\varepsilon x'$, where $\varepsilon = 1$ for the ellipse and $\varepsilon = -1$ for the hyperbola. It is easy to see that θ is a general-affine length, see (2.15). Hence, $k = 0$.

According to this example, we may call a nondegenerate curve is of *elliptic* (or *hyperbolic*) *type* if $\varepsilon = 1$ (or $\varepsilon = -1$).

Remark 2.7. The vector E_2, uniquely defined when $l \neq 0$, looks like a normal vector to the curve, and we call it the *general-affine normal.* By an analogy with affine spheres in equiaffine differential geometry, it is natural to call a curve, such that the map E_2 passes through one fixed point, a *general-affine circle.* Since $E_2 = x''$ and $x + \varepsilon E_2 = 0$ hold for an ellipse or a hyperbola, it is a general-affine circle. Conversely, let x be a general-affine circle. Then there exists a scalar function $r(t)$ and a fixed vector v such that $x + rE_2 = v$, and this implies that $\mathrm{d}x + \mathrm{d}rE_2 + r\,\mathrm{d}E_2 = 0$, and by identity (2.11), $(1 - \varepsilon r)\omega_s E_1 + (\mathrm{d}r - kr\omega_s)E_2 = 0$. Hence, $r = \varepsilon$ is constant and $k = 0$. Then by integrating the differential equation (2.12) when $k = 0$, we see that any general-affine circle is general-affinely congruent to (a part of) an ellipse or a hyperbola.

In order to obtain the curvature of a curve given relative to a parameter not necessarily a length parameter, we need a task, which we now briefly describe. Let $t \to x = x(t) \in \mathbb{A}^2$ be a nondegenerate curve such that the vectors x' and x'' are linearly independent. Then the derivative x''' is written as a linear combination of x' and x'': there are scalar functions $a = a(t)$ and $b = b(t)$ such that

$$x''' = ax'' + bx'. \tag{2.13}$$

Since $\mathrm{d}x = x'\,\mathrm{d}t$, the frame vector e_1 is a scalar multiple of x': $\mathrm{d}x = \omega e_1$, $e_1 = \lambda x'$ and $\omega = \lambda^{-1}\,\mathrm{d}t$ for a scalar function λ. Then the derivation $\mathrm{d}e_1 = \lambda(\lambda x'' + \lambda' x')\omega$ implies that the second frame vector is $e_2 = \lambda(\lambda x'' + \lambda' x')$. In order for the frame $\{e_1, e_2\}$ to be chosen as in Section 2.1, the vector $\mathrm{d}e_2$ must be a multiple of e_1. Since $\mathrm{d}e_2 = \{\lambda^2(\lambda a + 3\lambda')x'' + \lambda(\lambda^2 b + \lambda\lambda'' + \lambda'^2)x'\}\,\omega$, we have $\lambda a + 3\lambda' = 0$, i.e., $\lambda = \mathrm{e}^{-\int a(t)\,\mathrm{d}t/3}$ up to a positive constant multiple, and by definition,

$$l = -(\lambda^2 b + \lambda\lambda'' + \lambda'^2). \tag{2.14}$$

We now assume that $l \neq 0$ and recall that $\varepsilon = \mathrm{sign}(l)$. Then we have

$$\mathrm{d}s^2 = -\varepsilon\Big(b + \frac{2}{9}a^2 - \frac{1}{3}a'\Big)\,\mathrm{d}t^2. \tag{2.15}$$

If, in particular, t itself is a length parameter, then the equation is written as

$$x''' = ax'' + \Big(\frac{1}{3}a' - \frac{2}{9}a^2 - \varepsilon\Big)x', \tag{2.16}$$

and the curvature turns out to be

$$k = -\frac{2}{3}a. \tag{2.17}$$

When t is not necessarily a length parameter, we perform a change of parameter t to a new parameter $\sigma = \sigma(t)$ so that σ is a length parameter. For this purpose, we rewrite (2.13) relative to the variable

$$y(\sigma) = x(t).$$

A calculation shows that

$$\dddot{y}(\sigma) = A(\sigma)\ddot{y}(\sigma) + B(\sigma)\dot{y}(\sigma),$$

where

$$A(\sigma) = \Big(a - \frac{3\sigma''}{\sigma'}\Big)\frac{1}{\sigma'}, \tag{2.18}$$

$$B(\sigma) = \Big(b + a\frac{\sigma''}{\sigma'} - \frac{\sigma'''}{\sigma'}\Big)\frac{1}{\sigma'^2}. \tag{2.19}$$

Thus, we get the procedure of how to compute curvature:

Procedure for computing curvature.

Step 1. Given a curve $x(t)$, derive the differential equation (2.13).
Step 2. Compute $L = (-b - \frac{2}{9}a^2 + \frac{1}{3}a')$ and define ε by $\varepsilon = \operatorname{sign}(L)$.
Step 3. Compute the length parameter σ by solving $\mathrm{d}\sigma = \sqrt{\varepsilon L}\,\mathrm{d}t$.
Step 4. Compute A by (2.18); then $-\frac{2}{3}A$ is the curvature.

Remark 2.8. Due to Theorem 2.5, the curves with constant curvature are obtained by solving equation (2.16) for constant a. Table 1 is a list of curves with constant curvature $k = -\frac{2}{3}a$, which may be assumed to be nonpositive by Remark 2.4; α and γ are constants. Note that the case $k = -\infty$ corresponds to an exceptional curve parabola.

curvatures	curves with $\varepsilon = +1$
$k = 0$	$(t, -\sqrt{\alpha^2 - t^2})$
$-4 < k < 0$	$\mathrm{e}^{\gamma t}(\cos\alpha t, \sin\alpha t),\ \gamma \neq 0,\ \alpha \neq 0$
$k = -4$	$(t, t\log t),\ t > 0$
$-\infty < k < -4$	$(t, t^\alpha),\ \alpha \in (\frac{1}{2}, 1)$
curvatures	curves with $\varepsilon = -1$
$k = 0$	$(t, \sqrt{\alpha^2 + t^2})$
$k = -\sqrt{2}$	(t, e^t)
$-\infty < k < 0,\ k \neq -\sqrt{2}$	$(t, t^\alpha),\ \alpha \in (0, \frac{1}{2})$ or $\alpha \in (-1, 0)$

Table 1. Plane curves with constant general-affine curvature

2.2. Total curvature and general-affine vertices. Formula (2.11) implies the identity

$$\mathrm{d}\left(\log\left|\det\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}\right|\right) = -\frac{3}{2}k\omega_s, \tag{2.20}$$

where det is taken relative to a (any) unimodular structure of the space $\mathbb{R}^2$. This formula shows the following corollary immediately.

Corollary 2.9. *Assume that the curve C is nondegenerate and closed and has no affine inflection point. Then the total curvature $\int_C k\omega_s$ vanishes. In particular, such a curve has at least two general-affine flat points.*

As we will mention in Appendix B.1, any general-affine flat point, where $k = 0$ by definition, is nothing but a sextactic point. We know a classical theorem due to Mukhopadhayaya, also due to Herglotz and Rado, that the number of sextactic points of a strictly convex simply closed smooth curve is at least six; we refer, e.g. to [29], [31]. In other words, on such a curve there are at least six general-affine flat points. We refer also to [26] for a projective study of sextactic points.

Furthermore, as an analogue of the Euclidean plane curve, it is natural to introduce a notion of a *general-affine vertex*, where k is extremal. The corollary above says that any nondegenerate closed curve without affine inflection point has at least two general-affine vertices. The following example shows that the number 2 is the lowest possible value.

Example 2.10. The curve $x(t) = (\cos(nt)\cos(t), \cos(nt)\sin(t))$ is generally called a rose curve. Let us choose $n = \frac{1}{3}$ with the range $t \in [0, 3\pi]$. It satisfies the equation

$$x''' = -\frac{8\sin(\frac{1}{3}t)T}{3(1+4T^2)}x'' - \frac{4(7+8T^2)}{9(1+4T^2)}x',$$

where $T = \cos(\frac{1}{3}t)$. It is seen that $L > 0$, $\varepsilon = 1$ and the length parameter σ is defined by

$$\mathrm{d}\sigma = \frac{2\sqrt{256T^2+320T^4+69}}{9(1+4T^2)}\,\mathrm{d}t,$$

and the curvature

$$k(t) = 80\sin\left(\frac{2}{3}t\right)(4T^2-2T+3)(4T^2+2T+3)(320T^4+256T^2+69)^{-3/2}$$

is defined for all values t, and it vanishes at $t = 0, \frac{3}{2}\pi$. Moreover, it is easy to see that the number of general-affine vertices is two.

3. General-affine extremal plane curves and the associated differential equation

In Section 2.1, we have defined the general-affine length element ω_s, which naturally defines the general-affine length functional for plane curves, and in this section we consider the curves that are extremal with respect to this functional and we prove the variational formula, which is a nonlinear differential equation relative to the general-affine curvature, then discuss some special solutions with reference to Chazy equations. Furthermore, we compute the variational formula of the general-affine energy integral.

