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A Deformed Quon Algebra

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Abstract. The quon algebra is an approach to particle statistics in order to provide a theory in which the Pauli exclusion principle and Bose statistics are violated by a small amount. The quons are particles whose annihilation and creation operators obey the quon algebra which interpolates between fermions and bosons. In this paper we generalize these models by introducing a deformation of the quon algebra generated by a collection of operators $a_{i,k}$, $(i,k) \in \mathbb{N}^* \times [m]$, on an infinite dimensional vector space satisfying the deformed q-mutator relations $a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j}$. We prove the realizability of our model by showing that, for suitable values of q, the vector space generated by the particle states obtained by applying combinations of $a_{i,k}$'s and $a_{i,k}^\dagger$'s to a vacuum state $|0\rangle$ is a Hilbert space. The proof particularly needs the investigation of the new statistic cinv and representations of the colored permutation group.

1 Introduction

Let $\mathbb{R}(q)$ be the fraction field of the real polynomials with variable q. By a deformed quon algebra $\mathbf{A}$, we mean the free algebra

$$\mathbb{R}(q)\big[a_{i,k} \;\big|\; (i,k) \in \mathbb{N}^* \times [m]\big]$$

subject to the anti-involution $\dagger$ exchanging $a_{i,k}$ with $a_{i,k}^\dagger$, and to the commutation relation

$$a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta and

$$\beta_{k,l} = \begin{cases} 0 & \text{if } l-k \equiv m \mod m \\ 1 & \text{otherwise} \end{cases}.$$

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This algebra is a generalization of the quon algebra introduced by Greenberg [2], subject to the commutation relation $a_j a_i^\dagger = q a_i^\dagger a_j + \delta_{i,j}$ obeyed by the annihilation and creation operators of the quon particles, and generating a model of infinite statistics. Moreover, the quon algebra is a generalization of the classical Bose and Fermi algebras corresponding to the restrictions $q = 1$ and $q = -1$ respectively, as well as of the intermediate case $q = 0$ suggested by Hegstrom and investigated by Greenberg [1].

In a Fock-like representation, the generators of $\mathbf{A}$ are the linear operators $a_{i,k}, a_{i,k}^\dagger \colon \mathbf{V} \to \mathbf{V}$ on an infinite dimensional real vector space $\mathbf{V}$ satisfying the commutation relations

$$a_{j,l} a_{i,k}^\dagger - q a_{i,k}^\dagger a_{j,l} = q^{\beta_{k,l}} \delta_{i,j},$$

and the relations

$$a_{i,k}|0\rangle = 0,$$

where $a_{i,k}^\dagger$ is the adjoint of $a_{i,k}$, and $|0\rangle$ is a nonzero distinguished vector of $\mathbf{V}$. The $a_{i,k}$'s are the annihilation operators and the $a_{i,k}^\dagger$'s the creation operators.

Let $\mathbf{H}$ be the vector subspace of $\mathbf{V}$ generated by the particle states obtained by applying combinations of $a_{i,k}$'s and $a_{i,k}^\dagger$'s to $|0\rangle$, or

$$\mathbf{H} := \big\{ a|0\rangle \;\big|\; a \in \mathbf{A} \big\}.$$

The aim of this article is to prove the realizability of this model through the following theorem.

Theorem 1. *$\mathbf{H}$ is a Hilbert space for the bilinear form $(\cdot,\cdot) \colon \mathbf{H} \times \mathbf{H} \to \mathbb{R}(q)$ defined by*

$$\big(a|0\rangle, b|0\rangle\big) := \langle 0| a^\dagger\, b |0\rangle \quad \textit{with} \quad \langle 0|0\rangle = 1,$$

and for

$$-1 < q < 1 \textit{ if } m = 1 \quad \textit{and} \quad \frac{1}{1-m} < q < 1 \textit{ if } m > 1.$$

Theorem 1 is a generalization of the realizability of the quon algebra model in infinite statistics proved by Zagier [3, Theorem 1].

