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MATLIS DUAL OF LOCAL COHOMOLOGY MODULES

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Abstract. Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module such that $\mathfrak{a}M \neq M$ and $\operatorname{cd}(\mathfrak{a},M) - \operatorname{grade}(\mathfrak{a},M) \leqslant 1$, where $\operatorname{cd}(\mathfrak{a},M)$ is the cohomological dimension of M with respect to $\mathfrak{a}$ and $\operatorname{grade}(\mathfrak{a},M)$ is the M-grade of $\mathfrak{a}$. Let $D(-) := \operatorname{Hom}_R(-,E)$ be the Matlis dual functor, where $E := E(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. We show that there exists the following long exact sequence

$$\begin{aligned}0 &\longrightarrow H^{n-2}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \longrightarrow H^{n}_{\mathfrak{a}}(D(H^{n}_{\mathfrak{a}}(M))) \longrightarrow D(M)\\ &\longrightarrow H^{n-1}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \longrightarrow H^{n+1}_{\mathfrak{a}}(D(H^{n}_{\mathfrak{a}}(M)))\\ &\longrightarrow H^{n}_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))) \longrightarrow H^{n}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \longrightarrow \ldots,\end{aligned}$$

where $n := \operatorname{cd}(\mathfrak{a},M)$ is a non-negative integer, $x_1,\ldots,x_{n-1}$ is a regular sequence in $\mathfrak{a}$ on M and, for an R-module L, $H^i_{\mathfrak{a}}(L)$ is the ith local cohomology module of L with respect to $\mathfrak{a}$.

Keywords: local cohomology module; Matlis dual functor, filter regular sequence

MSC 2010: 13D45, 13D07

1. Introduction

Throughout the paper, $(R,\mathfrak{m})$ will denote a commutative Noetherian ring with nonzero identity, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module. We shall use $D(-)$ to denote the Matlis dual functor; thus $D(-) := \operatorname{Hom}_R(-,E)$, where $E := E(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. Also, we use $\mathbb{N}_0$ (or $\mathbb{N}$) to denote the set of non-negative (or positive) integers. Our terminology follows the textbook [1] on local cohomology.

There are some problems related to $D(H^i_{\mathfrak{a}}(M))$ (see for example conjecture $(*)$ in [2] and [3]), where $H^i_{\mathfrak{a}}(M)$ is the ith local cohomology module of M with respect to $\mathfrak{a}$. Recently, such modules have been studied by some authors, such as

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Hellus in [2], [3], [4], Hellus and Schenzel in [5], Khashyarmanesh in [7] and Schenzel in [10], and has led to some interesting results. In this direction, an interesting question is whether the module $D(H^i_{\mathfrak{a}}(R))$ is 'small' in the sense that, in certain cases, $H^i_{\mathfrak{a}}(D(H^i_{\mathfrak{a}}(R)))$ is either E or zero. By using the theory of D-modules, in certain situations, it was shown that $H^i_{\mathfrak{a}}(D(H^i_{\mathfrak{a}}(R)))$ is either E or zero (cf. [4], Theorems 3.1 and 3.2). The second author in [7] proved that for a non-negative integer n and an ideal $\mathfrak{a}$ of R with $\mathfrak{a}M \neq M$, if $H^i_{\mathfrak{a}}(M) = 0$ for every $i \neq n$, then $H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(M))) = D(M)$. In the above-mentioned results, the ideal $\mathfrak{a}$ satisfies the equality $\operatorname{grade}(\mathfrak{a}, R) = \operatorname{cd}(\mathfrak{a}, R)$ in [4] or $\operatorname{grade}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$ in [7], where $\operatorname{grade}(\mathfrak{a}, M)$ is the common length of maximal regular sequences in $\mathfrak{a}$ on M and $\operatorname{cd}(\mathfrak{a}, M)$ is the cohomological dimension of M with respect to $\mathfrak{a}$. Clearly, $\operatorname{cd}(\mathfrak{a}, M) \geqslant \operatorname{grade}(\mathfrak{a}, M)$. So the interesting question related to this context is how to determine a relation between the R-modules $H^i_{\mathfrak{a}}(D(H^i_{\mathfrak{a}}(M)))$ and $D(M)$ in the case that $\operatorname{cd}(\mathfrak{a}, M) \neq \operatorname{grade}(\mathfrak{a}, M)$. In this paper, we assume $\operatorname{cd}(\mathfrak{a}, M) - \operatorname{grade}(\mathfrak{a}, M) \leqslant 1$. In fact, we show that there exists an exact sequence involving the R-modules $H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(M)))$ and $D(M)$, where $n = \operatorname{cd}(\mathfrak{a}, M)$. Finally, as a consequence, we deduced the main result of [7].

