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ON THE RELATIONSHIPS BETWEEN STIELTJES TYPE INTEGRALS OF YOUNG, DUSHNIK AND KURZWEIL

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Dedicated to the memory of Štefan Schwabik

Abstract. In this paper we explain the relationship between Stieltjes type integrals of Young, Dushnik and Kurzweil for functions with values in Banach spaces. To this aim also several new convergence theorems will be stated and proved.

Keywords: Kurzweil integral; Young integral; Dushnik integral; Kurzweil-Stieltjes integral; Young-Stieltjes integral; Dushnik-Stieltjes integral; convergence theorem

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1. Introduction

Integral equations of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A]x + f(t) - f(t_0) \tag{1.1}$$

which are generalizations of systems of linear differential equations admit solutions that need not be continuous. Up to now such equations have been considered by several authors starting with Kurzweil (see [9]) and Hildebrandt (see [5]). For further contributions see e.g. [1], [8], [12], [10], [13], [18]–[24] and references therein. These papers worked with several different concepts of the Stieltjes type integral like Young's (Hildebrandt), Kurzweil's (Kurzweil, Schwabik and Tvrdý), Dushnik's

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(Hönig) or Lebesgue's (Ashordia, Meng and Zhang). Thus, an interesting question "what are the relationships between all these concepts?" arises.

It is known that (cf. [9], Theorem 1.2.1) the Kurzweil-Stieltjes integral is in finite dimensional setting equivalent to the Ward-Perron-Stieltjes one, while the relationship between the Ward-Perron-Stieltjes and the Lebesgue-Stieltjes integrals has been described in [16], Theorem VI.8.1. For more details, see Chapter 6 of [13]. Young integrals are discussed in detail in Section II.19 of the monograph [6] by Hildebrandt. Obviously, they are more general than the corresponding Riemann-Stieltjes integrals. The relationship between the Young and the Dushnik integrals is indicated by MacNerney (see [11], Theorem B). Finally, for scalar functions the relationship between the Young and the Kurzweil-Stieltjes integrals was considered in [18] and [17]. In particular, it was shown there that if $f\colon [a,b] \to \mathbb{R}$ is regulated and $g\colon [a,b] \to \mathbb{R}$ has a bounded variation, then the Young integral of f with respect to g on $[a,b]$ exists and coincides with the Kurzweil-Stieltjes integral of f with respect to g on $[a,b]$ (cf. Schwabik [18] and [17]).

Further, it is known that integration processes based on Riemann type sums, such as the Kurzweil integral, can be extended to Banach space-valued functions. Among other contributions it is worth highlighting the monograph by Schwabik and Ye (see [25]), which studies these types of integrals and their connections e.g. with the classical ones due to Bochner and Pettis. Concerning integrals of Stieltjes type, Hönig presented a quite complete study in [7] dealing with the Dushnik integral. In [20] and [23] Schwabik investigated the fundamental properties of the Kurzweil-Stieltjes integration in abstract spaces, although in those papers he called this integral "abstract Perron-Stieltjes integral". Some results regarding integral equations and generalized linear differential equations in Banach spaces involving the Kurzweil-Stieltjes integral can be found e.g. in [4], [3], [14], [21], and [22]. Moreover, Monteiro and Tvrdý in [14] extended the results obtained by Schwabik and completed the theory so that it was well applicable to proving results on the continuous dependence of solutions to generalized linear differential equations in a Banach space (see [15]).

The aim of this paper is to complete this schedule in an abstract setting. In addition, we will present also convergence results that are possibly new though not surprising. Let us emphasize that the proofs of all the assertions presented in this paper are based on rather elementary tools.

2. Preliminaries

The symbols like $\mathbb{R}$, $\mathbb{N}$, $[a,b]$, and (a,b) have their usual and traditional meaning. For a subset M of $[a,b]$, the symbol χ_M denotes, as usual, its characteristic function, i.e. $\chi_M(t) = 1$ if $t \in M$ and $\chi_M(t) = 0$ if $t \notin M$.

