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THE FAN GRAPH IS DETERMINED BY ITS SIGNLESS LAPLACIAN SPECTRUM

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Abstract. Given a graph G, if there is no nonisomorphic graph H such that G and H have the same signless Laplacian spectra, then we say that G is Q-DS. In this paper we show that every fan graph F_n is Q-DS, where $F_n = K_1 \vee P_{n-1}$ and $n \geqslant 3$.

Keywords: signless Laplacian spectrum; join graph; graph determined by its spectrum

MSC 2010: 05C50, 15A18

1. Introduction

Throughout this paper, G is an undirected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $N(u)$ and $d(u)$ be the neighbor set and the degree of vertex u, respectively. In the sequel, we enumerate the degrees in nonincreasing order, i.e., $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$, where $d(v_i) = d_i$ for $i \in \{1, 2, \ldots, n\}$. Sometimes we write $d_i(G)$ and $d_G(u)$ in place of d_i and $d(u)$, respectively, in order to indicate the dependence on G. As usual, K_n, P_n and C_n denote the complete graph, path and cycle of order n, respectively, and $G_1 \vee G_2$ denotes the *join graph* of two vertex disjoint graphs G_1 and G_2. In other words, $G_1 \vee G_2$ is the graph having vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv\colon\ u \in V(G_1),\ v \in V(G_2)\}$.

Let $A(G)$ and $D(G)$, respectively, be the adjacency matrix and the diagonal matrix of G. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. Denote by $\Phi(G, x)$ the Q-characteristic polynomials of graph G.

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It is easy to see that $Q(G)$ is positive semidefinite [2] and hence its eigenvalues can be arranged as

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G) \geqslant 0.$$

If there is no confusion, sometimes we simply write $\mu_i(G)$ as μ_i. In the following, let $S_Q(G)$ denote the spectra, i.e., the eigenvalues of $Q(G)$.

Two graphs are said to be Q-cospectral or L-cospectral if they have the same signless Laplacian or Laplacian spectrum, respectively. A graph G is said to be Q-DS or L-DS if H Q-cospectral or L-cospectral to G implies that $H = G$, respectively.

Which graphs are determined by their spectra? This question was proposed by Dam and Haemers in [11], and has drawn much attention recently. The literature contains dozens of results on this topic. For details, we refer the readers to [5], [8], [10], [11], [12] and the references therein.

The fan graph is denoted by $F_n = K_1 \vee P_{n-1}$. In [10], it was proved that F_n is L-DS for any $n \geqslant 3$. In this note, we will show that:

Theorem 1.1. *For any $n \geqslant 3$, F_n is Q-DS.*

2. Some properties for the Q-cospectral graph with F_n

Consider two sequences of real numbers: $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n$ and $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_m$ with $m < n$. The latter sequence is said to *interlace* the former whenever $\alpha_i \geqslant \beta_i \geqslant \alpha_{n-m+i}$ for $i = 1, 2, \ldots, m$.

Lemma 2.1 ([7]). *Let A be a symmetric matrix. If B is a principal submatrix of A, then the eigenvalues of B interlace the eigenvalues of A.*

When M is a real symmetric matrix of order n, we use $\Theta_1(M) \geqslant \Theta_2(M) \geqslant \ldots \geqslant \Theta_n(M)$ to denote its eigenvalues.

Corollary 2.1. *Let G be a graph with n vertices. If the least degree vertex v_n and second minimum degree vertex v_{n-1} are not adjacent, then $\mu_{n-1} \leqslant d_{n-1}$.*

Proof. Since $v_n v_{n-1} \notin E(G)$, $Q(G)$ contains

$$B = \begin{pmatrix} d_n & 0 \\ 0 & d_{n-1} \end{pmatrix}$$

as a principal submatrix. By Lemma 2.1, $\mu_{n-1} \leqslant \Theta_1(B) = d_{n-1}$. □

Let uv be an edge of G. Let $m(v)$ denote the average of the degrees of the vertices being adjacent to v, i.e., $m(v) = \sum_{w \in N(v)} d(w)/d(v)$.

Lemma 2.2 ([3]). *If G is a connected graph with at least one edge, then*

$$\mu_1(G) \leqslant \max\{d(v) + m(v)\colon\ v \in V(G)\} \leqslant d_1(G) + d_2(G).$$

Lemma 2.3 ([4], [8]). *For any connected graph G with n vertices, $\mu_n < d_n$ and*

$$\mu_2 \geqslant \frac{1}{2}\big(d_1 + d_2 - \sqrt{(d_1 - d_2)^2 + 4}\big) \geqslant d_2 - 1.$$

Furthermore, if $d_3 = d_2 \leqslant d_1 - 2$, then $\mu_2 \geqslant d_2$.