2. Background

First of all, let us recall a generalization of the concept of regular sequences, which we shall use in the paper. Let $\mathfrak{a}$ be an ideal of R. We say that a sequence $x_1, \ldots, x_n$ of elements of $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M if

$$\operatorname{Supp}_R(((x_1, \ldots, x_{i-1})M :_M x_i)/(x_1, \ldots, x_{i-1})M) \subseteq V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the filter regular sequence which has been studied in [11], [12] and has led to some interesting results (see also [6], [9]). Note that both concepts coincide if $\mathfrak{a}$ is the maximal ideal in the local ring. Also note that $x_1, \ldots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of Appendix 2 (ii) of [12] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$; so if $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $x_{n+1} \in \mathfrak{a}$ such that $x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for a positive integer n there exists an $\mathfrak{a}$-filter regular sequence on M of length n. The following theorem shows that filter regular sequences provide a nice method for studying local cohomology modules.

Proposition 2.1 (See [8], Proposition 1.2). *Let* $x_1, \dots, x_n$ $(n \geqslant 1)$ *be an* $\mathfrak{a}$*-filter regular sequence on* M*. Then for all integers* i *with* $0 \leqslant i \leqslant n-1$*, we have the isomorphism*

$$H^i_{\mathfrak{a}}(M) \cong H^i_{(x_1,\dots,x_n)}(M).$$

For an R-module N, the cohomological dimension of N with respect to $\mathfrak{a}$ is defined as

$$\operatorname{cd}(\mathfrak{a}, N) = \max\{i \in \mathbb{Z}\colon\ H^i_{\mathfrak{a}}(N) \neq 0\}.$$

Finally, for the convenience of the reader, we recall the following proposition which we shall use in the paper.

Proposition 2.2 (See [7], Proposition 2.4). *Suppose that* $(R, \mathfrak{m})$ *is a local ring, and let* j *be an integer such that* $j > \operatorname{cd}(\mathfrak{a}, M) \geqslant 0$*. Then for an* $\mathfrak{a}$*-filter regular sequence* $x_1, \dots, x_j$ *on* M *we have that* $H^n_{\mathfrak{a}}(D(H^j_{(x_1,\dots,x_j)}(M))) = 0$ *for all* $n \in \mathbb{N}_0$*.*

3. Main results

We begin with the following lemma.

Lemma 3.1. *Suppose that* $(R, \mathfrak{m})$ *is a local ring, and let* $n := \operatorname{cd}(\mathfrak{a}, M)$*. Let* $x_1, \dots, x_n$ *be an* $\mathfrak{a}$*-filter regular sequence on* M*. Then*

$$H^n_{\mathfrak{a}}(D(H^n_{(x_1,\dots,x_n)}(M))) \cong H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(M))).$$

