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ON KURZWEIL-STIELTJES EQUIINTEGRABILITY AND GENERALIZED BV FUNCTIONS

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Dedicated to the memory of Professor Štefan Schwabik (1941–2009)

Abstract. We present sufficient conditions ensuring Kurzweil-Stieltjes equiintegrability in the case when integrators belong to the class of functions of generalized bounded variation.

Keywords: Kurzweil-Stieltjes integral; generalized bounded variation; variational measure; Stieltjes derivative

MSC 2010: 26A39, 26A42, 26A45, 26A24

1. Introduction

The notion of equiintegrable sequences naturally appears in the study of convergence theorems for gauge type integrals. The idea behind this concept is a uniform integrability of the sequence in the sense that there exists a single gauge that works for every function from the sequence. There is a vast literature exploiting different aspects of this concept in the theory of Kurzweil-Henstock integrals, e.g. [6], [9], [19]. On the other hand, little is known concerning conditions ensuring equiintegrability for Stieltjes type integrals, see [1], [12].

The concept of generalized bounded variation, which goes back to [14], is a natural extension of the notion of variation to arbitrary sets, see also [7], Chapter 6. Functions having generalized bounded variation played an important role in the first considerations regarding descriptive characterizations of Stieltjes- and gauge-type integrals, see [22] and [2]. In [3], a given descriptive definition of the Kurzweil-Stieltjes

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integral with respect to integrators of generalized bounded variation relies on a variational measure approach. The notion of full variational measure associated to arbitrary functions, as introduced in [21], is consistent with the usual Lebesgue-Stieltjes measure in the case of functions of bounded variation. Notably, the applicability of variational measures was further explored by Štefan Schwabik in his last papers, see [16], [17], [18], where a whole theory regarding integral extensions was built.

In this paper we investigate conditions under which a sequence $\{f_n\}$ is Kurzweil-Stieltjes equiintegrable with respect to a function g of generalized bounded variation. Our approach to this class of functions follows the work in [3], relying on a characterization of generalized bounded variation by means of variational measures. Our main theorem, Theorem 4.6, explores a uniform differentiability type condition in connection to Kurzweil-Stieltjes equiintegrable sequences. Such type condition forms the basis of some convergence results for gauge integrals, in particular we have the following result found in [10]:

Theorem A. *Let $f_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of Kurzweil-Henstock integrable functions which converges to $f\colon [a,b] \to \mathbb{R}$ almost everywhere in $[a,b]$. Assume that the functions $F_n(t) = \int_a^t f_n(s)\,\mathrm{d}s$, $n \in \mathbb{N}$, are uniformly differentiable, that is, for every $\tau \in [a,b]$ and $\varepsilon > 0$, there exists $\varrho(\tau) > 0$ such that*

$$|F_n(s) - F_n(t) - f_n(\tau)(s-t)| \leqslant \varepsilon|s-t| \tag{1.1}$$

for every $n \in \mathbb{N}$ and $\tau - \varrho(\tau) < t \leqslant \tau \leqslant s < \tau + \varrho(\tau)$. Then f is Kurzweil-Henstock integrable and

$$\int_a^b f(s)\,\mathrm{d}s = \lim_{n\to\infty} \int_a^b f_n(s)\,\mathrm{d}s.$$

In our investigation of equiintegrability, using an appropriate notion of differentiability compatible with Stieltjes integrals, a condition analogous to (1.1) plays a role. As a consequence, we extend the result above to Kurzweil-Stieltjes integrable sequences. Further, we discuss the necessity/sufficiency of some uniform properties of the sequence of primitives in connection with equiintegrability.

2. Kurzweil-Stieltjes integral

Let $[a,b]$ be a fixed compact interval. Given $A \subset [a,b]$, a *system in A* is a finite collection of tagged intervals $S = \{(c_j, [a_j, b_j])\colon\ j = 1, \ldots, m\}$ satisfying

$$a \leqslant a_1 < b_1 \leqslant \ldots \leqslant a_m < b_m \leqslant b, \quad \text{and} \quad c_j \in [a_j, b_j] \cap A,\ \ j = 1, \ldots, m.$$

By $\nu(S)$ we denote the number of tagged intervals in a system S, and if no misunderstanding can arise we write simply $S = \{(c_j, [a_j, b_j])\}$. Given a gauge δ on A, i.e. $\delta\colon A \to \mathbb{R}_+$, we say that a system S in A is δ-*fine* if

$$[a_j, b_j] \subset (c_j - \delta(c_j), c_j + \delta(c_j)) \quad \text{for every } j = 1, \dots, \nu(S).$$

Further, note that a partition of the interval $[a, b]$ is a particular type of a system in $[a, b]$, namely $P = \{(\tau_j, [\alpha_{j-1}, \alpha_j])\colon j = 1, \dots, \nu(P)\}$ where $\alpha_0 = a$ and $\alpha_{\nu(P)} = b$.