Lemma 2.4 ([3]). *Let G be a graph with $U \subseteq V(G)$, where $U = \{u_1, u_2, \ldots, u_k\}$. If all vertices of U have the same set of neighbors, then G has at least $k-1$ signless Laplacian eigenvalues equal to $d_G(u_1)$.*

Lemma 2.5 ([6]). *If G_i is an r_i-regular graph on n_i vertices for $i \in \{1, 2\}$, then*

$$\Phi(G_1 \vee G_2, x) = \frac{\Phi(G_1, x - n_2)\Phi(G_2, x - n_1)}{(x - 2r_1 - n_2)(x - 2r_2 - n_1)} f(x),$$

where $f(x) = x^2 - (2(r_1 + r_2) + (n_1 + n_2))x + 2(2r_1r_2 + r_1n_1 + r_2n_2)$.

Lemma 2.6 ([6]). *If G_i is an r_i-regular graph on n_i vertices for $i \in \{1, 2, 3\}$, then*

$$\Phi(G_1 \vee (G_2 \cup G_3), x) = \frac{\Phi(G_1, x - n_2 - n_3)\Phi(G_2, x - n_1)\Phi(G_3, x - n_1)}{(x - 2r_1 - n_2 - n_3)(x - 2r_2 - n_1)(x - 2r_3 - n_1)} f(x),$$

where $g(x) = x^3 - (2(r_1+r_2+r_3)+2n_1+n_2+n_3)x^2 + \big((n_1+n_2+n_3)(n_1+2(r_2+r_3)) + 4(r_1(n_1+r_3)+r_2(r_1+r_3))\big)x - \big(2n_1(n_1r_1+n_2r_2+n_3r_3+2r_1(r_2+r_3)) + 4r_2r_3(2r_1+n_2+n_3)\big)$.

Let I_n be the identity matrix of order n.

Lemma 2.7. *If $S_Q(G) = S_Q(F_n)$ and $n \geqslant 5$, then $d_2(G) \leqslant 5$, $n-4 \leqslant d_1(G) \leqslant n-1$ and*

$$1 < \mu_{n-2}(G) \leqslant \mu_2(G) < 5 \leqslant n < \frac{1}{2}\big(n + 1 + \sqrt{n^2 - 2n + 9}\big) < \mu_1(G).$$

P r o o f. Note that $Q(F_n)$ contains $I_{n-1} + Q(P_{n-1})$ as its principal submatrix. By Lemma 2.1, we have $\mu_{n-2}(G) \geqslant 1 + \mu_{n-2}(P_{n-1}) > 1$ and $\mu_2(G) \leqslant \mu_1(I_{n-1} + Q(P_{n-1})) < 5$. Now, Lemma 2.3 implies that $d_2(G) - 1 \leqslant \mu_2(G) < 5$ and hence $d_2(G) \leqslant 5$.

Since each edge deletion from a connected graph G will strictly decrease the largest signless Laplacian eigenvalue (namely, $\mu_1(G)$), by Lemmas 2.5 and 2.6 we have

$$\mu_1(G) > \begin{cases} \mu_1\Big(K_1 \vee \Big(\dfrac{n-2}{2}K_2 \cup K_1\Big)\Big) & \text{when } n \text{ is even;} \\ \mu_1\Big(K_1 \vee \Big(\dfrac{n-1}{2}K_2\Big)\Big) & \text{when } n \text{ is odd.} \end{cases}$$

Note that

$$\begin{aligned} \mu_1\Big(K_1 \vee \Big(\frac{n-1}{2}K_2\Big)\Big) &= \frac{1}{2}\big(n + 2 + \sqrt{n^2 - 4n + 12}\big) \\ &> \frac{1}{2}\big(n + 1 + \sqrt{n^2 - 2n + 9}\big) = \mu_1\Big(K_1 \vee \Big(\frac{n-2}{2}K_2 \cup K_1\Big)\Big). \end{aligned}$$

Thus, we obtain the required inequality for $\mu_1(G)$.