P r o o f. Since $x_1, \dots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, there exists $x_{n+1} \in \mathfrak{a}$ such that $x_1, \dots, x_n, x_{n+1}$ form an $\mathfrak{a}$-filter regular sequence on M. (Note that the existence of such an element is explained in the beginning of the previous section.) Thus, by [7], Lemma 2.2, there exists an exact sequence

$$\begin{aligned} 0 \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{(x_1,\dots,x_n)}(M) \longrightarrow (H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}} \\ \longrightarrow H^{n+1}_{(x_1,\dots,x_{n+1})}(M) \longrightarrow 0, \end{aligned}$$

where for an R-module N, $N_{x_{n+1}}$ denotes the module of fractions of N with respect to the multiplicatively closed subset $\{x^i_{n+1}\colon\ i \in \mathbb{N}_0\}$. Now, by applying the Matlis dual functor $D(-)$ to it, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow D(H^{n+1}_{(x_1,\dots,x_{n+1})}(M)) \longrightarrow D((H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}}) \stackrel{f}{\longrightarrow} D(H^n_{(x_1,\dots,x_n)}(M)) \\ \longrightarrow D(H^n_{\mathfrak{a}}(M)) \longrightarrow 0. \end{aligned}$$

So, there are exact sequences

$$0 \longrightarrow D(H^{n+1}_{(x_1,\dots,x_{n+1})}(M)) \longrightarrow D((H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}}) \longrightarrow L \longrightarrow 0 \tag{3.1}$$

and

$$0 \longrightarrow L \longrightarrow D(H^n_{(x_1,\dots,x_n)}(M)) \longrightarrow D(H^n_{\mathfrak{a}}(M)) \longrightarrow 0, \tag{3.2}$$

where L is the image of f. On the other hand, since multiplication by x_{n+1} provides an automorphism on $(H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}}$, and also for arbitrary non-negative integer l, every element of $H^l_{\mathfrak{a}}(D((H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}}))$ is annihilated by some power of $\mathfrak{a}$, we conclude that

$$H^l_{\mathfrak{a}}(D((H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}})) = 0.$$

Then, sequence (3.1) in conjunction with Proposition 2.2 implies that $H^l_{\mathfrak{a}}(L) = 0$ for $l = n,\ n+1$. So the result now follows by applying the functor $\Gamma_{\mathfrak{a}}(-)$ on the short exact sequence (3.2). □

The next proposition is concerned with the concept of grade of an ideal. As we mentioned in Introduction, for an ideal $\mathfrak{a}$ of R with $\mathfrak{a}M \neq M$ we refer to the common length of all maximal regular sequence on M contained in $\mathfrak{a}$ as the M-grade of $\mathfrak{a}$ and we denote this non-negative integer by $\operatorname{grade}(\mathfrak{a}, M)$.

Proposition 3.2. *Suppose that $\mathfrak{a}$ is a proper ideal of local ring $(R, \mathfrak{m})$ such that* $\operatorname{grade}(\mathfrak{a}, M) \geqslant n-1$, *and let $x_1, \dots, x_{n-1}$ be a regular sequence in $\mathfrak{a}$ on M. Then we have the following statements:*

(i) $H^{n-2}_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\dots,x_{n-1})}(M))) = 0$;

(ii) $H^{n-1}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \cong D(M)$.

P r o o f. In view of [7], Lemma 2.2, there exists an exact sequence

$$\begin{aligned} 0 \longrightarrow H^{n-2}_{(x_1,\dots,x_{n-2})}(M) &\longrightarrow (H^{n-2}_{(x_1,\dots,x_{n-2})}(M))_{x_{n-1}} \\ &\longrightarrow H^{n-1}_{(x_1,\dots,x_{n-1})}(M) \longrightarrow 0. \end{aligned} \tag{3.3}$$

Since multiplication by x_{n-1} provides an automorphism on $(H^{n-2}_{(x_1,\dots,x_{n-2})}(M))_{x_{n-1}}$, and every element of $H^l_{\mathfrak{a}}(D((H^{n-2}_{(x_1,\dots,x_{n-2})}(M))_{x_{n-1}}))$ is annihilated by some power of $\mathfrak{a}$, we have that

$$H^l_{\mathfrak{a}}(D((H^{n-2}_{(x_1,\dots,x_{n-2})}(M))_{x_{n-1}})) = 0 \quad \forall\, l \in \mathbb{N}_0. \tag{3.4}$$