Given a pair of functions $f, g\colon [a, b] \to \mathbb{R}$ and a partition $P = \{(\tau_j, [\alpha_{j-1}, \alpha_j])\}$, let

$$S(F, \mathrm{d}g, P) = \sum_{j=1}^{\nu(P)} f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})).$$

We say that the Kurzweil-Stieltjes integral $\int_a^b f \, \mathrm{d}g$ exists if there exists a number $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$|S(f, \mathrm{d}g, P) - I| < \varepsilon \quad \text{for every } \delta\text{-fine partition } P \text{ of } [a, b].$$

We denote $I = \int_a^b f \, \mathrm{d}g$. The Kurzweil-Stieltjes integral has the usual properties of linearity, additivity with respect to adjacent intervals, etc. For a comprehensive study of this integral we refer to [12].

Accordingly, equiintegrability is defined as follows:

Definition 2.1. Let $f_n, g_n\colon [a, b] \to \mathbb{R}$ for $n \in \mathbb{N}$. We say that $\{f_n\}$ is equiintegrable with respect to $\{g_n\}$, if the integral $\int_a^b f_n \, \mathrm{d}g_n$ exists for each $n \in \mathbb{N}$, and for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\left| S(f_n, \mathrm{d}g_n, P) - \int_a^b f_n \, \mathrm{d}g_n \right| < \varepsilon$$

holds for every δ-fine partition P of $[a, b]$ and for every $n \in \mathbb{N}$.

The notion of equiintegrability yields the following basic convergence result.

Theorem 2.2. *Let $f_n, g_n\colon [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$ be such that $\{f_n\}$ is equiintegrable with respect to $\{g_n\}$, and assume that there exist $f, g\colon [a, b] \to \mathbb{R}$ satisfying*

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \text{and} \quad \lim_{n\to\infty} g_n(t) = g(t) \quad \text{for all } t \in [a, b].$$

Then the integral $\int_a^b f \, \mathrm{d}g$ exists and

$$\int_a^b f \, \mathrm{d}g = \lim_{n\to\infty} \int_a^b f_n \, \mathrm{d}g_n. \tag{2.1}$$

Moreover, if $\{g_n\}$ is uniformly convergent and g is bounded, then

$$\lim_{n\to\infty} \sup_{t\in[a,b]} \left| \int_a^t f_n \, \mathrm{d}g_n - \int_a^t f \, \mathrm{d}g \right| = 0. \tag{2.2}$$

The first convergence result in the theorem above can be obtained by the same arguments used to prove an analogous theorem for Kurzweil-Henstock integrals that can be found in any monograph devoted to such a theory, see e.g. [7]. The proof of the uniform convergence of the indefinite integrals in (2.2) is slightly more technical and relies on Saks-Henstock lemma, see [12], Theorem 6.8.2 for details. As has been shown in [12], Example 6.8.3, the uniform convergence of the sequence $\{g_n\}$ as well as the boundedness of the integrator g are essential assumptions to guarantee (2.2).

3. Generalized bounded variation

Generalized bounded variation is a natural extension of the notion of variation to arbitrary sets. Roughly speaking, a function $g\colon [a,b] \to \mathbb{R}$ is of generalized bounded variation on a set $A \subset [a,b]$ if A can be decomposed into a countable union of sets in which g has bounded variation, see e.g. [7], Chapter 6 for details. Herein we make use of an equivalent definition by means of variational measures. We therefore start this section by presenting the notion of variational measure associated to an arbitrary function.

Given $g\colon [a,b] \to \mathbb{R}$, $A \subset [a,b]$ and $\delta\colon A \to \mathbb{R}_+$, denote

$$W(g, A, \delta) = \sup \sum_{j=1}^{\nu(S)} |g(b_j) - g(a_j)|,$$

where the supremum is taken over all δ-fine systems $S = \{(c_j, [a_j, b_j])\}$ in A. The set function m_g defined by

$$m_g(A) = \inf\{W(g, A, \delta)\colon \delta\colon A \to \mathbb{R}_+\}$$

is the so-called *variational measure induced by g* (also known as Thomson variational measure, see [21]). It is worth mentioning that m_g defines a metric outer measure on $[a,b]$, and it coincides with the Lebesgue outer measure in the case when g is the identity function. For other properties of variational measures we refer to [21] (see also [3]).