Recall that a finite sequence $\boldsymbol{\alpha} = \{\alpha_0, \ldots, \alpha_{\nu(\boldsymbol{\alpha})}\}$ of points from $[a,b]$ is a *division* of $[a,b]$ if $a = \alpha_0 < \ldots < \alpha_{\nu(\boldsymbol{\alpha})} = b$. The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$. The couple $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ is a *tagged partition* of $[a,b]$ if $\boldsymbol{\alpha}$ is a division of $[a,b]$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\nu(P)})$ is such that $\xi_j \in [\alpha_{j-1}, \alpha_j]$ for all j. If $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ is a partition of $[a,b]$, the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\xi}$ are, respectively, denoted as α_j and ξ_j, while $\nu(P) = \nu(\boldsymbol{\alpha})$ is the number of the subintervals $[\alpha_{j-1}, \alpha_j]$ generated by the division $\boldsymbol{\alpha}$.

Let X be a Banach space. For any function $f\colon [a,b] \to X$ we set

$$\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|_X$$

and the *variation* $\operatorname{var}_a^b f$ of f over $[a,b]$ is given by

$$\operatorname{var}_a^b f = \sup_{\boldsymbol{\alpha} \in \mathcal{D}[a,b]} \sum_{j=1}^{\nu(\boldsymbol{\alpha})} \|f(\alpha_j) - f(\alpha_{j-1})\|_X.$$

If $\operatorname{var}_a^b f < \infty$ we say that f has a *bounded variation* on $[a,b]$. $\mathrm{BV}([a,b],X)$ denotes the Banach space of all functions $f\colon [a,b] \to X$ of bounded variation on $[a,b]$ equipped with the norm $\|f\|_{\mathrm{BV}} = \|f(a)\|_X + \operatorname{var}_a^b f$.

Recall that a function $f\colon [a,b] \to X$ is *regulated* on $[a,b]$ if it has one sided limits

$$\lim_{\tau \to t-} f(\tau) = f(t-) \in X \quad \text{and} \quad \lim_{\tau \to s+} f(\tau) = f(s+) \in X$$

for all $t \in (a,b]$ and $s \in [a,b)$. For every function f regulated on $[a,b]$ and points $t \in (a,b]$ and $s \in [a,b)$, we denote

$$\Delta^- f(t) = f(t) - f(t-) \quad \text{and} \quad \Delta^+ f(s) = f(s+) - f(s).$$

The set of all functions regulated on $[a,b]$ having values in X is denoted by $G([a,b],X)$.

Furthermore, a function $f\colon [a,b] \to X$ is a *finite step function* if there exist an $m \in \mathbb{N}$ and a division $\{s_0, \ldots, s_m\}$ of $[a,b]$ such that f is constant on every subinterval (s_{k-1}, s_k) for $k \in \{1, \ldots, m\}$. Equivalently, if

$$(2.1) \qquad f(t) = c + \sum_{k=0}^{m-1} \chi_{(s_k,b]}(t) c_k + \sum_{k=1}^{m} \chi_{[s_k,b]}(t) d_k + \chi_{[b]}(t) d_m \quad \text{for } t \in [a,b],$$

where c, c_k for $k \in \{0, \ldots, m-1\}$, and d_k for $k \in \{1, \ldots, m\}$ can be arbitrary elements of X and for $\tau = b$ by $\chi_{[\tau,b]}$ we understand the characteristic function of the one point set $[b] := \{b\}$.

It is known (cf. e.g. [7], Theorem 3.1) that $f\colon [a,b]\to X$ is regulated if and only if it is the uniform limit of finite step functions.

If X, Y and Z are Banach spaces, then, as usual, the symbols $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_Z$ stand for the norms in X, Y, Z, respectively. If there is a nontrivial (i.e. not identically zero) continuous bilinear mapping $B\colon X\times Y\to Z$ continuous in the sense that the inequality $\|B(x,y)\|_Z \leqslant \|x\|_X\|y\|_Y$ holds for all $x\in X$ and $y\in Y$, we say that the triple X, Y, Z is a *bilinear triple* with respect to B. In such a case, we will use the abbreviation xy instead of $B(x,y)$.

If $\mathcal{B}=(X,Y,Z)$ is a bilinear triple, then for functions $f\colon [a,b]\to X$, $g\colon [a,b]\to Y$ and a partition $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$, we set

$$S(f,\mathrm{d}g,P)=\sum_{j=1}^{\nu(P)} f(\xi_j)[g(\alpha_j)-g(\alpha_{j-1})],$$
$$S(\mathrm{d}f,g,P)=\sum_{j=1}^{\nu(P)} [f(\alpha_j)-f(\alpha_{j-1})]g(\xi_j)$$