Since $\mu_1(G) > n$ and $d_2(G) \leqslant 5$, we have $n-4 \leqslant d_1(G) \leqslant n-1$ by Lemma 2.2. $\square$

Lemma 2.8 ([2])**.** *If G is a graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} d_i = 2m, \quad \text{and} \quad \sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2.$$

If $S_Q(G) = S_Q(F_n)$, then $d_2 \leqslant 5$ and $d_1 \geqslant n-4$ by Lemma 2.7. Hereafter, we suppose that G has n_j vertices of degree j in $V(G) \setminus \{v_1\}$ for $j = 1, 2, \ldots, 5$. By Lemma 2.8 it follows that

$$\begin{cases} n_2 = \frac{1}{2}(n^2 - 9n + 12) - \frac{1}{2}d_1(d_1 - 7) - 3n_1 - n_5, \\ n_3 = -n^2 + 9n - 10 + d_1(d_1 - 6) + 3n_1 + 3n_5, \\ n_4 = \frac{1}{2}(n-1)(n-6) - \frac{1}{2}d_1(d_1 - 5) - n_1 - 3n_5. \end{cases} \tag{2.1}$$

Lemma 2.9. *If $S_Q(G) = S_Q(F_n)$, then G is connected with either $d_1(G) \geqslant n-2$ or $d_n(G) = 1$.*

P r o o f. From Lemma 2.7 it follows that $d_2(G) \leqslant 5$ and $n - 4 \leqslant d_1(G) \leqslant n - 1$. If $d_1(G) = n - 4$ and $d_n(G) \geqslant 2$, then $d_2(G) = 5$ by Lemmas 2.2 and 2.7. So, $n \geqslant 9$. In this case, $n_1 = 0$. From (2.1) we have $n_3 + n_4 = 15 - 2n < 0$, a contradiction. Thus,

$$d_1(G) = n - 4 \text{ implies that } d_n(G) = 1. \tag{2.2}$$

We first prove that G is connected. By contradiction, we assume that G is disconnected. Since F_n is nonbipartite, $\mu_n(G) = \mu_n(F_n) > 0$ (see [2], Proposition 2.1). Recall that $d_1(G) \geqslant n-4$. Thus, G contains a connected component with at least $n-3$ vertices. So, $G = G_1 \cup C_3$, where $d_1(G_1) = n-4$. Since $S_Q(G) = S_Q(F_n)$ and $S_Q(C_3) = \{4, 1, 1\}$,

$$d_{n-3}(G_1) > \mu_{n-3}(G_1) \geqslant \mu_{n-2}(G) > 1$$

by Lemmas 2.3 and 2.7. Now, we have $d_n(G) = 2$, contradicting (2.2).

We now show that either $d_1(G) \geqslant n-2$ or $d_n(G) = 1$. By contradiction, from (2.2) we assume that $d_n(G) \geqslant 2$ and $d_1(G) = n-3$. In this case, $n_1 = 0$ and $d_2(G) \geqslant 4$ by Lemmas 2.2 and 2.7. From (2.2) it follows that

$$n_3 + n_4 = 8 - n, \tag{2.3}$$

which implies that $n \in \{7, 8\}$.

If $n = 8$, then $n_3 = n_4 = 0$ by (2.3). Recall that $n_1 = 0$. Thus, $0 = n_3 = -7 + 3n_5$ by (2.1), a contradiction. Otherwise, $n = 7$. Now, (2.3) implies that $n_3 + n_4 = 1$, and hence $n_4 = 5 - 3n_5 \in \{0, 1\}$ by (2.1), a contradiction. □

Lemma 2.10. *If $d_1(G) \leqslant n-3$, then G and F_n are not Q-cospectral.*

Proof. By contradiction, we assume that $S_Q(G) = S_Q(F_n)$, and hence Lemmas 2.7 and 2.9 imply that G is connected with $d_2(G) \leqslant 5$ and $n-4 \leqslant d_1(G) \leqslant n-3$.

Case 1. $d_1(G) = n-4$. Suppose $\max\{d(v) + m(v)\colon v \in V(G)\}$ occurs at the vertex u_0. If $d(u_0) \leqslant 4$, by Lemmas 2.2 and 2.7 we have

$$\mu_1(G) \leqslant d(u_0) + m(u_0) \leqslant d(u_0) + d_1(G) = d(u_0) + n - 4 \leqslant n < \mu_1(F_n),$$

a contradiction.