A characterization of functions of generalized bounded variation by means of variational measures goes back to the seminal work [21]. A good account on the topic can be found in [8], and it inspires the following definition (cf. [8], Theorem 3.12).

Definition 3.1. Let $g\colon [a,b] \to \mathbb{R}$ and $A \subset [a,b]$ be given. We say that g is of generalized bounded variation on A if there exists a decomposition $A = \bigcup\limits_{k=0}^{\infty} A_k$ such that A_0 is countable and $m_g(A_k) < \infty$ for every $k \in \mathbb{N}$. If, in addition, g is bounded, then we may choose $A_0 = \emptyset$.

We will denote by $\mathrm{BVG}_*[a,b]$ the set of functions $g\colon [a,b] \to \mathbb{R}$ of generalized bounded variation on $[a,b]$. Clearly, the class $\mathrm{BVG}_*[a,b]$ encompasses the functions of bounded variation on $[a,b]$. In particular, if g is continuous and has bounded variation on $[a,b]$, then $\mathrm{var}_J(g) = m_g(J)$ for any subinterval $J \subseteq [a,b]$ (cf. [16], Lemma 3.2). Moreover, since a function g is continuous at a point $c \in [a,b]$ if and only if $m_g(\{c\}) = 0$, it follows that the set of discontinuity points of a function in $\mathrm{BVG}_*[a,b]$ is at most countable. However, such functions need not be regulated or even bounded, see [21] and [11].

We recall the following notion which is of major importance in the study of a fundamental theorem of calculus for Kurzweil-Stieltjes integrals (see [3], Section 4).

Definition 3.2. Let $g\colon [a,b] \to \mathbb{R}$ be given. We say that a function $F\colon [a,b] \to \mathbb{R}$ is *g-normal*, if $m_F(A) = 0$ whenever $m_g(A) = 0$, $A \subset [a,b]$.

The next two theorems summarize some properties of the Kurzweil-Stieltjes integral in connection to variational measures (for the proofs see [3], Proposition 2.9 and [11], Proposition 2.21, respectively).

Theorem 3.3. *Let $g\colon [a,b] \to \mathbb{R}$ be given. If $f\colon [a,b] \to \mathbb{R}$ is null except on a set $Z \subset [a,b]$ with $m_g(Z) = 0$, then $\int_a^t f\,\mathrm{d}g = 0$ for every $t \in [a,b]$.*

Theorem 3.4. *Let $f, g\colon [a,b] \to \mathbb{R}$ be such that the integral $\int_a^b f\,\mathrm{d}g$ exists, and let*

$$F(t) = \int_a^t f\,\mathrm{d}g, \quad t \in [a,b].$$

Then the function F is g-normal.

If, in addition, $g \in \mathrm{BVG}_*[a,b]$, then $F \in \mathrm{BVG}_*[a,b]$.

4. Equiintegrability

In this section we discuss equiintegrability with respect to integrators in the class $\mathrm{BVG}_*[a,b]$. Our investigation relies on assumptions concerning the sequence of primitives, like a uniform differentiability condition which features in convergence results for gauge integrals, see e.g. [10], Corollary 8.16 and [20], Theorem 7.6.3. For the sake of clarity, we divide this section into two parts. The first is devoted to fixing the notions that will later compose the hypotheses of our main result. In the second part we present our main theorem (Theorem 4.6) and further discuss its assumptions.

4.1. Preliminary definitions. Recall that a function $f\colon [a,b] \to \mathbb{R}$ is said to be regulated, if both one-sided limits $f(t+)$, $f(t-)$ exist at every point $t \in [a,b]$ with the convention $f(a-) = f(a)$, and $f(b+) = f(b)$. We denote by $G[a,b]$ the space of regulated functions. When it comes to compactness in this space of functions, the following concept is crucial (cf. [4]).

Definition 4.1. A set $\mathcal{A} \subset G[a,b]$ is called *equiregulated* if for every $\varepsilon > 0$ and every $t_0 \in [a,b]$ there exists $\eta > 0$ such that:

$$\begin{aligned} |f(t) - f(t_0+)| < \varepsilon \quad &\text{for all } t_0 < t < t_0 + \eta,\ f \in \mathcal{A}, \\ |f(t) - f(t_0-)| < \varepsilon \quad &\text{for all } t_0 - \eta < t < t_0,\ f \in \mathcal{A}. \end{aligned}$$

The notion of g-derivative was introduced in [13] for monotone functions g (see also [5]). Following the generalization of such a notion presented in [11], herein we will consider derivatives with respect to functions in $G_L[a,b]$; the subspace of $G[a,b]$ of all left continuous regulated functions.