3.1. Extremal plane curves relative to the length functional. We have shown that there exists a unique frame $e = \{x, E_1, E_2\}$ such that (2.11) holds. Recall that Ω denotes the 3×2 matrix in (2.11). Let $x_\eta(t)$ be a family of curves parametrized by η around $\eta = 0$ and $x_0 = x$. We assume that $x_\eta(t) = x(t)$ outside a compact set C and $x_0(t)$ is parametrized by general-affine arc length. For simplicity, we further assume that x_η has no affine inflection point, i.e., the invariant ω_s does not vanish anywhere for all η. The length functional L is given by

$$L(\eta) = \int_C \omega_s(\eta).$$

For the sake of brevity, we use the notation δ to denote the derivative with respect to η evaluated at $\eta = 0$:

$$\delta a = \left.\frac{\mathrm{d}a(\eta)}{\mathrm{d}\eta}\right|_{\eta=0}.$$

Then the curve x is called *general-affine extremal* if

$$\delta L = 0$$

for any compactly supported deformation of x.

We want to derive a differential equation for an affine extremal curve. Since $\{E_1, E_2\}$ are linearly independent, there exists a 3×2-matrix τ such that

$$\delta \begin{pmatrix} x \\ E_1 \\ E_2 \end{pmatrix} = \tau \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau_0^1 & \tau_0^2 \\ \tau_1^1 & \tau_1^2 \\ \tau_2^1 & \tau_2^2 \end{pmatrix}$$

holds. The components of Ω and τ are denoted by ω_α^β and τ_α^β, where $\alpha = 0, 1, 2$ and $\beta = 1, 2$. Since $\delta\, \mathrm{d}e = \mathrm{d}\delta e$ with $e = \{x, E_1, E_2\}$, we have

$$\delta\omega_\alpha^\beta - \mathrm{d}\tau_\alpha^\beta = \sum_{\gamma=1,2} \tau_\alpha^\gamma \omega_\gamma^\beta - \omega_\alpha^\gamma \tau_\gamma^\beta.$$

In terms of entries of Ω and τ, we have

$$\delta\omega_s - \mathrm{d}\tau_0^1 = -\Big(\frac{1}{2}k\tau_0^1 + \varepsilon\tau_0^2 + \tau_1^1\Big)\omega_s, \tag{3.1}$$

$$-\mathrm{d}\tau_0^2 = \Big(\tau_0^1 - \tau_1^2 - k\tau_0^2\Big)\omega_s, \tag{3.2}$$

$$-\frac{1}{2}\delta(k\omega_s) - \mathrm{d}\tau_1^1 = -\Big(\varepsilon\tau_1^2 + \tau_2^1\Big)\omega_s, \tag{3.3}$$

$$\delta\omega_s - \mathrm{d}\tau_1^2 = \Big(\tau_1^1 - \frac{1}{2}k\tau_1^2 - \tau_2^2\Big)\omega_s, \tag{3.4}$$

$$-\varepsilon\delta\omega_s - \mathrm{d}\tau_2^1 = \varepsilon\Big(\tau_1^1 + \frac{1}{2}\varepsilon k\tau_2^1 - \tau_2^2\Big)\omega_s, \tag{3.5}$$

$$-\delta(k\omega_s) - \mathrm{d}\tau_2^2 = (\tau_2^1 + \varepsilon\tau_1^2)\omega_s. \tag{3.6}$$

Here we use $\omega_0^1 = \omega_1^2 = -\varepsilon\omega_2^1 = \omega_s$, $\omega_0^2 = 0$, $\omega_1^1 = -\frac{1}{2}k\omega_s$ and $\omega_2^2 = -k\omega_s$. Adding (3.4) and $-\varepsilon\times$(3.5), we get

$$2\delta\omega_s - \mathrm{d}\tau_1^2 + \varepsilon\,\mathrm{d}\tau_2^1 = -\frac{1}{2}k(\tau_1^2 + \varepsilon\tau_2^1)\omega_s$$

adding (3.6) and $-2\times$(3.3), we have

$$2\,\mathrm{d}\tau_1^1 - \mathrm{d}\tau_2^2 = 3\varepsilon(\tau_1^2 + \varepsilon\tau_2^1)\omega_s. \tag{3.7}$$

Combining these equations, we have

$$2\delta\omega_s = \mathrm{d}\tau_1^2 - \varepsilon\,\mathrm{d}\tau_2^1 + \frac{\varepsilon}{6}k(-2\,\mathrm{d}\tau_1^1 + \mathrm{d}\tau_2^2). \tag{3.8}$$

Recall that the deformation is compactly supported. Then by using Stokes' theorem and integration by parts, we have

$$\delta L = -\frac{\varepsilon}{12}\int_C (-2\tau_1^1 + \tau_2^2)\,\mathrm{d}k.$$

We now compute $-2\tau_1^1 + \tau_2^2$ as follows. From (3.1) and (3.4), we have

$$\mathrm{d}\tau_0^1 - \mathrm{d}\tau_1^2 = (2\tau_1^1 - \tau_2^2 + \varepsilon\tau_0^2)\omega_s + \frac{1}{2}k(\tau_0^1 - \tau_1^2)\omega_s.$$

Inserting (3.2) into the above equation, we have

$$-2\tau_1^1 + \tau_2^2 = {\tau_0^2}'' - \frac{3}{2}k{\tau_0^2}' + \Big(\varepsilon + \frac{1}{2}k^2 - k'\Big)\tau_0^2. \tag{3.9}$$

Here $\{'\}$ denotes the derivative with respect to the general-affine arc length ω_s, i.e., $a' = \mathrm{d}a/\omega_s$ for a function a. Finally, using integration by parts again,

$$\delta L = -\frac{\varepsilon}{12}\int_C \Big(k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' + \varepsilon k'\Big)\tau_0^2\omega_s \tag{3.10}$$

holds. If we now take $x(t) + \eta\{v^1(t,\eta)E_1(t) + v^2(t,\eta)E_2(t)\}$ for the family of curves $x_\eta(t)$, where v^1 and v^2 are arbitrary smooth functions with compact support relative to t, then $\delta x = v^1(t,0)E_1 + v^2(t,0)E_2$. This means that $\tau_0^2 = v^2(t,0)$ is arbitrary for this family and vanishing of the integral implies the following proposition.

Proposition 3.1 ([18]). *A nondegenerate plane curve without affine inflection point is general-affine extremal relative to the length functional if and only if*

$$k''' + \frac{3}{2}kk'' + \Big(\varepsilon + \frac{1}{2}k' + \frac{1}{2}k^2\Big)k' = 0 \tag{3.11}$$

holds. In particular, any curves of constant general-affine curvature are extremal.

We remark here that the differential equation (3.11) was first given in [18], equation (33) though some modifications are necessary. The formula for $\varepsilon = 1$ was then rediscovered by Verpoort in [32], page 432 by making use of his general variational formula of equiaffine invariants: The differential equation is written in terms of both of the equiaffine curvature and the general-affine curvature, and looks much simpler than (3.11). As a result, he proved the following corollary, which we now state in our setting.

Corollary 3.2 ([32]). *Let $x(t) = (x_1(t), x_2(t))$ be a curve parametrized by general-affine parameter t. Assume that it is general-affine extremal. Then there exist constants c_1, c_2 and c_3 such that the general-affine curvature k can be written as*

$$k = c_1x_1 + c_2x_2 + c_3.$$

Proof. Let us consider the ordinary linear differential equation

$$z''' + \frac{3}{2}kz'' + \Big(\varepsilon + \frac{1}{2}k' + \frac{1}{2}k^2\Big)z' = 0$$

with unknown function z. We note that this has the same form as equation (2.12), therefore x_1 and x_2 are solutions. Also any constant is obviously a solution. On the other hand, if x is extremal, then equation (3.11) shows that $k(t)$ itself is a solution. Therefore k can be expressed as claimed. □

We also have the following property on the curvature integral.

Proposition 3.3. *The variation of total curvature on any compact interval always vanishes, i.e.*

$$\delta \int_C k\omega_s = 0.$$

Proof. Adding (3.3) and (3.6),

$$-\frac{3}{2}\delta(k\omega_s) - \mathrm{d}\tau_1^1 - \mathrm{d}\tau_2^2 = 0$$

holds. Then Stokes' theorem implies the proposition. □

Remark 3.4. It is known that Chazy [8] classified third order nonlinear ordinary differential equations of Painlevé type, i.e. the solutions only admit poles as movable singularities. Then Chazy equations are classified into 13 classes of equations from I to XIII and the full list of equations can be found in [1]. We here cite Chazy equations for IV, V and VI, which are respectively given by

$$\begin{aligned} k''' + 3kk'' + 3{k'}^2 + 3k^2k' - Sk' - S'k - T &= 0, \\ k''' + 2kk'' + 4{k'}^2 + 2k^2k' - 2Rk' - R'k &= 0, \\ k''' + kk'' + 5{k'}^2 + k^2k' - 3Qk' - Q'k + Q'' &= 0, \end{aligned}$$

where S, T, R, Q are certain analytic functions of t, see [1]. Then (3.11) is clearly seen to be of the form of the above Chazy equations with the coefficients of kk'', ${k'}^2$ and k^2k' replaced by the half integers $\frac{3}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$, respectively, and with S, T, R or Q chosen properly.