To prove Theorem 1, we begin by showing in Section 3 that

$$\mathcal{B} := \big\{ a_{i_1,k_1}^\dagger \dots a_{i_n,k_n}^\dagger |0\rangle \;\big|\; (i_u, k_u) \in \mathbb{N}^* \times [m],\, n \in \mathbb{N} \big\}$$

is a basis of $\mathbf{H}$, so that we can assume that

$$\mathbf{H} = \Big\{ \sum_{i=1}^{n} \lambda_i b_i \;\Big|\; n \in \mathbb{N}^*,\, \lambda_i \in \mathbb{R}(q),\, b_i \in \mathcal{B} \Big\}.$$

Denote by $\mathbb{U}_m$ the group of all m^{th} roots of unity, and $\mathfrak{S}_n$ the permutation group on $[n]$. We represent an element π of the colored permutation group of m colors $\mathbb{U}_m \wr \mathfrak{S}_n$ by

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ (\sigma(1), k_1) & (\sigma(2), k_2) & \dots & (\sigma(n), k_n) \end{pmatrix},$$

where $k_1, \ldots, k_n \in [m]$, and σ is a permutation of $[n]$. But we also adopt the notation $\pi = (\sigma, \alpha)$ meaning that $\sigma \in \mathfrak{S}_n$ and $\alpha\colon [n] \to [m]$ such that

$$\forall i \in [n], \pi(i) = \big(\sigma(i), \alpha(i)\big).$$

More generally, let I be a multiset of n elements in $\mathbb{N}^*$, and $\mathfrak{S}_I$ its permutation set. An element θ of the colored permutation set $\mathbb{U}_m \wr \mathfrak{S}_I$ is defined by $\theta := (\varphi, \epsilon)$ meaning that $\varphi \in \mathfrak{S}_I$ and $\epsilon\colon [n] \to [m]$ such that

$$\forall i \in [n], \theta(i) = \big(\varphi(i), \epsilon(i)\big).$$

Denote the infinite matrix associated to the bilinear form in Theorem 1 by

$$\mathbf{M} := \big((f,g)\big)_{f,g \in \mathcal{B}}.$$

Let $\left[{\mathbb{N}^* \atop n} \right]$ be the set of multisets of n elements in $\mathbb{N}^*$. We also prove in Section 3 that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[{\mathbb{N}^* \atop n} \right]} \mathbf{M}_I \quad \text{with} \quad \mathbf{M}_I = \Big(\langle 0 |\, a_{\vartheta(n)} \ldots a_{\vartheta(1)}\, a^\dagger_{\theta(1)} \ldots a^\dagger_{\theta(n)} \,| 0 \rangle \Big)_{\vartheta, \theta \in \mathbb{U}_m \wr \mathfrak{S}_I}.$$

For $m = 3$ for example, we have

$$\mathbf{M}_{[2]} = \begin{pmatrix}
1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 \\
q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 \\
q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 \\
q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 \\
q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 \\
q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 \\
q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 \\
q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 \\
q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1 & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q \\
q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & 1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 \\
q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 \\
q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q \\
q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 \\
q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 \\
q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q \\
q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q \\
q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q \\
q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1
\end{pmatrix}.$$

We need to introduce the statistic $\mathrm{cinv}\colon \mathbb{U}_m \wr \mathfrak{S}_n \to \mathbb{N}$ defined by

$$\mathrm{cinv}(\sigma, \alpha) := \#\big\{(i,j) \in [n]^2 \;\big|\; i < j, \sigma(i) > \sigma(j)\big\} + \#\big\{i \in [n] \big| \alpha(i) \neq m\big\}.$$

Still in Section 3, we prove that $\mathbf{M}_I$ is the representation of

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\mathrm{cinv}\, \pi} \pi \tag{1}$$

on the $\mathbb{U}_m \wr \mathfrak{S}_n$-module $\mathbb{R}[\mathbb{U}_m \wr \mathfrak{S}_I]$. Hence if the regular representation of (1), which is $\mathbf{M}_{[n]}$, is positive definite, then $\mathbf{M}_I$ is positive definite.