Given $g \in G_L[a,b]$ and a function $f\colon [a,b] \to \mathbb{R}$, the *derivative of f with respect to g* (or the *g-derivative*) at a point $t \in [a,b]$ is given by

$$\begin{aligned} f_g'(t) &= \lim_{s \to t} \frac{f(s) - f(t)}{g(s) - g(t)} \qquad \text{if } g \text{ is continuous at } t, \\ f_g'(t) &= \lim_{s \to t+} \frac{f(s) - f(t)}{g(s) - g(t)} \qquad \text{if } g \text{ is discontinuous at } t, \end{aligned}$$

provided the limit exists. Clearly, the definition above has sense only if the point t does not belong to the set

$$C_g = \{t \in [a,b]\colon\ g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}.$$

Using the notation from [13], we write $D_g = \{t \in [a,b]\colon\ g(t+) - g(t) \neq 0\}$. This notion of derivative is consistent with the Kurzweil-Stieltjes integral as shown in the following theorem borrowed from [11].

Theorem 4.2. *Let $g \in G_L[a,b] \cap \mathrm{BVG}_*[a,b]$, $f\colon [a,b] \to \mathbb{R}$ be such that the integral $\int_a^b f \,\mathrm{d}g$ exists, and let*

$$F(t) = \int_a^t f \,\mathrm{d}g, \quad t \in [a,b].$$

Then $F_g'(t) = f(t)$ for $t \in [a,b] \setminus N$, where $N \subset [a,b]$ with $m_g(N) = 0$.

Having in mind the notion of g-derivative, we formulate an analog of the uniform differentiability condition (1.1) as follows:

Definition 4.3. Let $g \in G_L[a,b] \cap \mathrm{BVG}_*[a,b]$, and let $F_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ be given. We say that the sequence $\{F_n\}$ is *uniformly g-differentiable* at $t_0 \in [a,b] \setminus C_g$, if for each $n \in \mathbb{N}$ the g-derivative $(F_n)'_g(t_0)$ exists, and for every $\varepsilon > 0$ there exists $\varrho(t_0) > 0$ such that one of the following inequalities holds:

(i) In the case when $t_0 \in D_g$, we have

$$|F_n(t) - F_n(t_0) - (F_n)'_g(t_0)(g(t) - g(t_0))| \leqslant \varepsilon |g(t) - g(t_0)|$$

for every $n \in \mathbb{N}$ and $t_0 < t < t_0 + \varrho(t_0)$.

(ii) In the case when g is continuous at t_0, we have

$$|F_n(s) - F_n(t) - (F_n)'_g(t_0)(g(s) - g(t))| \leqslant \varepsilon |g(s) - g(t)|$$

for every $n \in \mathbb{N}$ and $t_0 - \varrho(t_0) < t \leqslant t_0 \leqslant s < t_0 + \varrho(t_0)$.

The definition above somehow extends to g-derivatives the notion presented in [20], Definition 7.6.1, stating that the limit defining the derivative of each function of the sequence $\{F_n\}$ is uniform with respect to $n \in \mathbb{N}$.

Another concept important for the formulation of our main result is inspired by Definition 3.2 and reads as follows:

Definition 4.4. Let $g\colon [a,b] \to \mathbb{R}$ be given. We say that a sequence of functions $F_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, is *uniformly g-normal*, if for every $A \subset [a,b]$ with $m_g(A) = 0$, and for every $\varepsilon > 0$ there exists a gauge $\delta\colon A \to \mathbb{R}_+$ such that

$$\sum_{j=1}^{\nu(S)} |F_n(b_j) - F_n(a_j)| < \varepsilon$$

for every $n \in \mathbb{N}$ and for every δ-fine system $S = \{(c_j, [a_j, b_j])\}$ in A.

The definition above is related to the notion of uniformly negligible variation, cf. [20], Definition 7.6.2. Indeed, a uniformly g-normal sequence can be understood as a sequence which has uniformly negligible variation on sets with null g-variational measure.

4.2. Main results. Our next goal is to provide sufficient conditions for equiintegrability with respect to integrators of generalized bounded variation. To this end, we will make use of the following theorem which states that sets of null g-variational measure can be disregarded when investigating equiintegrability with respect to a function g.