If $5 \leqslant d(u_0) \leqslant n-4$, then

$$\begin{aligned} \mu_1(G) &\leqslant d(u_0) + m(u_0) \leqslant d(u_0) + \frac{2|E(G)| - d(u_0) - (n-1-d(u_0))d_n(G)}{d(u_0)} \\ &\leqslant d(u_0) + \frac{2(2n-3) - d(u_0) - (n-1-d(u_0))}{d(u_0)} = d(u_0) + \frac{3n-5}{d(u_0)}. \end{aligned} \tag{2.4}$$

When $n = 9$, by (2.4) we have $d(u_0) + m(u_0) \leqslant 9.4 < 9.6 < \mu_1(F_9)$, a contradiction.

When $n \geqslant 10$, it is easy to see that

$$d(u_0) + m(u_0) \leqslant \max\Big\{5 + \frac{3n-5}{5}, n - 4 + \frac{3n-5}{n-4}\Big\} < \frac{1}{2}\big(n + 1 + \sqrt{n^2 - 2n + 9}\big),$$

against Lemmas 2.2 and 2.7.

Case 2. $d_1(G) = n - 3$. We consider the following three subcases:

Subcase 2.1. $n = 7$. In this case, $d_1(G) = 4$ and hence $d_2(G) \leqslant 4$ and $n_5 = 0$ by Lemma 2.7. Now, (2.1) implies that $n_2 = 5 - 3n_1 \geqslant 0$ and $n_3 = 3n_1 - 4 \geqslant 0$, a contradiction.

Subcase 2.2. $n = 8$. In this case, by (2.1) it follows that $n_2 = 7 - 3n_1 - n_5$, $n_3 = 3n_1 + 3n_5 - 7$ and $n_4 = 7 - n_1 - 3n_5$. Thus, either $n_5 = n_2 = n_3 = 2$ and $n_1 = 1$, or $n_5 = 1$ and $n_4 = n_3 = n_1 = 2$. If $n_5 = 1$ and $n_4 = n_3 = n_1 = 2$, then Corollary 2.1 implies that $\mu_7(G) \leqslant 1 < 1.19 < \mu_7(F_8)$, a contradiction. Otherwise, $n_5 = n_2 = n_3 = 2$ and $n_1 = 1$.

We assume that there exist two vertices of degree five being not adjacent with each other, then by Lemma 2.1 we obtain $\mu_2(G) \geqslant 5 > \mu_2(F_8)$, a contradiction. Thus, every pair of vertices of degree five are adjacent. In this case, $Q(G)$ contains

$$B_1 = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

as a principal submatrix. By Lemma 2.1, $\mu_3(G) \geqslant \Theta_3(B_1) = 4 > \mu_3(F_8)$, a contradiction.

Subcase 2.3. $n \geqslant 9$. We first suppose that $0 \leqslant n_5 \leqslant 1$. By (2.1) we have $n_2 + n_3 = 8 - n + 2n_5$. Thus, $n_5 = 1$ and $9 \leqslant n \leqslant 10$. No matter if $n = 9$ or $n = 10$, it will yield a contradiction by (2.1).

Next we suppose that $n_5 \geqslant 2$. When $n \geqslant 10$, by Lemma 2.3 we have $\mu_2(G) \geqslant 5$, against Lemma 2.7. When $n = 9$, by (2.1) we have $n_3 + n_4 = 2n_1 - 1$, and hence $n_1 \geqslant 1$. Now, Lemma 2.3 implies that $\mu_9(G) < 1 = \mu_9(F_9)$, a contradiction. □

Lemma 2.11. *If $d_1(G) = n - 2$ and $n \geqslant 7$, then G and F_n are not Q-cospectral.*

P r o o f. By contradiction, we assume that $S_Q(G) = S_Q(F_n)$. By Lemma 2.9, G is connected. If $n_1 \geqslant 4$, then v_1 is adjacent to at least three vertices of degree one, as $d(v_1) = d_1(G) = n - 2$. By Lemmas 2.3 and 2.4 we have $\mu_{n-2}(G) \leqslant 1$, which contradicts Lemma 2.7. Thus, $0 \leqslant n_1 \leqslant 3$.

When $n = 7$, since $\mu_7(F_7) = 1$, we have $n_1 = 0$ by Lemma 2.3. By (2.1), it follows that $n_5 = 1$, $n_2 = 3$, and $n_3 = 2$. There are exactly five connected graphs of order 7 with $n_2 = 3$, $n_3 = 2$ and $n_5 = 1$ and $d_1(G) = 5$ (see [1], pages 217–223). It can be easily checked that none of them is Q-cospectral with F_7, a contradiction.