Example 3.5. We can see that the following $k(t)$ are solutions of (3.11):

$$k(t) = 3\sqrt{2}\tanh\big(\sqrt{2}(t-c)\big) \quad \text{and} \quad k(t) = 3\sqrt{2}\coth\big(\sqrt{2}(t-c)\big)$$

for $\varepsilon = 1$ (see [32], Example 14), and

$$k(t) = -3\sqrt{2}\tan(\sqrt{2}(t-c)), \quad k(t) = 3\sqrt{2}\cot(\sqrt{2}(t-c)) \quad \text{and} \quad k(t) = \pm\sqrt{2} + \frac{3}{t-c}$$

for $\varepsilon = -1$. For each of these solutions we can compute the associated plane curve by integrating the differential equation using computer software. Since the expression is not simple, we give here one example for the case $\varepsilon = -1$ and $k(t) = \sqrt{2} + 3/t$, and the curve is written as (x_1, x_2) for $t > 0$:

$$x_1 = 3\sqrt{2} + \frac{1}{t}, \quad x_2 = \frac{\sqrt{\pi}\big(1 + 3\sqrt{2}t\big)\operatorname{erfi}(\sqrt{3t}/2^{1/4})}{t} + \frac{2^{3/4}\sqrt{3}\exp(-3t/\sqrt{2})}{\sqrt{t}},$$

where erfi is the error function defined by

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{t^2}\, \mathrm{d}t.$$

3.2. Variation of energy integral. The integral

$$\int_C \frac{1}{2} k^2 \omega_s$$

can be regarded a general-affine energy integral. Its variational formula is interesting. More generally, one can consider a variational problem for the curvature functional:

$$F(\eta) = \int_C f(k)\omega_s$$

with η the variational parameter, where f is a smooth function of one variable. Then a straightforward computation similar to that in the previous subsection shows that the functional is critical if and only if

$$G'' + \frac{3}{2} k G' + \Big(\varepsilon + \frac{1}{2} k' + \frac{1}{2} k^2\Big) G = 0 \tag{3.12}$$

holds, where G is the function defined as

$$G = 4 \ddddot{f}(k) k'^3 + 12 \dddot{f}(k) k' k'' + \ddot{f}(k)(4k''' - k'k^2 + 16\varepsilon k') - \dot{f}(k) k k' + f(k) k'.$$

When $f = \frac{1}{2}k^2$, we see that

$$G = 4k''' - \frac{3}{2} k^2 k' + 16\varepsilon k'.$$

Even for this special f, integration of (3.12) seems to be highly complicated.

4. How to find plane curves with given general-affine curvature

Let us consider the nondegenerate curve given by a graph immersion $x(t) = (t, f(t))$. We will find the formula of the curvature given by the function f and show fundamental examples of graph immersions.

Since x is nondegenerate, $x'' = (0, f'') \neq 0$, we may assume $f'' > 0$. Since $x''' = (0, f''')$, the coefficients of the differential equation (2.13) are $a = f'''/f''$ and $b = 0$. Hence,

$$\lambda = \mathrm{e}^{-\int a(t)\, \mathrm{d}t/3} = (f'')^{-1/3}$$

up to a constant multiple. If we set $\mu = \lambda^2 = (f'')^{-2/3}$, then (2.14) and (2.15) imply

$$l = -\frac{1}{2}\mu'', \quad \mathrm{d}s^2 = -\varepsilon\frac{\mu''}{2\mu}\,\mathrm{d}t^2. \tag{4.1}$$

Hence, we have the formula

$$k = \frac{\mathrm{d}\log l}{\mathrm{d}s} = \sqrt{\frac{-2\varepsilon\mu}{\mu''}}\frac{\mu'''}{\mu''}. \tag{4.2}$$

Thus, we can see that the general-affine arc length and the curvature are represented by using f as follows:

Lemma 4.1. *The quantities* $\mathrm{d}s^2$ *and* k^2 *are expressed by use of the function* f *as:*

$$\begin{aligned}\mathrm{d}s^2 &= \frac{|3f''f'''' - 5(f''')^2|}{9(f'')^2}\,\mathrm{d}t^2,\\ k^2 &= \frac{|9(f'')^2f''''' - 45f''f'''f'''' + 40(f''')^3|^2}{|3f''f'''' - 5(f''')^2|^3}.\end{aligned} \tag{4.3}$$

As we will see in Appendix A.1, $-\mu''(=2l)$ equals the equiaffine curvature up to a multiplicative constant. The factor on the right-hand side of the first expression is known, see [3], page 14. The second expression of k^2 was already presented in [30], page 54 from a different point of view. We refer also to [25], page 343 and [9]. The differential polynomials in the numerator and denominator on the right-hand side of k^2 are generally known; Berzolari [2] stated that those go back to Monge, see [20].

Making use of expression (4.2), we study how to find a graph immersion of plane curves with given general-affine curvature k, by integrating the following nonlinear differential equation directly:

$$\mu(\mu''')^2 = -\varepsilon\frac{k^2}{2}(\mu'')^3. \tag{4.4}$$

We regard the function μ' of t as a function of μ and set

$$w(\mu) = \mu'(t) = \frac{\mathrm{d}\mu}{\mathrm{d}t}.$$

Then, by the chain rule, we have

$$\mu'' = w\dot{w}, \quad \mu''' = w\dot{w}^2 + w^2\ddot{w}.$$

Hence, equation (4.4) is written as

$$\mu w^2(\dot{w}^2 + w\ddot{w})^2 + \varepsilon\frac{k^2}{2}w^3\dot{w}^3 = 0, \tag{4.5}$$

which can be reduced to the Abel equation as follows:

(i) First reduction: We introduce s by setting

$$w(x) = \pm\exp\left(-\varepsilon\int s^2\,\mathrm{d}x\right).$$

Here we choose the sign properly, depending on the function w. Then we get the equation

$$8x(-\varepsilon\dot{s} + s^3)^2 - k^2s^4 = 0 \quad (x > 0). \tag{4.6}$$

Therefore, the original differential equation (4.4) is equivalent to

$$\varepsilon\dot{s} = \frac{k}{2\sqrt{2x}}s^2 + s^3, \tag{4.7}$$

which is an *Abel equation of the first kind.*

It is easy to see that for constant $k < 0$ with $\varepsilon = -1$ or $k \leqslant -4$ with $\varepsilon = 1$, the solution s of (4.7) can be explicitly obtained as

$$s(x) = \frac{a}{\sqrt{2x}} \quad \text{with } a = \frac{-k \pm \sqrt{-16\varepsilon + k^2}}{4}.$$

The corresponding curves are given in Table 1 and $k < 0$ and $\varepsilon = -1$ or $k \leqslant -4$ with $\varepsilon = 1$. Moreover, in the case of $k = 0$ (both $\varepsilon = \pm 1$), the solution s can be obtained as

$$s(x) = \frac{1}{\sqrt{\varepsilon(a - 2x)}},$$

where a is some constant. The corresponding curves are given in Table 1 with $k = 0$. On the contrary, in the case of $-4 < k < 0$ for $\varepsilon = 1$, the solution s of (4.7) is not easy to write down explicitly. The corresponding curves are logarithmic spirals given in Table 1.

(ii) Second reduction: We define s by

$$w(x) = \pm\exp\left(-\varepsilon\int s^{-2}\,\mathrm{d}x\right),$$

by choosing the sign properly. Then a straightforward computation shows that equation (4.5) is transformed into

$$-k^2 s^2 + 8x(\varepsilon + s\dot{s})^2 = 0,$$

which is equivalent to

$$s\dot{s} = \frac{k}{2\sqrt{2x}} s - \varepsilon. \tag{4.8}$$

This is a particular case of the *Abel equation of the second kind.* We refer to [27], Section 1.3.2 for integrable Abel equations.

Theorem 4.2. *For any general-affine plane curve with graph immersion* $(t, f(t))$, *there exists a function* s, *given as above, such that it satisfies the Abel equation of the first kind or second kind,* (4.7) *or* (4.8), *respectively. Conversely, for any given function* k, *a solution* s *of* (4.7) *or* (4.8) *gives rise to a plane curve of graph immersion* $(t, f(t))$ *with general-affine curvature* k.

5. General-affine curvature of space curves

In this section, we will give a general-affine treatment of space curves; we introduce several notions such as curvature, length parameter and ordinary differential equation associated with space curves from a general-affine point of view.

Let $x\colon t \mapsto x(t) \in \mathbb{A}^3$ be a curve in a 3-dimensional affine space with parameter t and let $e = \{e_1, e_2, e_3\}$ be a frame along x; it is a set of independent vectors of $\mathbb{A}^3$. The vector-valued 1-form $\mathrm{d}x$ is written as

$$\mathrm{d}x = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3, \tag{5.1}$$

and the dependence of e_i on the parameter is described by the equation

$$\mathrm{d}e_i = \sum_{j=1,2,3} \omega_i^j e_j, \tag{5.2}$$

where ω^j and ω_i^j are 1-forms as before in the case of plane curves and $1 \leqslant i,\, j \leqslant 3$. We call $\{\omega^i, \omega_i^j\}$ the coframe.