By applying the functor $D(-)$ to (3.3) in conjunction with (3.4), we have that

$$(3.5)\qquad H^l_{\mathfrak{a}}(D(H^{n-2}_{(x_1,\dots,x_{n-2})}(M))) \cong H^{l+1}_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\dots,x_{n-1})}(M))) \quad \forall\, l \in \mathbb{N}_0.$$

(i) By applying the telescoping method on (3.5), we have the following isomorphisms:

$$(3.6)\qquad \begin{aligned} H^{n-2}_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\dots,x_{n-1})}(M))) &\cong H^{n-3}_{\mathfrak{a}}(D(H^{n-2}_{(x_1,\dots,x_{n-2})}(M))) \\ &\cong \dots \\ &\cong H^1_{\mathfrak{a}}(D(H^2_{(x_1,x_2)}(M))) \\ &\cong \Gamma_{\mathfrak{a}}(D(H^1_{(x_1)}(M))). \end{aligned}$$

On the other hand, since x_1 is a nonzerodivisor on M, we have the exact sequence

$$0 \longrightarrow M \longrightarrow M_{x_1} \longrightarrow H^1_{(x_1)}(M) \longrightarrow 0,$$

which implies that the sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(D(H^1_{(x_1)}(M))) \longrightarrow \Gamma_{\mathfrak{a}}(D(M_{x_1})) \longrightarrow \Gamma_{\mathfrak{a}}(D(M))$$

is exact. Again $\Gamma_{\mathfrak{a}}(D(M_{x_1})) = 0$. Thus $\Gamma_{\mathfrak{a}}(D(H^1_{(x_1)}(M))) = 0$. The result now follows from (3.6).

(ii) Again the telescoping method on (3.5) shows that

$$(3.7)\qquad \begin{aligned} H^{n-1}_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\dots,x_{n-1})}(M))) &\cong H^{n-2}_{\mathfrak{a}}(D(H^{n-2}_{(x_1,\dots,x_{n-2})}(M))) \\ &\cong \dots \\ &\cong H^2_{\mathfrak{a}}(D(H^2_{(x_1,x_2)}(M))) \\ &\cong H^1_{\mathfrak{a}}(D(H^1_{(x_1)}(M)). \end{aligned}$$

Moreover, since x_1 is a nonzerodivisor, we have the exact sequence

$$0 \longrightarrow M \longrightarrow M_{x_1} \longrightarrow H^1_{(x_1)}(M) \longrightarrow 0.$$

Hence $H^1_{\mathfrak{a}}(D(H^1_{(x_1)}(M))) \cong \Gamma_{\mathfrak{a}}(D(M))$, and so by (3.7), we have that

$$H^{n-1}_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\dots,x_{n-1})}(M))) \cong \Gamma_{\mathfrak{a}}(D(M)).$$

Also, since M is finitely generated and every element of E is annihilated by some power of $\mathfrak{m}$, it is easy to see that $\Gamma_{\mathfrak{a}}(D(M)) \cong D(M)$. It follows that

$$H^{n-1}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \cong D(M),$$

as required. $\square$

Theorem 3.3. *Assume that R is a local ring. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}M \neq M$ and $\operatorname{cd}(\mathfrak{a}, M) - \operatorname{grade}(\mathfrak{a}, M) \leqslant 1$. Then there exists the following long exact sequence:*

$$\begin{aligned}0 &\longrightarrow H^{n-2}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \longrightarrow H^{n}_{\mathfrak{a}}(D(H^{n}_{\mathfrak{a}}(M))) \longrightarrow D(M)\\ &\longrightarrow H^{n-1}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \longrightarrow H^{n+1}_{\mathfrak{a}}(D(H^{n}_{\mathfrak{a}}(M))) \longrightarrow H^{n}_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\ldots,x_{n-1})}(M)))\\ &\longrightarrow H^{n}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) \longrightarrow \ldots,\end{aligned}$$

where $n := \operatorname{cd}(\mathfrak{a}, M)$ is a non-negative integer and $x_1, \ldots, x_{n-1}$ is a regular sequence in $\mathfrak{a}$ on M. Furthermore, if $H^{n-1}_{\mathfrak{a}}(D(H^{n-1}_{\mathfrak{a}}(M))) = 0$, then $D(M)$ is a homomorphic image of $H^{n}_{\mathfrak{a}}(D(H^{n}_{\mathfrak{a}}(M)))$.