When $n = 8$, if $n_5 \geqslant 1$, since $d_1 = 6$ and $d_2 = 5$, by Lemma 2.3 we have $\mu_2(G) > 4.38 > \mu_2(F_8)$, a contradiction. Otherwise, $n_5 = 0$. Now, (2.1) implies that

$n_3 = n_1 = 1$, $n_2 = 2$, and $n_4 = 3$. If the vertex of degree 6 is adjacent to at most two vertices of degree four, then by Lemma 2.2 we obtain

$$\begin{aligned}\mu(G) &\leqslant \max\left\{6 + \frac{(4+2)\times 2+3+1}{6}, 4 + \frac{6+4\times 2+3}{4}, 3 + \frac{6+4\times 2}{3}, 2 + \frac{6+4}{2}\right\}\\ &< 8.67 < \mu(F_8),\end{aligned}$$

a contradiction. Otherwise, the vertex of degree 6 is adjacent to three vertices of degree four, and hence $Q(G)$ contains B_1, B_2 or B_3 as a principle submatrix, where

$$B_1 = \begin{pmatrix} 6 & 1 & 1 & 1\\ 1 & 4 & 0 & 0\\ 1 & 0 & 4 & 0\\ 1 & 0 & 0 & 4\end{pmatrix}, \quad B_2 = \begin{pmatrix} 6 & 1 & 1 & 1\\ 1 & 4 & 1 & 0\\ 1 & 1 & 4 & 0\\ 1 & 0 & 0 & 4\end{pmatrix},$$
$$B_3 = \begin{pmatrix} 6 & 1 & 1 & 1\\ 1 & 4 & 1 & 1\\ 1 & 1 & 4 & 0\\ 1 & 1 & 0 & 4\end{pmatrix}, \quad B_4 = \begin{pmatrix} 6 & 1 & 1 & 1\\ 1 & 4 & 1 & 1\\ 1 & 1 & 4 & 1\\ 1 & 1 & 1 & 4\end{pmatrix}.$$

Note that $\Theta_3(B_1) = \Theta_3(B_3) = \Theta_3(B_4) = 4 > \mu_3(F_8)$ and $\Theta_2(B_2) > 4.46 > \mu_2(F_8)$. By Lemma 2.1, we get a contradiction. So, we may suppose that $n \geqslant 9$ in the following.

If $n_5 \geqslant 2$, since $n \geqslant 9$, $d_1(G) \geqslant 7$. By Lemma 2.3 we have $\mu_2(G) \geqslant 5$, contradicting Lemma 2.7. Otherwise, we may suppose that $0 \leqslant n_5 \leqslant 1$ in what follows.

By (2.1), it follows that

$$\text{(2.5)}\quad n_2 = n - 3 - n_5 - 3n_1, \quad n_3 = 6 - n + 3n_5 + 3n_1, \quad n_4 = n - 4 - 3n_5 - n_1.$$

Since $0 \leqslant n_1 \leqslant 3$, $0 \leqslant n_5 \leqslant 1$ and $n_3 \geqslant 0$, we have $9 \leqslant n \leqslant 18$ by (2.5).

Case 1. n is odd. In this case, $n \in \{9, 11, 13, 15, 17\}$. Computer aided calculations show that $\mu_n(F_n) = 1$ and hence $n_1 = 0$ by Lemma 2.3. Now, (2.5) implies that $n = 9$ with $n_2 = 5$, $n_4 = 2$ and $n_5 = 1$. By Lemma 2.3, $\mu_2(G) > 4.58 > \mu_2(F_9)$, a contradiction.

Case 2. n is even. In this case, $n \in \{10, 12, 14, 16, 18\}$. Computer aided calculations show that $\mu_{n-1}(F_n) > 1$. Thus, by Corollary 2.1, we can conclude that $0 \leqslant n_1 \leqslant 1$. If $n_5 = 0$, then $n_3 = 6 - n + 3n_1 \geqslant 0$ by (2.5), against $0 \leqslant n_1 \leqslant 1$ and $n \geqslant 10$. Otherwise, $n_5 = 1$. Since $n_3 = 9 - n + 3n_1 \geqslant 0$ and $0 \leqslant n_1 \leqslant 1$, we have $n \in \{10, 12\}$. When $n \in \{10, 12\}$, since $d_1 \geqslant 8$ and $d_2 = 5$, by Lemma 2.3 we have $\mu_2(G) > 4.69 > \mu_2(F_n)$, a contradiction. □