We prove in Section 4 that

$$\det \mathbf{M}_{[n]} = \Big(\big(1+(m-1)q\big)(1-q)^{m-1}\prod_{i=1}^{n-1}(1-q^{i^2+i})^{\frac{(n-i)}{(i^2+i)}}\Big)^{m^n n!}.$$

We particularly can infer that $\mathbf{M}_{[n]}$ is nonsingular for

$$-1 < q < 1 \text{ if } m = 1 \quad \text{and} \quad \frac{1}{1-m} < q < 1 \text{ if } m > 1.$$

Since $\mathbf{M}_{[n]}$ is the identity matrix of order $m^n n!$ if $q = 0$, we deduce by continuity that $\mathbf{M}_{[n]}$ is positive definite for the values of q mentioned above. For these suitable values of q, $\mathbf{M}$ is then a symmetric positive definite matrix or, in other terms, the bilinear form of Theorem 1 is an inner product on $\mathbf{H}$.

But before investigating the deformed quon algebra, it is necessary to recall some notions in representation theory and do some computations in Section 2.

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2 Representation Theory

We recall the useful notions on representation theory of group and do some calculations for the cyclic groups.

Take a group G and a finite-dimensional vector space V over a field $\mathbb{K}$. Let $g, h \in G$, $a, b \in \mathbb{K}$, and $u, v \in V$. Then V is a G-module if there is a multiplication $\cdot$ of elements of V by elements of G such that

- $u \cdot g \in V$.
- $(au + bv) \cdot g = a(u \cdot g) + b(v \cdot g)$,
- $u \cdot (gh) = (u \cdot g) \cdot h$,
- $u \cdot 1 = u$ where 1 is the neutral element of G.

Take an element x in the group algebra $\mathbb{K}[G]$. Suppose that $\{v_1, \dots, v_n\}$ is a basis of V, and that $v_j \cdot x = \sum_{i \in [n]} \mu_{i,j} v_i$. Then the representation of x on the G-module V is the matrix

$$R_V(x) := (\mu_{i,j})_{i,j \in [n]}.$$

In particular if $x = \sum_{g \in G} \lambda_g g \in \mathbb{K}[G]$ with $\lambda_g \in \mathbb{R}$, then the regular representation of x is

$$R_{\mathbb{K}[G]}(x) := \big(\lambda_{h^{-1}g}\big)_{g,h \in G}.$$

Lemma 1. *Let G be a finite group, $H \leq G$, and $x \in \mathbb{K}[H]$. Then,*

$$\det R_{\mathbb{K}[G]}(x) = \big(\det R_{\mathbb{K}[H]}(x)\big)^{|G:H|}.$$

Proof. Let $H = \{h_1, \dots, h_r\}$, and $\{g_1, \dots, g_k\}$ be a left coset representative set of H. On the ordered basis $(g_1h_1, \dots, g_1h_r, g_2h_1, \dots, g_2h_r, \dots, g_kh_1, \dots, g_kh_r)$ of $\mathbb{K}[G]$, we have

$$R_{\mathbb{K}[G]}(x) = R_{\mathbb{K}[H]}(x) \otimes I_{|G:H|},$$

where $I_{|G:H|}$ is the unit matrix of size $|G : H|$. □

Now consider the cyclic group Z_m of order m generated by γ, and take a variable z. We need the following equalities on the group algebra $\mathbb{R}(z)[Z_m]$.

Lemma 2. *We have*

$$\det R_{\mathbb{R}(z)[Z_m]}\big(1 + z \sum_{k\in[m-1]} \gamma^k\big) = \big(1 + (m-1)z\big)(1-z)^{m-1}.$$

Proof. The regular representation of $1 + z \sum_{k\in[m-1]} \gamma^k$ is the $m \times m$ circulant matrix with associated polynomial $f(x) = 1 + z \sum_{j\in[m-1]} x^j$. The determinant of this circulant matrix is $\prod_{i\in[m]} f(\zeta^i)$. If $i \in [m-1]$, then

$$\sum_{j\in[m-1]} \zeta^{ij} = \frac{1-\zeta^i}{1-\zeta^i} \sum_{j\in[m-1]} \zeta^{ij} = \frac{\zeta^i - 1}{1-\zeta^i} = -1.$$

Thus $f(1) = 1 + (m-1)z$, and $f(\zeta^i) = 1 - z$ for $i \in [m-1]$. □

Lemma 3. *We have*

$$\Big(1 + z \sum_{k\in[m-1]} \gamma^k\Big)^{-1} = \frac{1}{\big(1+(m-1)z\big)(1-z)}\Big(1 + (m-2)z - z \sum_{k\in[m-1]} \gamma^k\Big).$$