Proof. First of all, note that there exists a regular sequence $x_1, \ldots, x_{n-1}$ in $\mathfrak{a}$ on M, because $\operatorname{grade}(\mathfrak{a}, M) \geqslant n-1$. Also, there exists $x_n \in \mathfrak{a}$ such that $x_1, \ldots, x_{n-1}, x_n$ is an $\mathfrak{a}$-filter regular sequence on M. Hence, by [7], Lemma 2.2, there exists an exact sequence

$$\begin{aligned}0 &\longrightarrow H^{n-1}_{\mathfrak{a}}(M) \longrightarrow H^{n-1}_{(x_1,\ldots,x_{n-1})}(M) \longrightarrow (H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))_{x_n}\\ &\longrightarrow H^{n}_{(x_1,\ldots,x_n)}(M) \longrightarrow 0\end{aligned}$$

of local cohomology modules. So, we have the following exact sequence:

$$\begin{aligned}0 &\longrightarrow D(H^{n}_{(x_1,\ldots,x_n)}(M)) \longrightarrow D((H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))_{x_n}) \xrightarrow{g} D(H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))\\ &\longrightarrow D(H^{n-1}_{\mathfrak{a}}(M)) \longrightarrow 0.\end{aligned}$$

Now, by breaking the above exact sequence, we have the following short exact sequences:

$$0 \longrightarrow D(H^{n}_{(x_1,\ldots,x_n)}(M)) \longrightarrow D((H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))_{x_n}) \longrightarrow \operatorname{Im} g \longrightarrow 0 \tag{3.8}$$

and

$$0 \longrightarrow \operatorname{Im} g \longrightarrow D(H^{n-1}_{(x_1,\ldots,x_{n-1})}(M)) \longrightarrow D(H^{n-1}_{\mathfrak{a}}(M)) \longrightarrow 0. \tag{3.9}$$

Since multiplication by x_n provides an automorphism on $(H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))_{x_n}$, and every element of $H^{l}_{\mathfrak{a}}(D((H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))_{x_n}))$ is annihilated by some power of $\mathfrak{a}$, we have that

$$H^{l}_{\mathfrak{a}}(D((H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))_{x_n})) = 0 \quad \forall\, l \in \mathbb{N}_0.$$

Thus, by applying the functor $\Gamma_{\mathfrak{a}}(-)$ on (3.8) in conjunction with Lemma 3.1, we obtain the isomorphism $H^{n-1}_{\mathfrak{a}}(\operatorname{Im} g) \cong H^{n}_{\mathfrak{a}}(D(H^{n}_{\mathfrak{a}}(M)))$. Now, by applying the functor $\Gamma_{\mathfrak{a}}(-)$ on (3.9), the result follows from Proposition 3.2.

The final claim is then a consequence. □

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4 (Compare [7], Theorem 2.5). *Assume that R is a local ring. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}M \neq M$ and $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{grade}(\mathfrak{a}, M) = n$. Then there exist the following isomorphisms:*

(i) $H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(M))) \cong D(M)$,

(ii) $H^{n+i}_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(M))) \cong H^n_{\mathfrak{a}}(D(H^{n-1}_{(x_1,\ldots,x_{n-1})}(M)))$ *for every* $i \in \mathbb{N}$,

where $x_1, \ldots, x_{n-1}$ is a regular sequence in $\mathfrak{a}$ on M.

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