and

$$\begin{aligned} S_Y(f,\mathrm{d}g,P)&=\sum_{j=1}^{\nu(P)}\big(f(\alpha_{j-1})\Delta^+g(\alpha_{j-1})+f(\xi_j)[g(\alpha_j-)-g(\alpha_{j-1}+)]\\ &\qquad\qquad+f(\alpha_j)\Delta^-g(\alpha_j)\big)\quad\text{if } g \text{ is regulated}\\ S_Y(\mathrm{d}f,g,P)&=\sum_{j=1}^{\nu(P)}\big(\Delta^+f(\alpha_{j-1})g(\alpha_{j-1})+[f(\alpha_j-)-f(\alpha_{j-1}+)]g(\xi_j)\\ &\qquad\qquad+\Delta^-f(\alpha_j)g(\alpha_j)\big)\quad\text{if } f \text{ is regulated,}\end{aligned}$$

and define:

(i) The Young integral (Y) $\int_a^b f\,\mathrm{d}g$ (the Dushnik integral (D) $\int_a^b f\,\mathrm{d}g$) exists and equals $I\in Z$ if for every $\varepsilon>0$ there is a division $\boldsymbol{\alpha}_\varepsilon$ of $[a,b]$ such that

$$\|S_Y(f,\mathrm{d}g,P)-I\|_Z<\varepsilon\quad(\text{or } \|S(f,\mathrm{d}g,P)-I\|_Z<\varepsilon)$$

holds for all partitions $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ such that $\boldsymbol{\alpha}\supset\boldsymbol{\alpha}_\varepsilon$ and $\alpha_{j-1}<\xi_j<\alpha_j$ for all $j\in\{1,\dots,\nu(\boldsymbol{\alpha})\}$.

(ii) The Kurzweil-Stieltjes integral (K) $\int_a^b f\,\mathrm{d}g$ exists and equals $I\in Z$ if for every $\varepsilon>0$ there exists a function $\delta_\varepsilon\colon [a,b]\to(0,1)$ such that

$$\|S(f,\mathrm{d}g,P)-I\|_Z<\varepsilon$$

holds for all partitions $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ such that

$$[\alpha_{j-1},\alpha_j]\subset[\xi_j-\delta_\varepsilon(\xi_j),\xi_j+\delta_\varepsilon(\xi_j)]\quad\text{for all } j\in\{1,\dots,\nu(\boldsymbol{\alpha})\}.$$

Analogously,

(i) The Young integral (Y) $\int_a^b \mathrm{d}f\,g$ (the Dushnik integral (D) $\int_a^b \mathrm{d}f\,g$) exists and equals $I\in Z$ if for every $\varepsilon>0$ there is a division $\boldsymbol{\alpha}_\varepsilon$ of $[a,b]$ such that

$$\|S_Y(\mathrm{d}f,g,P)-I\|_Z<\varepsilon\quad(\|S(\mathrm{d}f,g,P)-I\|_Z<\varepsilon)$$

holds for all partitions $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ such that $\boldsymbol{\alpha}\supset\boldsymbol{\alpha}_\varepsilon$ and $\alpha_{j-1}<\xi_j<\alpha_j$ for all $j\in\{1,\dots,\nu(\boldsymbol{\alpha})\}$.

(ii) The Kurzweil-Stieltjes integral (K) $\int_a^b \mathrm{d}f\,g$ exists and equals $I\in Z$ if for every $\varepsilon>0$ there exists a function $\delta_\varepsilon\colon[a,b]\to(0,1)$ such that

$$\|S(\mathrm{d}f,g,P)-I\|_Z<\varepsilon$$

holds for all partitions $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ such that

$$[\alpha_{j-1},\alpha_j]\subset[\xi_j-\delta_\varepsilon(\xi_j),\xi_j+\delta_\varepsilon(\xi_j)]\quad\text{for all } j\in\{1,\dots,\nu(\boldsymbol{\alpha})\}.$$

The integral sums of the form S_Y have been introduced by Young in [26]. However, he considered the corresponding integrals only in the norm sense. So, it seems that the above refinement type definition is due to Hildebrandt (cf. e.g. [6]). The Dushnik integral got its name due to the thesis [2] by Dushnik and Kurzweil introduced his integral in [9].

An arbitrary function δ defined and positive on $[a,b]$ is said to be a *gauge* on $[a,b]$. For an arbitrary gauge δ on $[a,b]$, any tagged division $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ such that $[\alpha_{j-1},\alpha_j]\subset[\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)]$ for all $j\in\{1,\dots,\nu(\boldsymbol{\alpha})\}$ is said to be δ-*fine*.