Proof. The form of $1 + z \sum_{k\in[m-1]} \gamma^k$ gives us the intuition that its inverse has the form $x + y \sum_{k\in[m-1]} \gamma^k$. The calculation

$$\Big(1 + z \sum_{k\in[m-1]} \gamma^k\Big) \cdot \Big(x + y \sum_{k\in[m-1]} \gamma^k\Big)$$
$$= x + (m-1)zy + \Big(zx + \big(1 + (m-2)z\big)y\Big) \sum_{k\in[m-1]} \gamma^k$$

confirms the intuition since it leads us to solve the equation system

$$\begin{cases} x + (m-1)zy = 1 \\ zx + \big(1 + (m-2)z\big)y = 0 \end{cases}$$

to get the inverse of $1 + z \sum_{k\in[m-1]} \gamma^k$. We obtain

$$x = \frac{1 + (m-2)z}{\big(1 + (m-1)z\big)(1-z)} \quad \text{and} \quad y = -\frac{z}{\big(1 + (m-1)z\big)(1-z)}.$$ □

Lemma 4. *We have*

$$(1 - z\gamma)^{-1} = \frac{1}{1 - z^m} \sum_{i=0}^{m-1} z^i \gamma^i.$$

Proof. It comes from $(1 - z\gamma)(1 + z\gamma + \cdots + z^{m-1}\gamma^{m-1}) = 1 - z^m$. □

3 The Bilinear Form $(\cdot, \cdot)$

We first show that $\mathbf{H}$ is linearly generated by the particle states obtained by applying combinations of $a^\dagger_{i,k}$'s to $|0\rangle$. Then we prove that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[\mathbb{N}^* \atop n \right]} \mathbf{M}_I\,,$$

where $\mathbf{M}_I$ is a representation of $\displaystyle\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\operatorname{cinv} \pi} \pi$.

Lemma 5. *The vector space generated by our particle states is*

$$\mathbf{H} = \Big\{ \sum_{i=1}^{n} \lambda_i b_i \;\Big|\; n \in \mathbb{N}^*,\ \lambda_i \in \mathbb{R}(q),\ b_i \in \mathcal{B} \Big\}.$$

Proof. Let $(j, l) \in \mathbb{N}^* \times [m]$. We have,

$$\begin{aligned} a_{j,l}\, a^\dagger_{i_1,k_1} \dots a^\dagger_{i_r,k_r} &= q^r a^\dagger_{i_1,k_1} \dots a^\dagger_{i_r,k_r}\, a_{j,l} \\ &\quad + \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta_{-k_u,l}}\, a^\dagger_{i_1,k_1} \dots \widehat{a^\dagger_{i_u,k_u}} \dots a^\dagger_{i_r,k_r}, \end{aligned}$$

where the hat over the u^{th} term of the product indicates that this term is omitted. So

$$a_{j,l}\, a^\dagger_{i_1,k_1} \dots a^\dagger_{i_r,k_r}\, |0\rangle = \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta_{-k_u,l}}\, a^\dagger_{i_1,k_1} \dots \widehat{a^\dagger_{i_u,k_u}} \dots a^\dagger_{i_r,k_r} |0\rangle.$$

Thus one can recursively remove every annihilation operator $a_{j,l}$ of an element $a|0\rangle$ of $\mathbf{H}$. □

Lemma 6. *Let* $\big((j_1, l_1), \dots, (j_s, l_s)\big) \in (\mathbb{N}^* \times [m])^s$ *and* $\big((i_1, k_1), \dots, (i_r, k_r)\big) \in (\mathbb{N}^* \times [m])^r$*. If, as multisets,* $\{j_1, \dots, j_s\} \neq \{i_1, \dots, i_s\}$*, then*

$$\langle 0| a_{j_s,l_s} \dots a_{j_1,l_1}\, a^\dagger_{i_1,k_1} \dots a^\dagger_{i_r,k_r} |0\rangle = 0.$$