We assume in the following that the curve is nondegenerate in the sense that the vectors x', x'' and x''' are linearly independent and that $\omega^2 = \omega^3 = 0$ and $\omega_1^3 = 0$, so that e_1 is tangent to the curve and that $\{e_1, e_2\}$ is the first osculating space of the curve. We write $\omega^1 = \omega$ for simplicity.

Let $\widetilde{e} = \{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ be another choice of such a frame. Then it can be written as

$$\widetilde{e}_1 = \lambda e_1, \quad \widetilde{e}_2 = \mu e_1 + \nu e_2, \quad \widetilde{e}_3 = \alpha e_1 + \beta e_2 + \gamma e_3,$$

where $\lambda\nu\gamma \neq 0$. The associated coframe is written as $\widetilde{\omega}$ and $\widetilde{\omega}_i^j$, which satisfies

$$\mathrm{d}x = \widetilde{\omega}\widetilde{e}_1, \quad \mathrm{d}\widetilde{e}_i = \sum_{j=1,2,3} \widetilde{\omega}_i^j \widetilde{e}_j.$$

Then we have

$$\widetilde{\omega} = \lambda^{-1}\omega. \tag{5.3}$$

Since $\mathrm{d}\widetilde{e}_1$ is represented in two ways, one being

$$\mathrm{d}\widetilde{e}_1 = (\mathrm{d}\lambda)e_1 + \lambda(\omega_1^1 e_1 + \omega_1^2 e_2)$$

and the other being

$$\mathrm{d}\widetilde{e}_1 = \widetilde{\omega}_1^1(\lambda e_1) + \widetilde{\omega}_1^2(\mu e_1 + \nu e_2),$$

by comparing the coefficients of e_1 and e_2 in these expressions, we get

$$\nu\widetilde{\omega}_1^2 = \lambda\omega_1^2, \tag{5.4}$$
$$\lambda\widetilde{\omega}_1^1 + \mu\widetilde{\omega}_1^2 = \mathrm{d}\lambda + \lambda\omega_1^1. \tag{5.5}$$

Similarly, by considering $\mathrm{d}\widetilde{e}_2$, we have

$$\gamma\widetilde{\omega}_2^3 = \nu\omega_2^3, \tag{5.6}$$
$$\nu\widetilde{\omega}_2^2 + \beta\widetilde{\omega}_2^3 = \mathrm{d}\nu + \mu\omega_1^2 + \nu\omega_2^2, \tag{5.7}$$
$$\lambda\widetilde{\omega}_2^1 + \mu\widetilde{\omega}_2^2 + \alpha\widetilde{\omega}_2^3 = \mathrm{d}\mu + \mu\omega_1^1 + \nu\omega_2^1, \tag{5.8}$$

and by $\mathrm{d}\widetilde{e}_3$ we have

$$\gamma\widetilde{\omega}_3^3 = \mathrm{d}\gamma + \beta\omega_2^3 + \gamma\omega_3^3, \tag{5.9}$$
$$\nu\widetilde{\omega}_3^2 + \beta\widetilde{\omega}_3^3 = \mathrm{d}\beta + \alpha\omega_1^2 + \beta\omega_2^2 + \gamma\omega_3^2, \tag{5.10}$$
$$\lambda\widetilde{\omega}_3^1 + \mu\widetilde{\omega}_3^2 + \alpha\widetilde{\omega}_3^3 = \mathrm{d}\alpha + \alpha\omega_1^1 + \beta\omega_2^1 + \gamma\omega_3^1. \tag{5.11}$$

First note that from the generality assumption we have $\omega_1^2 \neq 0$ and $\omega_2^3 \neq 0$. Then by an appropriate choice of ν and γ, in view of (5.4) and (5.6), we can assume that $\widetilde{\omega}_1^2 = \widetilde{\omega}$ and $\widetilde{\omega}_2^3 = \widetilde{\omega}$. Hence, we can restrict our consideration to the case

$$\omega_1^2 = \omega \quad \text{and} \quad \omega_2^3 = \omega$$

in the following. In particular,

$$\nu = \lambda^2 \quad \text{and} \quad \gamma = \lambda^3 \tag{5.12}$$

are necessary. We next see that from (5.5), (5.7) and (5.9) we have

$$2\widetilde{\omega}_1^1 - \widetilde{\omega}_2^2 = 2\omega_1^1 - \omega_2^2 - 3\lambda^{-2}\mu\omega + \lambda^{-3}\beta\omega, \quad 3\widetilde{\omega}_1^1 - \widetilde{\omega}_3^3 = 3\omega_1^1 - \omega_3^3 - 3\lambda^{-2}\mu\omega - \lambda^{-3}\beta\omega.$$

Thus, an appropriate choice of the parameters μ and β makes the identities $\widetilde{\omega}_2^2 = 3\widetilde{\omega}_1^1$ and $\widetilde{\omega}_3^3 = 2\widetilde{\omega}_1^1$ hold. To keep this condition it is necessary to have $\mu = \beta = 0$. Now (5.8) can be rephrased as

$$\lambda\widetilde{\omega}_2^1 + \alpha\widetilde{\omega}_2^3 = \lambda^2\omega_2^1,$$

and we choose α so that $\widetilde{\omega}_2^1 = 0$. Thus, we can assume that $\omega_2^1 = 0$ and $\alpha = 0$. Moreover, (5.5) is

$$\widetilde{\omega}_1^1 = \lambda^{-1}\,\mathrm{d}\lambda + \omega_1^1,$$

and we can choose λ so that $\widetilde{\omega}_1^1 = 0$. Therefore $\omega_1^1 = 0$, and to keep this condition, λ is a non-zero constant. With these considerations, the last identities (5.10) and (5.11) turn out to be

$$\widetilde{\omega}_3^2 = \lambda\omega_3^2 \quad \text{and} \quad \widetilde{\omega}_3^1 = \lambda^2\omega_3^1,$$

respectively. We set

$$\omega_3^2 = -l\omega, \quad \omega_3^1 = -m\omega, \tag{5.13}$$

and similarly for $\widetilde{\omega}_3^2$ and $\widetilde{\omega}_3^1$. Then we have the covariance

$$\widetilde{l} = \lambda^2 l, \quad \text{and} \quad \widetilde{m} = \lambda^3 m. \tag{5.14}$$

Thus, we have seen that given a nondegenerate curve x, there exists a frame e with the coframe of the form

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \\ -m\omega & -l\omega & 0 \end{pmatrix}. \tag{5.15}$$

We remark here that, in the equiaffine treatment of space curves, the scalars l and m above are known to be absolute invariants, called the *equiaffine curvature* and the *equiaffine torsion*, respectively; we refer to Appendix A.2. In this paper we call the point, where $l = 0$, an *affine inflection point*.

In the following we assume $l \neq 0$ and let ε denote the sign of l:

$$\varepsilon = \mathrm{sign}(l).$$

It is an invariant of the curve. Then we define the *general-affine length element* by

$$\omega_s = \sqrt{\varepsilon l}\,\omega, \tag{5.16}$$

which is well-defined and independent of the frame in view of (5.14), and the parameter s for which $\mathrm{d}s = \omega_s$ holds is the *general-affine length parameter* determined up to an additive constant.

Definition 5.1. We call the scalar function k defined by

$$\frac{\mathrm{d}l}{l} = k\omega_s$$

the *first general-affine curvature*. In other words,

$$k = \frac{\mathrm{d}\log l}{\mathrm{d}s}. \tag{5.17}$$

We call the scalar function M defined by

$$M = \frac{m}{(\varepsilon l)^{3/2}} \tag{5.18}$$

the *second general-affine curvature* of the space curve.

Both curvatures defined above are absolute invariants.

We next define a new frame $\{E_1, E_2, E_3\}$ by setting

$$E_1 = \frac{1}{(\varepsilon l)^{1/2}}e_1, \quad E_2 = \frac{1}{\varepsilon l}e_2, \quad E_3 = \frac{1}{(\varepsilon l)^{3/2}}e_3.$$

It is easy to see that this frame does not depend on the choice of λ; hence, it is determined uniquely.

Thus we have proved the following:

Proposition 5.2. *Assume $l \neq 0$. Then the frame $\{E_1, E_2, E_3\}$ is uniquely defined from the immersion and it satisfies the Pfaffian equation*

$$(5.19) \qquad d\begin{pmatrix} x \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} = \Omega \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega_s & 0 & 0 \\ -\frac{1}{2}k\omega_s & \omega_s & 0 \\ 0 & -k\omega_s & \omega_s \\ -M\omega_s & -\varepsilon\omega_s & -\frac{3}{2}k\omega_s \end{pmatrix},$$

where ω_s is the general-affine length form, k and M are the first and second general-affine curvatures, respectively, and $\varepsilon = \operatorname{sign}(l)$.

By use of this choice of frame, we can see the following lemma, by a similar reasoning to that for Lemma 2.3.

Lemma 5.3. *The immersion x satisfies the ordinary differential equation*

$$(5.20) \quad x'''' + 3kx''' + \Big(\varepsilon + 2k' + \frac{11}{4}k^2\Big)x'' + \Big(M + \frac{1}{2}\varepsilon k + \frac{1}{2}k'' + \frac{7}{4}kk' + \frac{3}{4}k^3\Big)x' = 0,$$

relative to a general-affine length parameter.