Lemma 4.5. *Let $g\colon [a,b] \to \mathbb{R}$ and let $f_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence which is pointwise bounded. If $E \subset [a,b]$ is such that $m_g([a,b] \setminus E) = 0$, then $\{f_n\}$ is equiintegrable with respect to g if and only if $\{f_n \chi_E\}$ is equiintegrable with respect to g.*

Proof. Let $h_n := f_n \chi_E$, $n \in \mathbb{N}$. Note that for each $n \in \mathbb{N}$ the function $h_n - f_n$ is null except on $Z = [a,b] \setminus E$. Therefore, Lemma 3.3 ensures that $\int_a^b h_n \, \mathrm{d}g$ exists if and only if $\int_a^b f_n \, \mathrm{d}g$ exists; moreover, the integrals coincide. Moreover, for any partition $P = \{(\tau_j, [\alpha_{j-1}, \alpha_j])\}$ of $[a,b]$ and for $n \in \mathbb{N}$ we have

$$S(f_n, \mathrm{d}g, P) - \int_a^b f_n \, \mathrm{d}g = S(h_n, \mathrm{d}g, P) - \int_a^b h_n \, \mathrm{d}g + \sum_{\tau_j \in Z} f_n(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})).$$

This means that the result holds once we prove that for every $\varepsilon > 0$ there exists a gauge $\delta\colon Z \to \mathbb{R}_+$ such that

$$\left| \sum_{j=1}^{\nu(S)} f_n(c_j)(g(a_j) - g(b_j)) \right| < \varepsilon$$

for every $n \in \mathbb{N}$ and for every δ-fine system $S = \{(c_j, [a_j, b_j])\}$ in Z. To this end, let

$$Z_k = \{t \in Z\colon \ |f_n(t)| \leqslant k \text{ for every } n \in \mathbb{N}\}, \quad k \in \mathbb{N}.$$

Noting that $m_g(Z_k) \leqslant m_g(Z) = 0$, there is a gauge $\gamma_k\colon Z_k \to \mathbb{R}_+$ such that

$$W(g, Z_k, \gamma_k) < \frac{\varepsilon}{k 2^k}. \tag{4.1}$$

Moreover, since the sequence $\{f_n\}$ is pointwise bounded, $Z = \bigcup\limits_{k=1}^{\infty} Z_k$ and for each $t \in Z$ we can choose $\kappa(t) \in \mathbb{N}$ so that $t \in Z_{\kappa(t)}$ and $t \notin Z_m$ for $m \in \mathbb{N}$, $m < \kappa(t)$. Define $\delta(t) = \gamma_{\kappa(t)}(t)$, $t \in Z$, and let $S = \{(c_j, [a_j, b_j])\}$ be a δ-fine system in Z. Thus

$$\begin{aligned} \left| \sum_{j=1}^{\nu(S)} f_n(c_j)(g(a_j) - g(b_j)) \right| &\leqslant \sum_{k=1}^{\infty} \sum_{\substack{\kappa(c_j)=k \\ j=1,\dots,\nu(S)}} |f_n(c_j)(g(a_j) - g(b_j))| \\ &\leqslant \sum_{k=1}^{\infty} k \sum_{\substack{\kappa(c_j)=k \\ j=1,\dots,\nu(S)}} |g(a_j) - g(b_j)| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \end{aligned}$$

where the last inequality follows from (4.1), noting that $\{(c_j, [a_j, b_j])\colon \kappa(c_j) = k\}$ defines a γ_k-fine system on Z_k. This concludes the proof. $\square$

Now we present the main result of this paper.

Theorem 4.6. *Let $g \in G_L[a,b] \cap \mathrm{BVG}_*[a,b]$, and let $f_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ be such that the integrals $\int_a^b f_n \,\mathrm{d}g$ exist. Assume that the functions $F_n(t) = \int_a^t f_n \,\mathrm{d}g$, $n \in \mathbb{N}$, have the following properties:*

(i) *the sequence $\{F_n\}$ is equiregulated;*
(ii) *the sequence $\{F_n\}$ is uniformly g-normal;*
(iii) *there exists $Z \subset [a,b]$, with $m_g(Z) = 0$, such that $\{F_n\}$ is uniformly g-differentiable on $[a,b] \setminus Z$.*

If the sequence $\{f_n\}$ is pointwise bounded, then $\{f_n\}$ is equiintegrable with respect to g.

Proof. Denote $E = [a,b] \setminus Z$ and $h_n = f_n \chi_E$, $n \in \mathbb{N}$. In view of Theorem 4.5, it suffices to show that $\{h_n\}$ is equiintegrable with respect to g. To this end, given $\varepsilon > 0$, we will construct (in several steps) a gauge δ on $[a,b]$ satisfying the equiintegrability condition.