Proof. Suppose that v is the smallest integer in $[s]$ such that

$$j_v \notin \{i_1, \dots, i_r\} \setminus \{j_1, \dots, j_{v-1}\}.$$

Then

$$a_{j_s,l_s} \dots a_{j_1,l_1} a^\dagger_{i_1,k_1} \dots a^\dagger_{i_r,k_r} = P a_{j_v,l_v} \dots a_{j_1,l_1} + Q a_{j_v,l_v} \quad \text{with} \quad P, Q \in \mathbf{A}.$$

We deduce that

$$a_{j_s,l_s}\dots a_{j_1,l_1}a^\dagger_{i_1,k_1}\dots a^\dagger_{i_r,k_r}|0\rangle = Pa_{j_v,l_v}\dots a_{j_1,l_1}|0\rangle + Qa_{j_v,l_v}|0\rangle = 0.$$

In the same way, suppose that u is the smallest integer in $[r]$ such that i_u does not belong to the multiset $\{j_1,\dots,j_s\}\setminus\{i_1,\dots,i_{u-1}\}$. Then

$$a_{j_s,l_s}\dots a_{j_1,l_1}\, a^\dagger_{i_1,k_1}\dots a^\dagger_{i_r,k_r} = a^\dagger_{i_1,k_1}\dots a^\dagger_{i_u,k_u}\, P' + a^\dagger_{i_u,k_u}\, Q' \text{ with } P',Q'\in\mathbf{A}.$$

And $\langle 0|\, a_{j_s,l_s}\dots a_{j_1,l_1}\, a^\dagger_{i_1,k_1}\dots a^\dagger_{i_r,k_r} = \langle 0|\, a^\dagger_{i_1,k_1}\dots a^\dagger_{i_u,k_u}\, P' + \langle 0|\, a^\dagger_{i_u,k_u}\, Q' = 0.$ □

We just then need to investigate the product $\langle 0|\, a_{j_n,l_n}\dots a_{j_1,l_1}\, a^\dagger_{i_1,k_1}\dots a^\dagger_{i_n,k_n}\,|0\rangle$, where $(j_1,\dots,j_n)$ is a permutation of $(i_1,\dots,i_n)$. Consider a multiset I of n elements in $\mathbb{N}^*$.

Lemma 7. *Let $\theta,\vartheta\in\mathbb{U}_m\wr\mathfrak{S}_I$. Then,*

$$\langle 0|\, a_{\vartheta(n)}\dots a_{\vartheta(1)}\, a^\dagger_{\theta(1)}\dots a^\dagger_{\theta(n)}\,|0\rangle = \sum_{\substack{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n\\ \vartheta=\theta\pi}} q^{\operatorname{cinv}\pi}.$$

Proof. Let $(j_1,\dots,j_n)$ be a permutation of $(i_1,\dots,i_n)$. Then,

$$\begin{aligned}
&a_{j_n,l_n}\dots a_{j_1,l_1}a^\dagger_{i_1,k_1}\dots a^\dagger_{i_n,k_n}|0\rangle\\
&\quad= \sum_{\substack{(u_1,\dots,u_n)\in[n]^n\\ i_{u_1}=j_1,\dots,i_{u_n}=j_n}} \prod_{s\in[n]} q^{u_s-1-\#\left\{r\in[s-1]\,\middle|\,u_r<u_s\right\}} q^{\beta_{k_{u_s},l_s}}|0\rangle\\
&\quad= \sum_{\substack{(u_1,\dots,u_n)\in[n]^n\\ i_{u_1}=j_1,\dots,i_{u_n}=j_n}} \prod_{s\in[n]} q^{\#\left\{r\in[s-1]\,\middle|\,u_r>u_s\right\}} q^{\beta_{k_{u_s},l_s}}|0\rangle\\
&\quad= \sum_{\substack{(u_1,\dots,u_n)\in[n]^n\\ i_{u_1}=j_1,\dots,i_{u_n}=j_n}} q^{\#\left\{(r,s)\in[n]^2\,\middle|\,r<s,u_r>u_s\right\}+\sum_{s\in[n]}\beta_{k_{u_s},l_s}}|0\rangle\\
&\quad= \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \forall s\in[n],j_s=i_{\sigma(s)}}} q^{\#\left\{(r,s)\in[n]^2\,\middle|\,r<s,\sigma(r)>\sigma(s)\right\}+\sum_{s\in[n]}\beta_{k_{\sigma(s)},l_s}}|0\rangle\\
&\quad= \sum_{\substack{\pi=(\sigma,\alpha)\in\mathbb{U}_m\wr\mathfrak{S}_n\forall s\in[n],\\ j_s=i_{\sigma(s)},l_s\equiv k_{\sigma(s)}+\alpha(s)\pmod m}} q^{\operatorname{cinv}\pi}|0\rangle.
\end{aligned}$$