Theorem 2.1. *If $S_Q(G) = S_Q(F_n)$ and $n \geqslant 7$, then G and F_n have the same degree sequence.*

P r o o f. Since $S_Q(G) = S_Q(F_n)$, by Lemmas 2.7, 2.9–2.11, G is connected with $d_2(G) \leqslant 5$ and $d_1(G) = n-1$. When $n = 7$, if $n_5 \geqslant 1$, since $d_1 = 6$ and $d_2 = 5$, by Lemma 2.3 we have $\mu_2(G) > 4.38 > \mu_2(F_7)$, a contradiction. Thus, $n_5 = 0$ and so G and F_7 have the same degree sequence by (2.1).

Now, we consider the case of $n \geqslant 8$. By Lemmas 2.3 and 2.7, we have $0 \leqslant n_5 \leqslant 1$. From (2.1) it follows that $n_4 + n_1 + 3n_5 = 0$, and hence G and F_n share the same degree sequence. □

Corollary 2.2. *If $S_Q(G) = S_Q(F_n)$ and $n \geqslant 7$, then either $G \cong K_1 \vee (C_b \cup P_{n-1-b})$ or $G \cong F_n$, where $3 \leqslant b \leqslant n-3$.*

P r o o f. From Theorem 2.1 and Lemma 2.9, if G is Q-cospectral with F_n, then G is connected and so $G \cong K_1 \vee (C_{k_1} \cup C_{k_2} \cup \ldots \cup C_{k_t} \cup P_a)$, where $k_1 + k_2 + \ldots + k_t = n - 1 - a$ and $a \geqslant 2$.

If $t \geqslant 2$, since $Q(G)$ contains $I_{n-1} + Q(C_{k_1} \cup C_{k_2} \cup \ldots \cup C_{k_t} \cup P_a)$ as its principal submatrix, by Lemma 2.1 we have $\mu_2(G) \geqslant 1 + \mu_2(C_{k_1} \cup C_{k_2} \cup \ldots \cup C_{k_t} \cup P_a) = 5$, which contradicts Lemma 2.7. Thus, $0 \leqslant t \leqslant 1$ and hence the result follows. □

3. The proof of Theorem 1.1

Let $\Theta(G)$ be the largest eigenvalue of $A(G) + \alpha D(G)$, where $\alpha \geqslant 0$. To complete the proof of Theorem 1.1, it suffices to show that $K_1 \vee (C_b \cup P_{n-1-b})$ and F_n are not Q-cospectral by Corollary 2.2 for $n \geqslant 7$ (since the case of $3 \leqslant n \leqslant 6$ can be checked easily). In what follows, we will prove the following more general result:

Theorem 3.1. *For any $\alpha \geqslant 0$ and $3 \leqslant b \leqslant n-3$, we have*

$$\Theta(K_1 \vee (C_b \cup P_{n-1-b})) > \Theta(F_n).$$

To prove Theorem 3.1, we need the following famous property of $\Theta(G)$.

Lemma 3.1 (See [9], page 18). *Let G be a connected graph. If $\varphi = (\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_n))^\top$ is a unit vector defined on $V(G)$, then*

$$\Theta(G) \geqslant \varphi^\top (A(G) + \alpha D(G))\varphi = 2 \sum_{uv \in E(G)} \varphi(u)\varphi(v) + \alpha \sum_{i=1}^{n} d(v_i)\varphi^2(v_i),$$

where the equality holds if and only if φ is an eigenvector corresponding to $\Theta(G)$.

When $\alpha \geqslant 0$ and G is connected, there is a unique positive unit eigenvector corresponding to $\Theta(G)$ (see [9], page 21), and we call such an eigenvector the *Perron vector* of G hereafter. In what follows, let $V(F_n) = \{w_1, w_2, \ldots, w_n\}$ with $d(w_n) = n-1$ and $w_j w_{j+1} \in E(F_n)$ for $1 \leqslant j \leqslant n-2$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ be the Perron vector of F_n such that x_i corresponds to the vertex w_i, where $1 \leqslant i \leqslant n$. We call $w_p w_{p+1}$ a *special edge* if $p \leqslant \lfloor \frac{1}{2}(n-3) \rfloor$ and $x_p = x_{p+1}$, and we call w_p a *special vertex* if $p \leqslant \lfloor \frac{1}{2}(n-1) \rfloor$ and $x_p = x_{p-2}$.