Step 1. Since $m_g(Z) = 0$, by condition (ii) there exists $\gamma\colon Z \to \mathbb{R}_+$ such that

$$\sum_{j=1}^{\nu(S)} |F_n(b_j) - F_n(a_j)| < \varepsilon$$

for every $n \in \mathbb{N}$ and for every γ-fine system $S = \{(c_j, [a_j, b_j])\}$ in Z.

Step 2. Let $[a,b] = \bigcup\limits_{k=1}^{\infty} X_k$ be a decomposition with $m_g(X_k) < \infty$. Without loss of generality, assume that $X_j \cap X_k = \emptyset$ for $j \neq k$, and denote $E_k = E \cap X_k$, $k \in \mathbb{N}$. Noting that $m_g(E_k) < \infty$, choose $\gamma_k\colon E_k \to \mathbb{R}_+$ such that

$$W(g, E_k, \gamma_k) < m_g(E_k) + 1.$$

Given $t \in E$, we know by Theorem 4.2 that $(F_n)'_g(t) = f_n(t)$ for all $n \in \mathbb{N}$. Moreover, we can find a unique $k \in \mathbb{N}$ such that $t \in E_k$. Taking

$$\varepsilon_k = \frac{\varepsilon}{2^k(m_g(E_k) + 1)},$$

by condition (iii) there exists $0 < \varrho(t) < \gamma_k(t)$ such that: if $t \in E_k \cap D_g$ we have

$$|F_n(v) - F_n(t) - f_n(t)(g(v) - g(t))| \leqslant \varepsilon_k |g(v) - g(t)|$$

for every $n \in \mathbb{N}$ and $t < v < t + \varrho(t)$; otherwise, if $t \in E_k \setminus D_g$, we have

$$|F_n(v) - F_n(u) - f_n(t)(g(v) - g(u))| \leqslant \varepsilon_k |g(v) - g(u)|$$

for every $n \in \mathbb{N}$ and $t - \varrho(t) < u \leqslant t \leqslant v < t + \varrho(t)$.

Step 3. Consider an enumeration of $D_g = \{d_l\}$. Due to the left continuity of g, for each $l \in \mathbb{N}$ there exists $\eta_l > 0$ such that

$$|g(s) - g(d_l)| \leqslant \frac{\varepsilon}{2^{l+1}(M_l + 1)}, \quad d_l - \eta_l < s \leqslant d_l,$$

where $M_l = \sup\{|f_n(d_l)|\colon\ n \in \mathbb{N}\}$. Moreover, condition (i) ensures that for each $l \in \mathbb{N}$ we can choose $\eta_l > 0$ so that the inequality

$$|F_n(s) - F_n(d_l)| \leqslant \frac{\varepsilon}{2^{l+1}}, \quad d_l - \eta_l < s \leqslant d_l \quad \text{for all } n \in \mathbb{N},$$

is also satisfied.

Step 4. Now, consider $\delta\colon\ [a, b] \to \mathbb{R}_+$ given by

$$\delta(t) = \begin{cases} \gamma(t) & \text{if } t \in Z, \\ \varrho(t) & \text{if } t \in E \setminus D_g, \\ \min\{\eta_l, \varrho(t)\} & \text{if } t = d_l \in D_g \text{ for some } l \in \mathbb{N}. \end{cases}$$

Fixing an arbitrary $n \in \mathbb{N}$ and given a δ-fine partition $P = \{(\tau_j, [\alpha_{j-1}, \alpha_j])\}$ of $[a, b]$ we have

$$\begin{aligned} |S(h_n, \mathrm{d}g, P) - F_n(b)| \leqslant & \sum_{\tau_j \in Z} |F_n(\alpha_j) - F_n(\alpha_{j-1})| \\ & + \sum_{\tau_j \in E} |f_n(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) - F_n(\alpha_j) + F_n(\alpha_{j-1})| \end{aligned}$$