We obtain the result by replacing $a_{j_n,l_n}\dots a_{j_1,l_1}$ and $a^\dagger_{i_1,k_1}\dots a^\dagger_{i_n,k_n}$ by $a_{\vartheta(n)}\dots a_{\vartheta(1)}$ and $a^\dagger_{\theta(1)}\dots a^\dagger_{\theta(n)}$ respectively. □

For example, take $m=4$,

$$\vartheta = \begin{pmatrix} 1 & 2 & 3\\ (2,4) & (5,1) & (2,4)\end{pmatrix} \quad\text{and}\quad \theta = \begin{pmatrix} 1 & 2 & 3\\ (5,2) & (2,3) & (2,1)\end{pmatrix}.$$

Then

$$\begin{aligned}\langle 0|a_{2,4}a_{5,1}a_{2,4}a^{\dagger}_{5,2}a^{\dagger}_{2,3}a^{\dagger}_{2,1}|0\rangle\\ &= q^{\operatorname{cinv}\left(\begin{smallmatrix}1 & 2 & 3\\ (2,1) & (1,3) & (3,3)\end{smallmatrix}\right)} + q^{\operatorname{cinv}\left(\begin{smallmatrix}1 & 2 & 3\\ (3,3) & (1,3) & (2,1)\end{smallmatrix}\right)}\\ &= q^4 + q^5\end{aligned}$$

Define the multiplication of an element $\theta = (\varphi, \epsilon)$ of $\mathbb{U}_m \wr \mathfrak{S}_I$ by an element $\pi = (\sigma, \alpha)$ of $\mathbb{U}_m \wr \mathfrak{S}_n$ by

$$\theta\cdot\pi = (\psi, \eta) \in \mathbb{U}_m\wr\mathfrak{S}_I \quad \text{with} \quad \forall i \in [n],\ \psi(i) = \varphi\sigma(i),\ \eta(i) \equiv \epsilon\sigma(i)+\alpha(i) \mod m.$$

Consider the vector space of linear combinations of colored permutations

$$\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I] := \Big\{ \sum_{\theta\in\mathbb{U}_m\wr\mathfrak{S}_I} z_\theta \theta \ \Big|\ z_\theta \in \mathbb{R}(q)\Big\}.$$

One can easily check that, relatively to the multiplication $\cdot$, $\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I]$ is a $\mathbb{U}_m \wr \mathfrak{S}_n$-module.

Proposition 1. *We have*

$$\mathbf{M}_I = R_{\mathbb{R}(q)[\mathbb{U}_m\wr\mathfrak{S}_I]}\Big(\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\Big).$$

Proof. Using Lemma 7, we obtain for $\theta \in \mathbb{U}_m \wr \mathfrak{S}_I$

$$\begin{aligned}\theta \cdot \sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi} &= \sum_{\vartheta\in\mathbb{U}_m\wr\mathfrak{S}_I} \Big(\sum_{\substack{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n\\ \vartheta=\theta\pi}} q^{\operatorname{cinv}\pi}\Big)\vartheta\\ &= \sum_{\vartheta\in\mathbb{U}_m\wr\mathfrak{S}_I} \langle 0|\, a_{\vartheta(n)} \dots a_{\vartheta(1)}\, a^{\dagger}_{\theta(1)} \dots a^{\dagger}_{\theta(n)}\, |0\rangle\vartheta\,. \end{aligned}$$ □

4 The Determinant of $\mathrm{M}_{[n]}$

We compute the determinant and the inverse of the regular representation of

$$\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi\,.$$

Consider the subgroup $\mathfrak{C}_n$ of $\mathbb{U}_m \wr \mathfrak{S}_n$ defined by

$$\mathfrak{C}_n := \big\{\pi = (\sigma, \alpha) \in \mathbb{U}_m \wr \mathfrak{S}_n \ \big|\ \forall i \in [n],\ \sigma(i) = i\big\}\,.$$

For $i \in [n]$, let ξ_i be the colored permutation

$$\begin{pmatrix} 1 & 2 & \dots & i & \dots & n\\ (1,m) & (2,m) & \dots & (i,1) & \dots & (n,m)\end{pmatrix}$$

in $\mathfrak{C}_n$. We need the following lemma.