In the definition of the curvature, we had an ambiguity of orientation of the chosen parameter: the change of the parameter from t to $-t$, transforms the equation to

$$x'''' - 3kx''' + \Big(\varepsilon - 2k' + \frac{11}{4}k^2\Big)x'' + \Big(-M - \frac{1}{2}\varepsilon k - \frac{1}{2}k'' + \frac{7}{4}kk' - \frac{3}{4}k^3\Big)x' = 0.$$

Namely, the transform $(k, M) \mapsto (-k, -M)$ keeps the form of the equation.

Thus, up to this ambiguity, we have the following theorem.

Theorem 5.4. *Given functions $k(t)$ and $M(t)$ of a parameter t, and $\varepsilon = \pm 1$, there exists a nondegenerate space curve $x(t)$ for which t is a general-affine length parameter, k is the first general-affine curvature, M is the second general-affine curvature, and ε is the sign of l, uniquely up to a general-affine transformation.*

Analogously to the case of plane curves, we have the following property on the total general-affine curvature:

Corollary 5.5. *Assume that the curve C is nondegenerate and closed, and has no affine inflection point. Then the total curvature $\int_C k\omega_s$ vanishes. In particular, such a curve has at least two general-affine flat points.*

The computation of general-affine curvatures of space curves given relative to parameter not necessarily a length parameter, is done similarly as for plane curves. Let $t \mapsto x = x(t) \in \mathbb{A}^3$ be a nondegenerate curve such that the vectors x', x'' and x''' are linearly independent and write the vector x'''' as a linear combination of x', x'' and x''':

$$x'''' = ax''' + bx'' + cx', \tag{5.21}$$

where $a = a(t)$, $b = b(t)$ and $c = c(t)$ are scalar functions. Let $e_1 = \lambda x'$ be the first frame vector and set $\omega = \lambda^{-1}\,\mathrm{d}t$. We will determine λ so that we get a normalized frame: $\mathrm{d}e_1 = e_2\omega$, $\mathrm{d}e_2 = e_3\omega$ and $\mathrm{d}e_3$ has no e_3-component. A computation shows that $e_2 = \lambda^2 x'' + \lambda\lambda' x'$ and $e_3 = (\lambda^3 x''' + 3\lambda^2\lambda' x'' + (\lambda^2\lambda'' + \lambda\lambda'^2)x')$, and $\mathrm{d}e_3 \equiv (\lambda^3 a + 6\lambda^2\lambda')x'''$ modulo (x', x''). Then $\lambda = \mathrm{e}^{-(\int a(t)\,\mathrm{d}t)/6}$ corresponds to the choice of the frame. Then $\mathrm{d}e_3$ is written as $\mathrm{d}e_3 = (\lambda^2 b + 7\lambda'^2 + 4\lambda\lambda'')\omega e_2 + (\lambda^3 c - \lambda^2\lambda' b - 6\lambda'^3 + \lambda^2\lambda''')\omega e_1$. By the definition in (5.13), we have $l = -(\lambda^2 b + 7\lambda'^2 + 4\lambda\lambda'')$ and $m = -\lambda^3 c + \lambda^2\lambda' b + 6\lambda'^3 - \lambda^2\lambda'''$. We now assume that $l \neq 0$ and recall that $\varepsilon = \mathrm{sign}(l)$. Then a straightforward computation shows that the length element is written as

$$\mathrm{d}s^2 = -\varepsilon\Big(b + \frac{11}{36}a^2 - \frac{2}{3}a'\Big)\,\mathrm{d}t^2. \tag{5.22}$$

If, in particular, t itself is a length parameter, then we have $b = -\varepsilon - \frac{11}{36}a^2 + \frac{2}{3}a'$ and by definition, the first curvature k is

$$k = -\frac{1}{3}a, \tag{5.23}$$

while the second curvature M turns out to be

$$M = -c + \frac{1}{6}a\varepsilon + \frac{1}{6}a'' - \frac{7}{36}aa' + \frac{1}{36}a^3. \tag{5.24}$$

For the actual computation, the next lemma will be useful.

Lemma 5.6. *Given a nondegenerate curve $x(t)$ satisfying* (5.21) *let $y(\sigma) = x(t)$ be a change of parameter from t to $\sigma(t)$. Then the vector y satisfies a differential equation*

$$\ddot{\ddot{y}} = A(\sigma)\dddot{y} + B(\sigma)\ddot{y} + C(\sigma)\dot{y}, \tag{5.25}$$

where $\{\dot{}\}$ *denotes derivation relative to* σ *and*

$$A(\sigma) = \Big(a - 6\frac{\sigma''}{\sigma'}\Big)\frac{1}{\sigma'}, \tag{5.26}$$

$$B(\sigma) = \Big(b + 3a\frac{\sigma''}{\sigma'} - 3\Big(\frac{\sigma''}{\sigma'}\Big)^2 - 4\frac{\sigma'''}{\sigma'}\Big)\frac{1}{\sigma'^2}, \tag{5.27}$$

$$C(\sigma) = \Big(c + b\frac{\sigma''}{\sigma'} + a\frac{\sigma'''}{\sigma'} - \frac{\sigma''''}{\sigma'}\Big)\frac{1}{\sigma'^3}. \tag{5.28}$$

Thus, we get the procedure of how to compute curvature:

Procedure for computing curvatures.

Step 1. Given a curve $x(t)$, derive the differential equation (5.21).
Step 2. Compute $L = -\big(b + \frac{11}{36}a^2 - \frac{2}{3}a'\big)$ and define ε by $\varepsilon = \operatorname{sign}(L)$.
Step 3. Compute the length parameter σ by solving $\mathrm{d}\sigma = \sqrt{\varepsilon L}\,\mathrm{d}t$.
Step 4. Compute A, B, and C by (5.26), (5.27) and (5.28); then $-\frac{1}{3}A$ is the first curvature k and $-C + \frac{1}{6}A\varepsilon + \frac{1}{6}\ddot{A} - \frac{7}{36}A\dot{A} + \frac{1}{36}A^3$ is the second curvature M.

Remark 5.7. The curves with constant k and M have a special interest, because such a curve is an orbit of a 1-parameter subgroup of general-affine motions. In [30], pages 36–39, a classification of such groups is given. In the present setting, it is enough to solve the differential equation $x'''' = ax''' + bx'' + cx'$ with constant coefficients a, b and c to get such curves.

For later reference, we give one example:

Example 5.8. For curves given by $x = (t, \mathrm{e}^{at}, t\mathrm{e}^{at})$ with a real constant a, the equation is

$$x'''' = 2ax''' - a^2x'',$$

and $\varepsilon = -1$, $k = -\sqrt{2}\operatorname{sign}(a)$, $M = \sqrt{2}\operatorname{sign}(a)$. In particular, the identity $M - k\varepsilon = 0$ holds. We refer to Appendix B.2.

6. General-affine extremal space curves and the associated differential equations

In Section 5, we have defined the general-affine length parameter and general-affine curvatures under the condition that the curve has no affine inflection point. In this section, we obtain the condition under which a space curve is extremal relative to the length functional and, in particular, show that any curve with constant general-affine curvatures is extremal.

Let $x_\eta(t)$ be a family of curves parametrized by η around $\eta = 0$ and $x_0 = x$. We assume that $x_\eta(t) = x(t)$ outside a compact set C, and that the invariant ω_s does not vanish anywhere for all η. Then x_η and the corresponding frame $\{E_1, E_2, E_3\}$ satisfy the equation in (5.19). Then the length functional L is given by

$$L(\eta) = \int_C \omega_s(\eta)$$

and the curve $x = x_0$ is said to be *general-affine extremal* if

$$\delta L = \frac{\mathrm{d}L}{\mathrm{d}\eta}\Big|_{\eta=0} = 0$$

holds for any compactly supported deformation of x.