Clearly $\{(\tau_j, [\alpha_{j-1}, \alpha_j])\colon\ \tau_j \in Z\}$ is a γ-fine system in Z, hence the definition of γ yields

$$\sum_{\tau_j \in Z} |F_n(\alpha_j) - F_n(\alpha_{j-1})| < \varepsilon.$$

Moreover, using the estimates from Step 2 we obtain

$$\begin{aligned} &\sum_{\tau_j \in E \setminus D_g} |f_n(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) - F_n(\alpha_j) + F_n(\alpha_{j-1})| \\ &\qquad \leqslant \sum_{k=1}^{\infty} \sum_{\tau_j \in E_k \setminus D_g} |f_n(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) - F_n(\alpha_j) + F_n(\alpha_{j-1})| \\ &\qquad \leqslant \sum_{k=1}^{\infty} \varepsilon_k \sum_{\tau_j \in E_k \setminus D_g} |g(\alpha_j) - g(\alpha_{j-1})| \leqslant \sum_{k=1}^{\infty} \varepsilon_k (m_g(E_k) + 1) \leqslant \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}; \end{aligned}$$

therefore

$$\sum_{\tau_j \in E \setminus D_g} |f_n(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) - F_n(\alpha_j) + F_n(\alpha_{j-1})| < \varepsilon. \tag{4.2}$$

By a similar reasoning we get

$$\sum_{\tau_j \in E\cap D_g} |f_n(\tau_j)(g(\alpha_j) - g(\tau_j)) - F_n(\alpha_j) + F_n(\tau_j)| < \varepsilon.$$

Having this in mind and using the inequalities from Step 3 we can estimate the sum over $\tau_j \in E \cap D_g$ as follows:

$$\begin{aligned}
&\sum_{\tau_j \in E\cap D_g} |f_n(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) - F_n(\alpha_j) + F_n(\alpha_{j-1})| \\
&\qquad < \varepsilon + \sum_{\tau_j \in E\cap D_g} |f_n(\tau_j)(g(\tau_j) - g(\alpha_{j-1})) - F_n(\tau_j) + F_n(\alpha_{j-1})| \\
&\qquad \leqslant \varepsilon + \sum_{l=1}^{\infty} \sum_{\substack{\tau_j \in E\cap D_g \\ \tau_j = d_l}} (M_l|g(\tau_j) - g(\alpha_{j-1})| + |F_n(\tau_j) - F_n(\alpha_{j-1})|) \\
&\qquad < \varepsilon + 2\sum_{l=1}^{\infty} \frac{\varepsilon}{2^{l+1}} < 2\varepsilon.
\end{aligned}$$

This together with (4.2) yields

$$\sum_{\tau_j \in E} |f_n(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) - F_n(\alpha_j) + F_n(\alpha_{j-1})| < 3\varepsilon.$$

Summarizing, for every δ-fine partition P of $[a,b]$ we have

$$|S(h_n, \mathrm{d}g, P) - F_n(b)| < 4\varepsilon \quad \text{for every } n \in \mathbb{N},$$

and this concludes the proof. □

To our knowledge, Theorem 4.6 provides original conditions for equiintegrability even in the special case of integrators g of bounded variation. Besides, as we can see in the proof, if g is assumed to be continuous, condition (i) can be suppressed.

By combining Theorems 4.6 and 2.2 we obtain a convergence result for Kurzweil-Stieltjes integrals which extends Theorem A (see also [10], Corollary 8.16). More precisely:

Corollary 4.7. *Let $g \in G_L[a,b] \cap \mathrm{BVG}_*[a,b]$ and $f_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ be such that the integrals $\int_a^b f_n \,\mathrm{d}g$ exist, and assume that there exists $f\colon [a,b] \to \mathbb{R}$ such that*

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \textit{for all } t \in [a,b].$$

If the functions $F_n(t) = \int_a^t f_n \,\mathrm{d}g$, $n \in \mathbb{N}$ *satisfy conditions* (i)–(iii) *from Theorem* 4.6, *then the integral* $\int_a^b f \,\mathrm{d}g$ *exists and*

$$\int_a^b f \,\mathrm{d}g = \lim_{n\to\infty} \int_a^b f_n \,\mathrm{d}g_n.$$

Note that, while Theorem A relies simply on the uniform differentiability, two extra conditions are required to obtain its counterpart in the Kurzweil-Stieltjes setting. Remarkably, both the conditions, (i) and (ii), correspond to properties which can be expected from the primitives associated to sequences which are equiintegrable. The relation between equiintegrability and equiregulatedness has been already observed in [15]. More precisely, we have the following result which is a particular case of [15], Proposition 3.4.

Theorem 4.8. *Let* $g \in G[a,b]$ *and let* $f_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ *be a sequence which is pointwise bounded. If* $\{f_n\}$ *is equiintegrable with respect to* g, *then the sequence of functions* $F_n(t) = \int_a^t f_n \,\mathrm{d}g$, $n \in \mathbb{N}$, *is equiregulated.*

As we can see in [20], Theorem 7.6.3, a uniform normality condition plays a role when the question of convergence is considered in abstract settings. Next we show that condition (ii) in Theorem 4.6 is indeed a natural assumption.