Lemma 3.2. *If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ is the Perron vector of F_n and $n \geqslant 3$, then $x_j = x_{n-j}$ holds for any $1 \leqslant j \leqslant \lfloor \frac{1}{2}(n-1) \rfloor$ and F_n contains neither a special edge nor a special vertex.*

P r o o f. In the proof of this result, we simplify $\Theta(F_n)$ as Θ. Let P be the permutation matrix that reverses the order of the vertices in the sequence $w_1, w_2, \ldots, w_{n-1}$, where the vertex w_n is fixed. Since $\mathbf{x}$ is an eigenvector for the eigenvalue Θ, so is $P\mathbf{x}$. Note that $\mathbf{x}$ is the unique positive unit vector corresponding to Θ and $P\mathbf{x}$ is also a positive unit vector corresponding to Θ. Thus, $P\mathbf{x} = \mathbf{x}$ and hence

$$x_j = x_{n-j} \text{ holds for any } 1 \leqslant j \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{3.1}$$

Now, from $(A(F_n) + \alpha D(F_n))\mathbf{x} = \Theta\mathbf{x}$ we have

$$\begin{cases} \Theta x_1 = 2\alpha x_1 + x_2 + x_n, \\ \Theta x_i = 3\alpha x_i + x_{i-1} + x_{i+1} + x_n \quad \text{for } i \in \left\{2, 3, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor\right\}. \end{cases} \tag{3.2}$$

From Lemma 3.1 we easily get $\Theta(G) \geqslant \alpha d(w_n) = \alpha(n-1)$ by setting $\varphi(w_n) = 1$ and $\varphi(w_j) = 0$ for $1 \leqslant j \leqslant n-1$. Now, by (3.2), it follows that $(\Theta + 1 - 2\alpha)(x_2 - x_1) = \alpha x_2 + x_3 > 0$, and hence $x_2 > x_1$.

First, we assume that $w_p w_{p+1}$ is a special edge of F_n, then $x_p = x_{p+1}$ and $p \leqslant \lfloor \frac{1}{2}(n-3) \rfloor$. In this case, we have $p \geqslant 2$ by $x_2 > x_1$. Let $G_1 = F_n + w_{p-1}w_{n-p-1} + w_p w_{n-p} - w_{p-1}w_p - w_{n-p-1}w_{n-p}$. Then, $G_1 \cong F_n$. By (3.1) and Lemma 3.1,

$$0 = \Theta(G_1) - \Theta(F_n) \geqslant 2(x_{p-1} - x_{n-p})(x_{n-p-1} - x_p) = 0,$$

and hence $\mathbf{x}$ is also a Perron vector of G_1. In this case, $x_{p-1} = x_{n-p} = x_p = x_{p+1}$. Now, let $G_2 = F_n + w_{p-2}w_{n-p-1} + w_{p-1}w_{n-p} - w_{p-2}w_{p-1} - w_{n-p-1}w_{n-p}$. Then, $G_2 \cong F_n$. Similarly, we have $x_{p-2} = x_{p-1} = x_p = x_{p+1}$ (since $x_{p-1} = x_{p+1} = x_{n-1-p}$). By repeating the above process, we have $x_1 = x_2$, a contradiction. Thus, we can conclude that F_n contains no special edge.

Secondly, we assume that w_p is a special vertex of F_n, then $x_p = x_{p-2} = x_{n-p} = x_{n+2-p}$ and $x_{p-1} = x_{n+1-p}$, where $p \leqslant \lfloor \frac{1}{2}(n-1) \rfloor$. Since $p \leqslant \lfloor \frac{1}{2}(n-1) \rfloor$, $w_p \neq w_{n-p}$. Let $G_3 = F_n + w_{p-1}w_{n+1-p} + w_{n-p}w_{p-2} - w_{p-1}w_{p-2} - w_{n+1-p}w_{n-p}$. Then, $G_3 \cong F_n$. By Lemma 3.1,

$$0 = \Theta(G_3) - \Theta(F_n) \geqslant 2(x_{p-1} - x_{p-2})^2 \geqslant 0,$$

and hence $x_{p-1} = x_{p-2}$. Now, $w_{p-1}w_{p-2}$ is a special edge of F_n, a contradiction. $\square$

P r o o f of Theorem 3.1. In the proof of this result, we rewrite $K_1 \vee (C_b \cup P_{n-b-1})$ as G. Without loss of generality, we suppose that n is even, as the case of n being odd can be dealt with by a similar method.