In what follows we will write more simply Kurzweil integral instead of Kurzweil-Stieltjes integral. Moreover, throughout the rest of the paper, we always assume that (X,Y,Z) is a bilinear triple with respect to some given and fixed nontrivial bilinear continuous mapping.

3. Main results

Our main goal is the following assertion.

Theorem 3.1. *Suppose that f and g are regulated on $[a,b]$ and at least one of them has a bounded variation on $[a,b]$. Then*

(i) *all the integrals* (K) $\int_a^b f\,\mathrm{d}g$, (Y) $\int_a^b f\,\mathrm{d}g$ *and* (D) $\int_a^b \mathrm{d}f\,g$ *exist and*

$$(\mathrm{K})\int_a^b f\,\mathrm{d}g = (\mathrm{Y})\int_a^b f\,\mathrm{d}g = f(b)g(b) - f(a)g(a) - (\mathrm{D})\int_a^b \mathrm{d}f\,g \tag{3.1}$$

and

(ii) *all the integrals* (K) $\int_a^b \mathrm{d}f\,g$, (Y) $\int_a^b \mathrm{d}f\,g$ *and* (D) $\int_a^b f\,\mathrm{d}g$ *exist and*

$$(\mathrm{K})\int_a^b \mathrm{d}f\,g = (\mathrm{Y})\int_a^b \mathrm{d}f\,g = f(b)g(b) - f(a)g(a) - (\mathrm{D})\int_a^b f\,\mathrm{d}g. \tag{3.2}$$

To prove Theorem 3.1, we will need several auxiliary results. First, we will consider some simple special cases.

Lemma 3.2. *Equalities* (3.1) *and* (3.2) *hold if* $f\colon [a,b]\to X$ *is a finite step function and* $g\colon [a,b]\to Y$ *is regulated or* $f\colon [a,b]\to X$ *is regulated and* $g\colon [a,b]\to Y$ *is a finite step function.*

P r o o f. a) First, assume that $g\in G([a,b],Y)$, $\widetilde{x}\in X$ and let the functions f_τ and f_σ be defined on $[a,b]$ by

$$f_\tau = \chi_{[\tau,b]}\widetilde{x} \quad \text{for } \tau\in[a,b] \quad \text{and} \quad f_\sigma = \chi_{(\sigma,b]}\widetilde{x} \quad \text{for } \sigma\in[a,b).$$

Obviously,

$$\begin{aligned} &(\mathrm{K})\int_a^b f_\tau\,\mathrm{d}g = (\mathrm{Y})\int_a^b f_\tau\,\mathrm{d}g = \widetilde{x}(g(b)-g(a)) && \text{if } \tau = a,\\ &(\mathrm{D})\int_a^b \mathrm{d}f_\tau\,g = 0 && \text{if } \tau = a. \end{aligned} \tag{3.3}$$

In addition, we claim that

$$\begin{aligned} &(\mathrm{K})\int_a^b f_\tau\,\mathrm{d}g = (\mathrm{Y})\int_a^b f_\tau\,\mathrm{d}g = \widetilde{x}(g(b)-g(\tau-)) && \text{if } \tau\in(a,b],\\ &(\mathrm{D})\int_a^b \mathrm{d}f_\tau\,g = \widetilde{x}g(\tau-) && \text{if } \tau\in(a,b],\\ &(\mathrm{K})\int_a^b f_\sigma\,\mathrm{d}g = (\mathrm{Y})\int_a^b f_\sigma\,\mathrm{d}g = \widetilde{x}(g(b)-g(\sigma+)) && \text{if } \sigma\in[a,b),\\ &(\mathrm{D})\int_a^b \mathrm{d}f_\sigma\,g = \widetilde{x}g(\sigma+) && \text{if } \sigma\in[a,b). \end{aligned} \tag{3.4}$$

Indeed, let $\tau \in (a,b)$ and let an arbitrary tagged partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of $[a,b]$ such that $\xi_j \in (\alpha_{j-1}, \alpha_j)$ for all $j \in \{1, \dots, \nu(P)\}$ be given. Without any loss of generality[1] we may assume that $\tau = \alpha_k$ for some $k \in \{1, \dots, \nu(P)-1\}$. Therefore

$$\begin{aligned} S_Y(f_\tau, \mathrm{d}g, P) &= \widetilde{x}\Big(\Delta^- g(\tau) + \sum_{j=k+1}^{\nu(\boldsymbol{\alpha})} [\Delta^+ g(\alpha_{j-1}) + g(\alpha_j-) - g(\alpha_{j-1}+) + \Delta^- g(\alpha_j)]\Big) \\ &= \widetilde{x}(g(b) - g(\tau-)). \end{aligned}$$