Theorem 4.9. *Let* $g \in G_L[a,b] \cap \mathrm{BVG}_*[a,b]$, *and let* $f_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ *be a sequence which is pointwise bounded. If* $\{f_n\}$ *is equiintegrable with respect to* g, *then the sequence of functions* $F_n(t) = \int_a^t f_n \,\mathrm{d}g$, $n \in \mathbb{N}$, *is uniformly* g*-normal.*

Proof. Let $[a,b] = \bigcup_{l=1}^{\infty} X_l$ be a decomposition of $[a,b]$ with $m_g(X_l) < \infty$. Without loss of generality, assume that $X_j \cap X_l = \emptyset$ for $j \neq l$, and define

$$E_{k,l} := \{t \in X_l\colon\ |f_n(t)| \leqslant k \text{ for every } n \in \mathbb{N}\}, \quad k,l \in \mathbb{N}.$$

Since $\{f_n\}$ is pointwise bounded, we have $[a,b] = \bigcup_{l=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,l}$.

Let $A \subset [a,b]$ be such that $m_g(A) = 0$, and let $\varepsilon > 0$ be given. For each $k,l \in \mathbb{N}$ such that $A \cap E_{k,l} \neq \emptyset$, the fact that $m_g(A \cap E_{k,l}) \leqslant m_g(A) = 0$ guarantees that there exists a gauge $\gamma_{k,l} : A \cap E_{k,l} \to \mathbb{R}_+$ such that

$$W(g, A \cap E_{k,l}, \gamma_{k,l}) < \frac{\varepsilon}{k 2^k 2^l}. \tag{4.3}$$

The equiintegrability together with the Saks-Henstock lemma (see [1], Lemma 7.2) ensures that there exists a gauge $\gamma\colon [a,b] \to \mathbb{R}_+$ such that

$$\sum_{j=1}^{\nu(S)} \bigl|f_n(c_j)(g(b_j)-g(a_j)) - F_n(b_j) + F_n(a_j)\bigr| < \varepsilon \tag{4.4}$$

for every $n \in \mathbb{N}$ and for every γ-fine system $S = \{(c_j,[a_j,b_j])\}$ in $[a,b]$.

Denote $M(t) = \sup\limits_n |f_n(t)|$, $t \in [a,b]$ (which is finite due to the pointwise boundedness of the sequence). For each $t \in A$, there exist uniquely determined numbers $\kappa(t), l(t) \in \mathbb{N}$, such that $t \in X_{l(t)}$ and $\kappa(t) = \min\{k \in \mathbb{N}\colon M(t) \leqslant k\}$. In other words, $t \in E_{\kappa(t),l(t)}$.

Define $\delta\colon A \to \mathbb{R}_+$ by $\delta(t) = \min\{\gamma(t), \gamma_{\kappa(t),l(t)}(t)\}$, $t \in A$, and let $S = \{(c_j,[a_j,b_j])\}$ be an arbitrary δ-fine system in A. For each $n \in \mathbb{N}$, by (4.4) we have

$$\begin{aligned}
\sum_{j=1}^{\nu(S)} |F_n(b_j)-F_n(a_j)| &\leqslant \sum_{j=1}^{\nu(S)} |F_n(b_j)-F_n(a_j)-f_n(c_j)(g(b_j)-g(a_j))| \\
&\quad + \sum_{j=1}^{\nu(S)} |f_n(c_j)(g(b_j)-g(a_j))| \\
&\leqslant \varepsilon + \sum_{j=1}^{\nu(S)} M(c_j)|g(b_j)-g(a_j)| \\
&< \varepsilon + \sum_{p=1}^{\infty}\sum_{k=1}^{\infty} k \sum_{\substack{\kappa(c_j)=k,\, l(c_j)=p \\ j=1,\dots,\nu(S)}} |g(b_j)-g(a_j)|.
\end{aligned}$$

Since $\{(c_j,[a_j,b_j])\colon \kappa(c_j)=k,\ l(c_j)=p\}$ defines a $\gamma_{k,p}$-fine system in $A \cap E_{k,l}$, using the inequality (4.3) we get

$$\sum_{j=1}^{\nu(S)} |F_n(b_j)-F_n(a_j)| < 2\varepsilon, \quad n \in \mathbb{N},$$

wherefrom we conclude that $\{F_n\}$ is uniformly g-normal. $\square$

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