We now consider the variation

$$\delta \begin{pmatrix} x_\eta \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} = \tau \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \tau = (\tau_\alpha^\beta)_{0\leqslant\alpha\leqslant3, 1\leqslant\beta\leqslant3}.$$

Then the compatibility condition $\mathrm{d}\delta = \delta\,\mathrm{d}$ implies that

$$\delta\omega_\alpha^\beta - \mathrm{d}\tau_\alpha^\beta = \sum_{\gamma=1,2,3} \tau_\alpha^\gamma \omega_\gamma^\beta - \omega_\alpha^\gamma \tau_\gamma^\beta,$$

where we set the entries of Ω in (5.19) as $(\omega_\alpha^\beta)_{0\leqslant\alpha\leqslant3,1\leqslant\beta\leqslant3}$. Then they are explicitly given by

$$\delta\omega_s - \mathrm{d}\tau_0^1 = \Big(-\frac{1}{2}k\tau_0^1 - M\tau_0^3 - \tau_1^1\Big)\omega_s, \tag{6.1}$$

$$-\mathrm{d}\tau_0^2 = (\tau_0^1 - k\tau_0^2 - \varepsilon\tau_0^3 - \tau_1^2)\omega_s, \tag{6.2}$$

$$-\mathrm{d}\tau_0^3 = \Big(\tau_0^2 - \frac{3}{2}k\tau_0^3 - \tau_1^3\Big)\omega_s, \tag{6.3}$$

$$\delta\omega_s - \mathrm{d}\tau_1^2 = \Big(\tau_1^1 - \frac{1}{2}k\tau_1^2 - \varepsilon\tau_1^3 - \tau_2^2\Big)\omega_s, \tag{6.4}$$

$$-\mathrm{d}\tau_1^3 = \Big(\tau_1^2 - k\tau_1^3 - \tau_2^3\Big)\omega_s, \tag{6.5}$$

$$-\mathrm{d}\tau_2^1 = \Big(\frac{1}{2}k\tau_2^1 - M\tau_2^3 - \tau_3^1\Big)\omega_s, \tag{6.6}$$

$$\delta\omega_s - \mathrm{d}\tau_2^3 = \Big(\tau_2^2 - \frac{1}{2}k\tau_2^3 - \tau_3^3\Big)\omega_s, \tag{6.7}$$

$$-\varepsilon\delta\omega_s - \mathrm{d}\tau_3^2 = \Big(-\varepsilon(\tau_3^3 - \tau_2^2) + \tau_3^1 + \frac{1}{2}k\tau_3^2 + M\tau_1^2\Big)\omega_s, \tag{6.8}$$

$$-\frac{1}{2}\delta(k\omega_s) - \mathrm{d}\tau_1^1 = (-M\tau_1^3 - \tau_2^1)\omega_s, \tag{6.9}$$

$$-\delta(k\omega_s) - \mathrm{d}\tau_2^2 = (\tau_2^1 - \varepsilon\tau_2^3 - \tau_3^2)\omega_s, \tag{6.10}$$

$$(6.11)\qquad -\frac{3}{2}\delta(k\omega_s) - \mathrm{d}\tau_3^3 = (\tau_3^2 + M\tau_1^3 + \varepsilon\tau_2^3)\omega_s,$$

$$(6.12)\qquad -\delta(M\omega_s) - \mathrm{d}\tau_3^1 = (k\tau_3^1 + M(\tau_1^1 - \tau_3^3) + \varepsilon\tau_2^1)\omega_s.$$

Adding (6.7) and $-\varepsilon\times$(6.8), we have

$$2\delta\omega_s - \mathrm{d}\tau_2^3 + \varepsilon\,\mathrm{d}\tau_3^2 = \Big(-\frac{1}{2}k(\tau_2^3 + \varepsilon\tau_3^2) - \varepsilon\tau_3^1 - \varepsilon M\tau_1^2\Big)\omega_s.$$

Then by Stokes' theorem we have

$$(6.13)\qquad 2\delta\int_C \omega_s = \int_C \Big(-\frac{1}{2}k(\tau_2^3 + \varepsilon\tau_3^2) - \varepsilon\tau_3^1 - \varepsilon M\tau_1^2\Big)\omega_s.$$

Next, from (6.9)+(6.10)−(6.11) we get

$$(-\tau_2^2 - \tau_1^1 + \tau_3^3)' = -2(\varepsilon\tau_2^3 + \tau_3^2) - 2M\tau_1^3,$$

which is written as

$$(6.14)\qquad -\frac{1}{2}k(\tau_3^2 + \varepsilon\tau_2^3) = -\frac{1}{4}\varepsilon k(\tau_1^1 + \tau_2^2 - \tau_3^3)' + \frac{1}{2}\varepsilon k M\tau_1^3.$$

Here $\{'\}$ denotes d/ω_s. Equation (6.6) implies

$$(6.15)\qquad \tau_3^1 = \tau_2^{1\prime} + \frac{1}{2}k\tau_2^1 - M\tau_2^3$$

and (6.10)+(6.11)−5×(6.9) gives

$$(6.16)\qquad (-\tau_2^2 - \tau_3^3 + 5\tau_1^1)' = 6(\tau_2^1 + M\tau_1^3).$$

Then by use of (6.5) and (6.16), (6.15) can be rephrased as

$$(6.17)\qquad \tau_3^1 = \tau_2^{1\prime} + \frac{1}{12}k(5\tau_1^1 - \tau_2^2 - \tau_3^3)' - M\Big(\tau_1^{3\prime} + \tau_1^2 - \frac{1}{2}k\tau_1^3\Big).$$

Finally, (6.14) and (6.17) implies that

$$(6.18)\qquad \begin{aligned} 2\delta\int_C \omega_s &= \frac{\varepsilon}{12}\int_C (-k(8\tau_1^1 + 2\tau_2^2 - 4\tau_3^3)' + 12M\tau_1^{3\prime})\omega_s \\ &= \frac{\varepsilon}{12}\int_C (k'(8\tau_1^1 + 2\tau_2^2 - 4\tau_3^3) + 12M\tau_1^{3\prime})\omega_s. \end{aligned}$$

Here we use integration by parts for the second equality.

We now compute $-6\times(6.1)+2\times(6.4)+4\times(6.7)$. A straightforward computation shows that

$$\begin{aligned} 8\tau_1^1 + 2\tau_2^2 - 4\tau_3^3 &= (6\tau_0^1 - 2\tau_1^2 - 4\tau_2^3)' - 3k\tau_0^1 - 6M\tau_0^3 + k\tau_1^2 + 2\varepsilon\tau_1^3 + 2k\tau_2^3 \\ &= 6X' - 4Y' - 3kX + 2kY - 6M\tau_0^3 + 2\varepsilon\tau_1^3. \end{aligned}$$

Here $X = \tau_0^1 - \tau_1^2$ and $Y = \tau_2^3 - \tau_1^2$. Thus, (6.18) can be again rephrased by using integration by parts, as

$$(6.19) \qquad 24\varepsilon\delta \int_C \omega_s = \int_C \{(-6k'' - 3k'k)X + (4k'' + 2k'k)Y - 6k'M\tau_0^3 + (2\varepsilon k' - 12M')\tau_1^3\}\omega_s.$$

Then by (6.5) and (6.2) we have

$$X = \tau_0^1 - \tau_1^2 = -\tau_0^{2\prime} + k\tau_0^2 + \varepsilon\tau_0^3, \quad Y = \tau_2^3 - \tau_1^2 = \tau_1^{3\prime} - k\tau_1^3.$$

Finally, making use of (6.3) to erase the τ_1^3-term, we can see that the τ_0^2 part of the integrand of (6.19) is computed as

$$(6.20) \qquad -10k''' - 15k''k - 5k'k^2 - 5k'^2 + 2\varepsilon k' - 12M'.$$

Similarly, the τ_0^3 part of the integrand of (6.19) can be computed as

$$(6.21) \qquad 4k'''' + 12k'''k + (11k^2 + 10k' - 8\varepsilon)k'' + 7k'^2k - 6\varepsilon k'k + 3k'k^3 - 6k'M + 12M'' + 18M'k.$$

Theorem 6.1. *A nondegenerate space curve without affine inflection point is general-affine extremal if and only if the following pair of ordinary differential equations is satisfied:*

$$(6.22) \qquad k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' - \frac{1}{5}\varepsilon k' + \frac{6}{5}M' = 0$$

and

$$(6.23) \qquad k'' + \frac{2}{3}k'k + \frac{5}{6}\varepsilon k'M - \frac{3}{2}\varepsilon kM' - \varepsilon M'' = 0.$$

In particular, all space curves which have constant general-affine curvatures are general-affine extremal.

Proof. Equation $(6.20) = 0$ is the differential equation (6.22) and, by inserting this equation into $(6.21) = 0$, we have the differential equation (6.23). $\square$

Example 6.2. Extremal curves with constant M.
First, assume $M = 0$. Then (6.23) can be easily integrated as

$$k(t) = -3a\tan(at) \quad \text{and} \quad 3a\tanh(at),$$

where a is a constant. Inserting this expression into (6.22), we get solutions

$$k(t) = -3a\tan(at), \quad a = \sqrt{\frac{2}{5}} \quad \text{when} \quad \varepsilon = 1$$

and

$$k(t) = 3a\tanh(at), \quad a = \sqrt{\frac{2}{5}} \quad \text{when} \quad \varepsilon = -1.$$

Second, assume M is a nonzero constant. Then

$$k(t) = -\frac{5}{4}\varepsilon M + 3a\tanh(at) \tag{6.24}$$

is a solution of (6.23) and it satisfies (6.22) if and only if

$$a^2(80a^2 - 125\varepsilon^2 M^2 + 32\varepsilon) = 0.$$

Thus, except for a constant solution, we have $k(t)$ above in (6.24), where

$$a = \frac{\sqrt{-32\varepsilon + 125M^2}}{4\sqrt{5}}.$$

If we started with $-\frac{5}{4}\varepsilon M - 3a\tan(at)$, another solution of (6.23), then a turns out to be

$$a = \frac{\sqrt{32\varepsilon - 125M^2}}{4\sqrt{5}},$$

which gives the same curvature function in (6.24) by choosing the constant M properly.