If $\tau = b$, then $S_Y(f_\tau, \mathrm{d}g, P) = \widetilde{x}(g(b) - g(b-))$ for each tagged partition P. To summarize, we have

$$(\mathrm{Y}) \int_a^b f_\tau \, \mathrm{d}g = \widetilde{x}(g(b) - g(\tau-)) \quad \text{for } \tau \in (a,b].$$

To determine the Dushnik integral $(\mathrm{D}) \int_a^b \mathrm{d}f_\tau \, g$, assume that $\tau \in (a,b]$ and $\varepsilon > 0$ are given and $\eta > 0$ is such that

$$\|g(t) - g(\tau-)\|_Y < \frac{\varepsilon}{\|\widetilde{x}\|_X} \quad \text{for all } t \in (\tau - \eta, \tau).$$

Let $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ be an arbitrary tagged partition of $[a,b]$ such that $\xi_j \in (\alpha_{j-1}, \alpha_j)$ for $j \in \{1, \dots, \nu(P)\}$ and $\tau = \alpha_k$ for some $k \in \{1, \dots, \nu(P)\}$. In addition, assume that $\tau - \eta < \alpha_{k-1} < \xi_k < \tau$. Then $S(\mathrm{d}f_\tau, g, P) = \widetilde{x} g(\xi_k)$ and

$$\|S(\mathrm{d}f_\tau, g, P) - \widetilde{x} g(\tau-)\|_Z = \|\widetilde{x}\|_X \|g(\xi_k) - g(\tau-)\|_Y < \varepsilon.$$

Hence,

$$(\mathrm{D}) \int_a^b \mathrm{d}f_\tau \, g = \widetilde{x} g(\tau-).$$

b) Again, let $\varepsilon > 0$ and $\tau \in (a,b)$ be given and let $\eta > 0$ be such that

$$|g(\tau-) - g(t)| < \frac{\varepsilon}{\|\widetilde{x}\|_X} \quad \text{for } t \in (\tau - \eta, \tau).$$

Define

$$\delta(t) = \begin{cases} \frac{1}{4}(\tau - t) & \text{if } t \in [a, \tau), \\ \eta & \text{if } t = \tau, \\ \frac{1}{4}(t - \tau) & \text{if } t \in (\tau, b]. \end{cases}$$

[1] Recall that both the Young and the Dushnik integrals are the refinement type integrals.

Then $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ is a δ-fine partition of $[a,b]$ only if $\tau=\xi_k$ for some $k\in\{1,\dots,\nu(P)\}$. In addition, we may assume that $\tau=\alpha_k$ and $\alpha_{k-1}\in(\tau-\eta,\tau)$. For any such δ-fine partition $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ we have

$$\begin{aligned}\|S(f_\tau,\mathrm{d}g,P)-\widetilde{x}[g(b)-g(\tau-)]\|_Z&=\|\widetilde{x}\|_X\left\|\sum_{j=k}^{\nu(P)}[g(\alpha_j)-g(\alpha_{j-1})]-[g(b)-g(\tau-)]\right\|_Y\\&=\|\widetilde{x}\|_X\|[g(b)-g(\alpha_{k-1})]-[g(b)-g(\tau-)]\|_Y\\&=\|\widetilde{x}\|_X\|g(\tau-)-g(\alpha_{k-1})\|_Y<\varepsilon.\end{aligned}$$

Therefore

$$(\mathrm{K})\int_a^b f_\tau\,\mathrm{d}g=\widetilde{x}(g(b)-g(\tau-)).$$

To summarize, the formulas on the first line of (3.4) are true. Similarly, we can justify also the relations given on the second line of (3.4).