Lemma 8. *We have*

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]}\Big(\sum_{\xi\in\mathfrak{C}_n} q^{\operatorname{cinv}\xi}\xi\Big) = \Big(\big(1+(m-1)q\big)\big(1-q\big)^{m-1}\Big)^{m^n n!}.$$

Proof. Remark that

$$\sum_{\xi\in\mathfrak{C}_n} q^{\operatorname{cinv}\xi}\xi = \prod_{i\in[n]}\big(1+q\sum_{k\in[m-1]}\xi_i^k\big).$$

Then, using Lemma 1 and Lemma 2, we obtain

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]}\big(1+q\sum_{k\in[m]}\xi_i^k\big) = \Big(\big(1+(m-1)q\big)\big(1-q\big)^{m-1}\Big)^{m^{n-1}n!}. \qquad \square$$

Now we can compute the determinant of $\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi$.

Theorem 2. *We have*

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]}\Big(\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi\Big)$$
$$= \Big(\big(1+(m-1)q\big)(1-q)^{m-1}\prod_{i=1}^{n-1}(1-q^{i^2+i})^{\frac{(n-i)}{(i^2+i)}}\Big)^{m^n n!}.$$

Proof. Every $\pi\in\mathbb{U}_m\wr\mathfrak{S}_n$ has a decomposition $\pi=\sigma\xi$ such that

$$\sigma\in\mathfrak{S}_n,\ \xi\in\mathfrak{C}_n,\ \text{and}\ \operatorname{cinv}\pi=\operatorname{cinv}\sigma+\operatorname{cinv}\xi.$$

Then,

$$\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi = \Big(\sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{cinv}\sigma}\sigma\Big)\Big(\sum_{\xi\in\mathfrak{C}_n} q^{\operatorname{cinv}\xi}\xi\Big).$$

It is known that [3, Theorem 2]

$$\det R_{\mathbb{R}(q)[\mathfrak{S}_n]}\Big(\sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{cinv}\sigma}\sigma\Big) = \prod_{i=1}^{n-1}(1-q^{i^2+i})^{\frac{(n-i)n!}{(i^2+i)}}.$$

We finally obtain the result by using Lemma 1 and Lemma 8. $\square$

For $k\in[n]$, denote by $t_{k,n}$ the permutation $(n\ n-1\ \ldots\ k)$ in cycle notation. Let

$$\gamma_n = \overrightarrow{\prod_{k\in[n-1]}} 1-q^{n-k}t_{k,n} \quad \text{and} \quad \varepsilon_n = \overleftarrow{\prod_{k\in[n]}} \frac{\sum_{i=0}^{n-k} q^{(n-k+2)i}\,t_{k,n}^i}{1-q^{(n-k+1)(n-k+2)}}.$$

Furthermore, let

$$\rho_k = \frac{1+(m-2)q-q\sum_{i\in[m-1]}\xi_k^i}{\big(1+(m-1)q\big)(1-q)}.$$

We finish with the inverse of $\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi$.

Proposition 2. *We have*

$$\Big(\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\operatorname{cinv} \pi} \pi\Big)^{-1} = \prod_{i \in [n]} \rho_i \cdot \overleftarrow{\prod_{i \in [n-1]}} \gamma_{i+1} \varepsilon_i \,.$$

Proof. We obtain

$$\Big(\sum_{\xi \in \mathfrak{C}_n} q^{\operatorname{cinv} \xi} \xi\Big)^{-1} = \prod_{i \in [n]} \rho_i$$

by means of Lemma 3. Then [3, Proposition 2] and Lemma 4 permit us to write

$$\Big(\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{cinv} \sigma} \sigma\Big)^{-1} = \overleftarrow{\prod_{i \in [n-1]}} \gamma_{i+1} \varepsilon_i \,. \qquad \square$$

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