We here recall the invariant θ_3 given equation (B.7):

$$\theta_3 = \frac{1}{4}(M - \varepsilon k).$$

Then the differential equations (6.22) and (6.23) are written as

$$k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' + \varepsilon k' + \frac{24}{5}\theta_3' = 0, \tag{6.25}$$

$$\theta_3'' + \frac{3}{2}k\theta_3' - \frac{5}{6}k'\theta_3 = 0. \tag{6.26}$$

Since $\theta_3 = 0$ characterizes a curve belonging to a linear complex, see Section B.2, in view of equation (3.11), we have the following corollary.

Corollary 6.3. *The general-affine extremal space curve belongs to a linear complex if and only if $M = \varepsilon k$ and k satisfies the differential equation* (3.11).

Since the differential equation (3.11) is the equation for an extremal plane curve, we have the following method of constructing an extremal space curve belonging to a linear complex:

Corollary 6.4. *Let k be the general-affine curvature of an extremal plane curve without affine inflection point. Let ε denote the sign of this curve. Then the set $\{k, M, \varepsilon\}$, where $M = \varepsilon k$, defines a space curve that is general-affine extremal and belongs to a linear complex.*

Thanks to Example 3.5, we can give concrete examples of such curves in Corollary 6.4. The explicit integration of the associated differential equation can be carried out with computer assistance. For example, when $\varepsilon = -1$ and $k(t) = \sqrt{2} + 3/t$, we get the curve (x_1, x_2, x_3) for $t > 0$, where

$$x_1 = \frac{1}{t}, \quad x_2 = 2^{1/4}\sqrt{\pi}\,\mathrm{erf}\Big(\frac{\sqrt{3t}}{2^{1/4}}\Big) - \frac{1-\sqrt{2}3t}{(3t)^{3/2}}\exp\Big(-\frac{3}{\sqrt{2}}t\Big),$$
$$x_3 = \int\Big\{\frac{6}{t^2}\int H(t)\,\mathrm{d}t + \frac{1}{t^{5/2}(\sqrt{2}+6t)}\exp\Big(-\frac{3}{\sqrt{2}}t\Big)\Big\}\,\mathrm{d}t$$

with

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t \quad \text{and} \quad H(t) = \frac{1}{\sqrt{t}(\sqrt{2}+6t)^2}\exp\Big(-\frac{3}{\sqrt{2}}t\Big).$$

Appendix

A. From equiaffine to general-affine. Since the group of equiaffine motions $\mathrm{SA}(n) = \mathrm{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ is a subgroup of the general-affine group $\mathrm{GA}(n) = \mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$, any general-affine invariant is obviously an equiaffine invariant. In this appendix, we give the expression of the general-affine length parameter and the general-affine curvature by use of the equiaffine length parameter and the equiaffine curvature for plane curves and space curves.

A.1 General-affine invariants in terms of equiaffine invariants for plane curves. Let us recall coframe (2.7) written as

$$\text{(A.1)} \qquad \mathrm{d}\begin{pmatrix} x \\ e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \\ -l\omega & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

In the equiaffine treatment, it is enough to consider only the unimodular change of frame, i.e., $\lambda\nu = 1$. Because we restrict our consideration to the case where

$\nu = \lambda^2$, we then have $\lambda = 1$. Thus, l is an absolute invariant because of (2.6). The scalar l is usually denoted by k_a and called the *equiaffine curvature* of a plane curve. The parameter t for which $\omega = \mathrm{d}t$ holds is called the *equiaffine length parameter*. Let $x(t)$ be a curve with equiaffine length parameter t. Then it is easy to see by (A.1) that x satisfies

$$x''' = -k_a x'. \tag{A.2}$$

On the other hand, when the curve is given by a graph immersion $x(t) = (t, f(t))$, where t is a parameter that is not necessarily equiaffine, the equiaffine length element is $(f'')^{1/3}\,\mathrm{d}t$ and the equiaffine curvature is $k_a = -\frac{1}{2}\mu''$ for $\mu = (f'')^{-2/3}$. As we have seen in (4.1), the equiaffine curvature is nothing but l up to a constant multiple. We refer to [3], pages 13–14.

Now we consider the curve in view of the group GA(2). Then the differential equation (A.2) above shows that the general-affine length element is

$$\mathrm{d}s = |k_a|^{1/2}\,\mathrm{d}t.$$

We rewrite the differential equation using the parameter $s = s(t)$: we set $y(s) = x(t)$ and let $\dot{\{\}}$ denote the derivation relative to s. Then we get the equation

$$\dddot{y} = A(s)\ddot{y} + B(s)\dot{y},$$

where $A(s)$ and $B(s)$ are given by (2.18) and (2.19). For simplicity, we set $K = |k_a|$. Since $s' = \mathrm{d}s/\mathrm{d}t = K^{1/2}$ and $s'' = \frac{1}{2}K'K^{-1/2}$, we see that $A(s) = -\frac{3}{2}K'K^{-3/2}$. Therefore the general-affine curvature of $x(t)$ is

$$k = K'K^{-3/2}.$$

The quantity of this form was already treated in [3], page 24 by dimension considerations to get an invariant relative to similarity transformation.

A.2 General-affine invariants in terms of equiaffine invariants for space curves. For space curves, we have coframe (5.14):

$$\mathrm{d}\begin{pmatrix} x \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \\ -m\omega & -l\omega & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{A.3}$$

In the equiaffine treatment, it is enough to consider only the unimodular change of frames, i.e., $\lambda\nu\gamma = 1$ and $e = (e_1, e_2, e_3)$ takes values in $\mathrm{SL}(3, \mathbb{R})$. By (5.12) we have

$\lambda = \pm 1$. This means that the scalar l is an absolute invariant and the scalar m is an invariant determined up to ± 1 by (5.14). As remarked in Section 5, the scalar l is usually called the *equiaffine curvature* of the space curve and the scalar m is called the *equiaffine torsion*; we refer to the books [3], [30]. The invariant l measures how the space curve differs from the osculating cubic parabola, which is defined to be the curve $(t, t^2/2, t^3/6)$ relative to certain affine coordinates.

The parameter t for which $\omega = \mathrm{d}t$ holds is called an *equiaffine length parameter.* Then equation (A.3) implies that the immersion $x(t)$ satisfies the differential equation

$$x'''' + lx'' + mx' = 0, \tag{A.4}$$

which is written in the form of equation (5.21), where $a = 0$, $b = -l$, $c = -m$. By (5.16), the general-affine length parameter σ is determined of the equiaffine curvature l as

$$\mathrm{d}\sigma^2 = K\,\mathrm{d}t$$

and by (5.17) and (5.18), the first and second general-affine curvatures are given as

$$k = K'K^{-3/2}, \qquad M = mK^{-3/2}$$

in terms of equiaffine curvature and equiaffine torsion. Relative to the parameter σ, the map $y(\sigma) = x(t)$ is seen to satisfy equation (5.25) whose coefficients are given by

$$\begin{aligned} A(\sigma) &= -\frac{3K'}{K^{3/2}}, \quad B(\sigma) = -\varepsilon + \frac{K'}{4K^3} - \frac{2K''}{K^2}, \\ C(\sigma) &= K^{-3/2}\Big(-m - \frac{\varepsilon K'}{2} - \frac{K'''}{2K} + \frac{3K'K''}{4K^2} - \frac{3K'^3}{8K^3}\Big), \end{aligned}$$

which follows from (5.26)–(5.28).

The curves for which l and m are constant can be obtained by solving equation (A.4); for the classification, see [30], page 75. We remark also that Blaschke [3] gave a variational formula of the equiaffine length and showed that extremal curves of this variation are the curves with $l = m = 0$; hence, the cubic parabola.

B. From general-affine to projective. It was Halphen in [12] who began a systematic study of projective curves using of ordinary differential equations. Later, Wilczynski gave a classical treatment of projective curves in the book (see [33]). Also, the books by Lane (see [15]) are standard references for this subject. In this appendix, we recall the definition of the projective length element and the projective curvature, and give the expressions of such invariants in terms of general-affine invariants.

B.1 Projective invariants in terms of general-affine invariants for plane curves. A nondegenerate curve in $\mathbb{P}^2$ with parameter t is described by an ordinary differential equation of the form

$$y''' + P_2 y' + P_3 y = 0, \tag{B.1}$$

such that a set of three independent solutions, say, x^1, x^2, x^3 defines a map $t \mapsto [x^1, x^2, x^3] \in \mathbb{P}^2$, where [ ] denotes homogeneous coordinates. For this equation, the form

$$P^{1/3}\,\mathrm{d}t, \quad \text{where} \quad P = P_3 - \frac{1}{2}P_2', \tag{B.2}$$

is called the projective length element. Furthermore, when t itself is a projective length parameter, the equation can be transformed by a certain change of variables from y to $z = \lambda y$ into the equation of the form

$$z''' + 2k_p z' + (1 + k_p')z = 0, \tag{B.3}$$

which is called the Halphen canonical form. Then the coefficient k_p is called the *projective curvature* and is given by the formula

$$k_p = P^{-2/3}\left(\frac{1}{2}P_2 - \frac{1}{3}\frac{P''}{P} + \frac{7}{18}\left(\frac{P'}{P}\right)^2\right); \tag{B.4}$$

we refer to [7], page 71. In particular, when k_p is constant, the curve is called an anharmonic curve and is obtained by integrating the differential equation $z''' + 2k_p z' + z = 0$; we refer to [33], pages 86–91. The list of anharmonic curves is the same as the list of plane curves with constant general-affine curvature in Remark 2.8, up to projective equivalence.