Having in mind (3.3) and (3.4) we can verify that the formulas

$$(\mathrm{K})\int_a^b f_\tau\,\mathrm{d}g=(\mathrm{Y})\int_a^b f_\tau\,\mathrm{d}g=f_\tau(b)g(b)-f_\tau(a)g(a)-(\mathrm{D})\int_a^b \mathrm{d}f_\tau\,g$$

and

$$(\mathrm{K})\int_a^b f_\sigma\,\mathrm{d}g=(\mathrm{Y})\int_a^b f_\sigma\,\mathrm{d}g=f_\sigma(b)g(b)-f_\sigma(a)g(a)-(\mathrm{D})\int_a^b \mathrm{d}f_\sigma\,g$$

are true for $\tau\in[a,b]$ and $\sigma\in[a,b)$. Consequently, since by (2.1) every finite step function $f\colon[a,b]\to X$ is a finite linear combination of functions of the type $\{f_\tau,f_\sigma\}$, the relation (3.1) follows.

c) Now, assume that $f\colon[a,b]\to Y$, $\widetilde{y}\in Y$ and the functions g_τ and g_σ are defined on $[a,b]$ by

$$g_\tau=\chi_{[\tau,b]}\widetilde{y}\quad\text{for }\tau\in[a,b]\quad\text{and}\quad g_\sigma=\chi_{(\sigma,b]}\widetilde{y}\quad\text{for }\sigma\in[a,b).$$

Obviously the relations

$$\begin{aligned}&(\mathrm{K})\int_a^b f\,\mathrm{d}g_\tau=(\mathrm{Y})\int_a^b f\,\mathrm{d}g_\tau=0\quad\text{if }\tau=a,\\&(\mathrm{D})\int_a^b \mathrm{d}f\,g_\tau=(f(b)-f(a))\widetilde{y}\qquad\text{if }\tau=a\end{aligned}\tag{3.5}$$

are true for an arbitrary function $f\colon [a,b]\to X$. Moreover, analogously to part a) of this proof we can show that the relations

$$
\begin{aligned}
&(\mathrm{K})\int_a^b f\,\mathrm{d}g_\tau = (\mathrm{Y})\int_a^b f\,\mathrm{d}g_\tau = f(\tau)\widetilde{y} && \text{if } \tau\in(a,b],\\
&(\mathrm{D})\int_a^b \mathrm{d}f\,g_\tau = (f(b)-f(\tau))\widetilde{y} && \text{if } \tau\in(a,b],\\
&(\mathrm{K})\int_a^b f\,\mathrm{d}g_\sigma = (\mathrm{Y})\int_a^b f\,\mathrm{d}g_\sigma = f(\tau)\widetilde{y} && \text{if } \sigma\in[a,b),\\
&(\mathrm{D})\int_a^b \mathrm{d}f\,g_\sigma = (f(b)-f(\tau))\widetilde{y} && \text{if } \sigma\in[a,b)
\end{aligned}
\tag{3.6}
$$

are true, as well. Thus

$$(\mathrm{K})\int_a^b f\,\mathrm{d}g_\tau = (\mathrm{Y})\int_a^b f\,\mathrm{d}g_\tau = f(b)g_\tau(b)-f(a)g_\tau(a)-(\mathrm{D})\int_a^b \mathrm{d}f\,g_\tau$$

and

$$(\mathrm{K})\int_a^b f\,\mathrm{d}g_\sigma = (\mathrm{Y})\int_a^b f\,\mathrm{d}g_\sigma = f(b)g_\sigma(b)-f(a)g_\sigma(a)-(\mathrm{D})\int_a^b \mathrm{d}f\,g_\sigma,$$

wherefrom the relation (3.1) again follows.

d) The proof of relation (3.2) under the assumptions of the lemma is quite analogous and we believe that we can skip it. □

Estimates needed later are summarized in the following lemma.

Lemma 3.3. *Let $f\colon [a,b]\to X$, $g\colon [a,b]\to Y$ and a partition P of $[a,b]$ be given. Then the estimates*

$$
\begin{aligned}
&\|S(f,\mathrm{d}g,P)\|_Z \leqslant \|f\|_\infty \operatorname{var}_a^b g,\\
&\|S(\mathrm{d}f,g,P)\|_Z \leqslant (\operatorname{var}_a^b f)\|g\|_\infty,\\
&\|S(f,\mathrm{d}g,P)\|_Z \leqslant (\|f(a)\|_X+\|f(b)\|_X+\operatorname{var}_a^b f)\|g\|_\infty,\\
&\|S(\mathrm{d}f,g,P)\|_Z \leqslant \|f\|_\infty(\|g(a)\|_Y+\|g(b)\|_Y+\operatorname{var}_a^b g),
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&\|S_Y(f,\mathrm{d}g,P)\|_Z \leqslant \|f\|_\infty \operatorname{var}_a^b g,\\
&\|S_Y(f,\mathrm{d}g,P)\|_Z \leqslant (\|f(a)\|_X+\|f(b)\|_X+\operatorname{var}_a^b f)\|g\|_\infty
\end{aligned}
\tag{3.8}
$$

if g is regulated on $[a,b]$ and

$$
\begin{aligned}
&\|S_Y(\mathrm{d}f,g,P)\|_Z \leqslant (\operatorname{var}_a^b f)\|g\|_\infty,\\
&\|S_Y(\mathrm{d}f,g,P)\|_Z \leqslant \|f\|_\infty(\|g(a)\|_Y+\|g(b)\|_Y+\operatorname{var}_a^b g)
\end{aligned}
\tag{3.9}
$$

if f is regulated on $[a,b]$ are true.