Let $n-1 = 2k+1$, where k is a positive integer. By Lemma 3.2, $x_{k+1-j} = x_{k+j+1}$ holds for any $j \in \{1, 2, \ldots, k\}$. If $b = 2s+1$ (s is a positive integer), then $k \geqslant s+1$ due to $n - b \geqslant 3$. It is easy to see that $G \cong F_n + w_{k+1-s}w_{k+s+1} + w_{k-s}w_{k+s+2} - w_{k+1-s}w_{k-s} - w_{k+s+1}w_{k+s+2}$. By Lemma 3.2, $x_{k+1-s} \neq x_{k-s}$. Now, $\Theta(G) - \Theta(F_n) \geqslant \mathbf{x}^\top(A(G) + \alpha D(G))\mathbf{x} - \mathbf{x}^\top(A(F_n) + \alpha D(F_n))\mathbf{x} = 2(x_{k+1-s} - x_{k-s})^2 > 0$ by Lemma 3.1.

Otherwise, $b = 2s$. In this case, since $n - b \geqslant 3$, we have $k \geqslant s+1$ as both k and s are positive integers. Thus it is easy to see that $G \cong F_n + w_{k+1-s}w_{k+s} + w_{k-s}w_{k+s+1} - w_{k+1-s}w_{k-s} - w_{k+s}w_{k+s+1}$. In this case, $\Theta(G) - \Theta(F_n) \geqslant \mathbf{x}^\top(A(G) + \alpha D(G))\mathbf{x} - \mathbf{x}^\top(A(F_n) + \alpha D(F_n))\mathbf{x} = 0$ by Lemma 3.1. From Lemma 3.1, if $\Theta(G) = \Theta(F_n)$, then $\mathbf{x}$ is also a Perron vector of $\Theta(G)$. Since $\Theta(G)x_{k+1-s} = \Theta(F_n)x_{k+1-s}$, we have $x_{k-s} = x_{k+s} = x_{k+2-s}$. In this case, w_{k+2-s} is a special vertex of F_n, against Lemma 3.2. This completes the proof of the theorem. $\square$

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References

[1] *D. M. Cvetković, M. Doob, I. Gutman, A. Torgašev*: Recent Results in the Theory of Graph Spectra. Annals of Discrete Mathematics 36, North-Holland, Amsterdam, 1988. zbl MR doi

[2] *D. Cvetković, P. Rowlinson, S. K. Simić*: Signless Laplacians of finite graphs. Linear Algebra Appl. *423* (2007), 155–171. zbl MR doi

[3] *K. Ch. Das*: The Laplacian spectrum of a graph. Comput. Math. Appl. *48* (2004), 715–724. zbl MR doi

[4] *K. Ch. Das*: On conjectures involving second largest signless Laplacian eigenvalue of graphs. Linear Algebra Appl. *432* (2010), 3018–3029. zbl MR doi

[5] *K. Ch. Das, M. Liu*: Complete split graph determined by its (signless) Laplacian spectrum. Discrete Appl. Math. *205* (2016), 45–51. zbl MR doi

[6] *M. A. A. de Freitas, N. M. M. de Abreu, R. R. Del-Vecchio, S. Jurkiewicz*: Infinite families of Q-integral graphs. Linear Algebra Appl. *432* (2010), 2352–2360. zbl MR doi

[7] *W. H. Haemers*: Interlacing eigenvalues and graphs. Linear Algebra Appl. *226–228* (1995), 593–616. zbl MR doi

[8] *M. Liu*: Some graphs determined by their (signless) Laplacian spectra. Czech. Math. J. *62* (2012), 1117–1134. zbl MR doi
[9] *M. Liu, B. Liu*: Extremal Theory of Graph Spectrum. Mathematical Chemistry Monographs 22, University of Kragujevac and Faculty of Science Kragujevac, Kragujevac, 2018.
[10] *X. Liu, Y. Zhang, X. Gui*: The multi-fan graphs are determined by their Laplacian spectra. Discrete Math. *308* (2008), 4267–4271. zbl MR doi
[11] *E. R. van Dam, W. H. Haemers*: Which graphs are determined by their spectrum? Linear Algebra Appl. *373* (2003), 241–272. zbl MR doi
[12] *J. Wang, H. Zhao, Q. Huang*: Spectral characterization of multicone graphs. Czech. Math. J. *62* (2012), 117–126. zbl MR doi

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