In the general-affine setting we had the differential equation (2.16), which can be transformed into the equation of the form (B.1) by changing x into $y = \mathrm{e}^{-(\int a\,\mathrm{d}t)/3}x$. The result is

$$y''' = \left(\frac{1}{9}a^2 - \frac{2}{3}a' - \varepsilon\right)y' + \left(\frac{1}{9}aa' - \frac{1}{3}a'' - \frac{1}{3}a\varepsilon\right)y.$$

Hence, we can see that

$$P = -\frac{1}{3}a\varepsilon;$$

this implies that the projective length element is $a^{1/3}\,\mathrm{d}t$ up to a scalar multiple, while $-\frac{2}{3}a$ is the general-affine curvature. In particular, the point where the general-affine curvature vanishes is the point where the invariant P vanishes, which is classically called a *sextactic point*.

We remark that in [28], Sasaki showed how to obtain the projective length parameter and the projective curvature directly from the equiaffine curvature.

B.2 Projective invariants in terms of general-affine invariants for space curves. A space curve in $\mathbb{P}^3$ is given by the immersion $t \mapsto x(t) \in \mathbb{A}^4$ satisfying an ordinary differential equation of the form

$$x'''' + 4p_1 x''' + 6p_2 x'' + 4p_3 x' + p_4 x = 0.$$

By multiplying the indeterminate x by nonzero factor to the equation is transformed into the equation

$$x''''' + 6P_2 x'' + 4P_3 x' + P_4 x = 0, \tag{B.5}$$

where

$$P_2 = p_2 - p_1^2 - p_1', \quad P_3 = p_3 - 3p_1 p_2 + 2p_1^3 - p_1'',$$
$$P_4 = p_4 - 4p_1 p_3 + 6p_1^2 p_2 - 6p_1' p_2 - 3p_1^4 + 6p_1^2 p_1' + 3(p_1')^2 - p_1'''.$$

Then the two forms $\theta_3\,\mathrm{d}t^3$ and $\theta_4\,\mathrm{d}t^4$, where

$$\theta_3 = P_3 - \frac{3}{2}P_2', \quad \theta_4 = P_4 - \frac{9}{5}P_2'' - \frac{81}{25}P_2^2 - 2\theta_3', \tag{B.6}$$

are fundamental invariant forms: we refer to [15]. Provided that $\theta_3 \neq 0$, the parameter s defined as

$$\mathrm{d}s = \theta_3^{1/3}\,\mathrm{d}t$$

is called the *projective length parameter*. Relative to this parameter, we can define projective curvatures; we refer to Appendix C. When $\theta_3 \equiv 0$, the curve x has a special property that the curve formed by the tangent vectors to the curve x, which lies in the 5-dimensional projective space, which consists of lines in $\mathbb{P}^3$, is degenerate in the sense that it belongs to a 4-dimensional hyperplane. Such a curve was said to belong to a *linear complex* and is named Gewindekurve in [3].

Given a nondegenerate curve $x(t)$ in the affine space $\mathbb{A}^3$, which is described by the differential equation (5.20), we associate a curve in $\mathbb{P}^3$ by a mapping $t \mapsto (1, x(t)) \in \mathbb{A}^4$, where 1 is a constant function. Then the projective invariants are computed by the definition above. In fact, a straightforward computation shows that

$$\theta_3 = \frac{1}{4}(M - \varepsilon k), \tag{B.7}$$

$$\theta_4 = -\frac{3}{4}kM - \frac{1}{2}M' + \frac{1}{5}\varepsilon k' + \frac{3}{10}\varepsilon k^2 - \frac{9}{100}. \tag{B.8}$$

In particular, when $\theta_3 \neq 0$, the projective length parameter s is given as above by use of the general-affine curvatures k and M. When $M = \varepsilon k$, the curve belongs to a linear complex. Example 5.8 in Section 5 is such a case.

C. Projective treatment of curves. A study of basic notions such as projective length parameter and projective curvature by use of moving frames was originally done by Cartan in [6], [7], Chapter 2. In this appendix, for comparison with the general-affine treatment, we will recall a little of it.

C.1 Projective invariants of plane curves. By a projective plane curve we mean a nondegenerate immersion into $\mathbb{P}^2$: $t \mapsto \underline{x}(t) \in \mathbb{P}^2$. We denote its lift to the affine space $\mathbb{A}^3$ by $t \mapsto x(t) \in \mathbb{A}^3 - \{0\}$. Let $e = \{e_0, e_1, e_2\}$ be a frame along x; at each point of x it is a set of independent vectors of $\mathbb{A}^3$ which depend smoothly on the parameter. We choose $e_0 = x$ for simplicity. Then we have the equation of motions of the frame

$$\mathrm{d}e = \Omega e, \qquad \Omega = \begin{pmatrix} \omega_0^0 & \omega_0^1 & \omega_0^2 \\ \omega_1^0 & \omega_1^1 & \omega_1^2 \\ \omega_2^0 & \omega_2^1 & \omega_2^2 \end{pmatrix}.$$

It is natural to assume that $\omega_0^0 + \omega_1^1 + \omega_2^2 = 0$ and that the space generated by e_0 and e_1 is the space generated by the vector $x(t)$ and the tangent vector $x'(t)$; $\omega := \omega_0^1$ is nontrivial and $\omega_0^2 = 0$. Once we choose a frame e with the required property, then another frame $\widetilde{e}$ is given as

$$\widetilde{e} = ge, \qquad g = \begin{pmatrix} \lambda & 0 & 0 \\ \mu & \alpha & 0 \\ \nu & \beta & \gamma \end{pmatrix}, \tag{C.1}$$

where $\lambda\alpha\gamma = 1$. The equation for $\widetilde{e}$ is written as

$$\mathrm{d}\widetilde{e} = \widetilde{\omega}\widetilde{e}, \qquad \widetilde{\omega} = g\omega g^{-1} + \mathrm{d}g \cdot g^{-1}.$$

Now we assume that the curve is nondegenerate and that the projective length element is definable, i.e., the invariant P defined in (B.2) does not vanish. Then it is shown that there is a unique frame such that the coframe has the form

$$\Omega = \begin{pmatrix} 0 & \omega & 0 \\ -k_p\omega & 0 & \omega \\ -\omega & -k_p\omega & 0 \end{pmatrix}, \tag{C.2}$$

where ω is the projective length element. The scalar k_p thus defined is nothing but the projective curvature. It is known that a curve is critical relative to the length functional if and only if

$$k_p''' + 8k_pk_p' = 0.$$

We refer to [6] for the induction of this equation and to [23] for a discussion of this equation in a broader framework.

C.2 Projective invariants of space curves. In Appendix B.2, we introduced the differential equation (B.5), which describes any nondegenerate space curve in projective space: $t \mapsto x(t) \in \mathbb{P}^3$, and defined two invariant forms $\theta_3\,\mathrm{d}t^3$ and $\theta_4\,\mathrm{d}t^4$ in (B.6). We now assume $\theta_3 \neq 0$ and choose the parameter t so that $\theta_3 = 1$; namely, let t be a projective length parameter. Then the equation is rewritten as

$$x'''' + 6P_2x'' + 2(2 + 3P_2')x' + P_4x = 0, \tag{C.3}$$

which is called the *Halphen canonical form* for space curves, and the two scalars P_2 and P_4 (or θ_4 instead of P_4) are called the *projective curvatures*; we refer to [10], page 26. We here set

$$k = \frac{3}{5}P_2, \tag{C.4}$$

$$\theta = \theta_4, \tag{C.5}$$

and call them the *first projective curvature* and the *second projective curvature*, respectively.

With this preparation, Bol in [4], Section 39 gave a choice of frame by using the invariants θ_3 and θ_4. For the differential equation (B.5), it is given as $\{x, u, y, z\}$, where

$$u = x', \quad y = u' + 3kx, \quad z = y' + 4ku + 2\theta_3 x,$$

which defines the frame equation when $\theta_3 = 1$ as

$$\mathrm{d}\begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3k & 0 & 1 & 0 \\ -2 & -4k & 0 & 1 \\ -\theta & -2 & -3k & 0 \end{pmatrix} \mathrm{d}t \begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix}. \tag{C.6}$$

The form ω is called the projective length element and defines the projective length functional for space curves. Kimpara in [14] showed that the functional is extremal if and only if

$$\begin{cases} k''' + 16kk' - \frac{1}{2}\theta' = 0, \\ k'''' + 16kk'' + 16(k')^2 + 6k' - \frac{1}{2}\theta'' = 0. \end{cases}$$

From these equations, the following theorem holds.

Theorem 6.5 ([14], [3], pages 233–234)**.** *A space curve without inflection points is projective extremal relative to the length functional if and only if both curvatures are constant.*

The proof is given by the fact that the subtraction of the derivative of the first equation from the second equation of (C.7) yields $k' = 0$. From this is follows that $\theta' = 0$, which implies the result.

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