Moreover, the estimates

$$\left\| \int_a^b f \, \mathrm{d}g \right\|_Z \leqslant \|f\|_\infty \operatorname{var}_a^b g, \tag{3.10}$$
$$\left\| \int_a^b f \, \mathrm{d}g \right\|_Z \leqslant (\|f(a)\|_X + \|f(b)\|_X + \operatorname{var}_a^b f)\|g\|_\infty$$

and

$$\left\| \int_a^b \mathrm{d}f \, g \right\|_Z \leqslant (\operatorname{var}_a^b f)\|g\|_\infty, \tag{3.11}$$
$$\left\| \int_a^b \mathrm{d}f \, g \right\|_Z \leqslant \|f\|_\infty(\|g(a)\|_Y + \|g(b)\|_Y + \operatorname{var}_a^b g)$$

hold for each of the three integrals under consideration, whenever it exists.

Proof. For the Kurzweil integral these inequalities are well-known, cf. [20], Proposition 10 and [14], Lemma 3.1. Since the set of admissible partitions for the Dushnik integral is contained in that for the Kurzweil integral, it follows immediately that relations (3.7), (3.10) and (3.11) are true also for the Dushnik integral. So, it remains to consider the Young integral. Assume that g is regulated.

a) If $a \leqslant \alpha \leqslant \xi \leqslant \beta \leqslant b$, then

$$\begin{aligned}
&\|f(\alpha)\Delta^- g(\alpha) + f(\xi)[g(\beta-) - g(\alpha+)] + f(\beta)\Delta^- g(\beta)\|_Z \\
&\qquad \leqslant \|f\|_\infty(\|\Delta^- g(\alpha)\|_Y + \|g(\beta-) - g(\alpha+)\|_Y + \|\Delta^- g(\beta)\|_Y) \leqslant \|f\|_\infty \operatorname{var}_a^b g,
\end{aligned}$$

wherefrom it is easy to deduce that the estimate

$$\|S_Y(f, \mathrm{d}g, P)\|_Z \leqslant \|f\|_\infty \operatorname{var}_a^b g$$

holds for every partition P of $[a,b]$. This means that the inequalities on the first lines of (3.8) and (3.10) are true also for the Young integral.

b) Observe that

$$\begin{aligned}
&f(\alpha)[g(\alpha+) - g(\alpha)] + f(\xi)[g(\beta-) - g(\alpha+)] + f(\beta)[g(\beta) - g(\beta-)] \\
&\qquad = [f(\alpha) - f(\xi)]g(\alpha+) + [f(\xi) - f(\beta)]g(\beta-) + f(\beta)g(\beta) - f(\alpha)g(\alpha)
\end{aligned}$$

holds for all $\alpha, \xi, \beta \in [a,b]$ such that $a \leqslant \alpha \leqslant \xi \leqslant \beta \leqslant b$. Having this in mind we can see that the estimate

$$\|S_Y(f, \mathrm{d}g, P)\|_Z \leqslant (\|f(a)\|_X + \|f(b)\|_X + \operatorname{var}_a^b f)\|g\|_\infty$$

is true for every partition P of $[a,b]$. Consequently, the second inequalities in (3.8) and (3.10) are true also for the Young integral.

Similarly we can verify the inequalities (3.9) and (3.11) for the Young integral when f is regulated. $\square$

The next convergence results are also true for all the three integrals under consideration. For the Kurzweil integral of scalar functions the proof is available e.g. in Chapter 6 of [13]. The idea is pretty transparent and, as we will see below, it is applicable also in the abstract situation including the Young and Dushnik integrals. First, we notice that in both situations the sequences of integrals depending on n are Cauchy sequences in the Banach space Z and therefore they have a limit $I \in Z$. Further, assumptions on the convergence of functions involved, the estimates given in Lemma 3.3 and the existence of the integrals $\int_a^b f_n \, \mathrm{d}g$ and/or $\int_a^b f \, \mathrm{d}g_n$ imply that the limit integrals exist and equal I. Frankly speaking, parts (ii) of Theorems 3.4–3.7 will not be needed later. They are included just for the sake of completeness of our convergence results.

Theorem 3.4. *Let $f, f_n\colon [a,b] \to X$ for $n \in \mathbb{N}$, and $g\colon [a,b] \to Y$ be such that the integrals $\int_a^b f_n \, \mathrm{d}g$ exist for all $n \in \mathbb{N}$. Suppose that at least one of the following conditions is satisfied:*

(i) *The sequence f_n converges on $[a,b]$ uniformly to f and g has a bounded variation on $[a,b]$.*

(ii) $\lim\limits_{n\to\infty} \|f_n - f\|_{\mathrm{BV}} = 0$ *and g is bounded on $[a,b]$.*

Then the integral $\int_a^b f \, \mathrm{d}g$ exists as well, and

$$\lim_{n\to\infty} \int_a^b f_n \, \mathrm{d}g = \int_a^b f \, \mathrm{d}g.$$

P r o o f. Both the integral and the sum symbols now may refer to any of those three integrals we are considering in this paper.

We claim that in both cases (i) and (ii) the sequence $\bigl\{\int_a^b f_n \, \mathrm{d}g\bigr\}$ satisfies the Cauchy condition. In case (i), we have by (3.10) from Lemma 3.3

$$\left\| \int_a^b f_n \, \mathrm{d}g - \int_a^b f_m \, \mathrm{d}g \right\|_Z = \left\| \int_a^b (f_n - f_m) \, \mathrm{d}g \right\|_Z \leqslant \|f_n - f_m\|_\infty \operatorname{var}_a^b g \tag{3.12}$$

for all $m, n \in \mathbb{N}$. Since $\operatorname{var}_a^b g$ is finite and $\{f_n\}$ is uniformly convergent, the right-hand side of (3.12) will be arbitrarily small if m, n are sufficiently large.

In case (ii), we use (3.10) from Lemma 3.3 to get

$$\begin{aligned}\left\| \int_a^b f_n \, \mathrm{d}g - \int_a^b f_m \, \mathrm{d}g \right\|_Z = \left\| \int_a^b (f_n - f_m) \, \mathrm{d}g \right\|_Z &\leqslant 2\|f_n - f_m\|_{\mathrm{BV}} \|g\|_\infty \\ &\leqslant 2(\|f_n - f\|_{\mathrm{BV}} + \|f - f_m\|_{\mathrm{BV}}) \|g\|_\infty\end{aligned}$$

for all $m, n \in \mathbb{N}$. Since $\|f - f_m\|_{\mathrm{BV}} \to 0$ for $m \to \infty$ and $\|f - f_n\|_{\mathrm{BV}} \to 0$ for $n \to \infty$ and $\|g\|_\infty$ is finite, the right-hand side of the last relation will become arbitrarily small for m, n sufficiently large.

Hence, in both cases, there exists $I \in Z$ such that

$$\lim_{n\to\infty} \int_a^b f_n \, \mathrm{d}g = I. \tag{3.13}$$

To show that $\int_a^b f \, \mathrm{d}g = I$, let $\varepsilon > 0$ be given. We claim that there exists an $n_1 \in \mathbb{N}$ such that

$$\|S(f - f_n, \mathrm{d}g, P)\|_Z < \varepsilon \tag{3.14}$$

if $n \geqslant n_1$ and P is an arbitrary partition of $[a, b]$.

In case (i), this follows from (3.7) in Lemma 3.3, which yields

$$\|S(f - f_n, \mathrm{d}g, P)\|_Z \leqslant \|f - f_n\|_\infty \operatorname{var}_a^b g.$$

In case (ii), we use (3.8) in Lemma 3.3 to get

$$\|S(f - f_n, \mathrm{d}g, P)\| \leqslant 2\|f - f_n\|_{\mathrm{BV}} \|g\|_\infty.$$

These estimates show the validity of (3.14). By (3.13), there exists an $n_0 \geqslant n_1$ such that

$$\left\| \int_a^b f_{n_0} \, \mathrm{d}g - I \right\|_Z < \varepsilon.$$

Now, in the case of the Kurzweil integral, we can choose a gauge δ_ε on $[a, b]$ such that

$$\left\| S(f_{n_0}, \mathrm{d}g, P) - \int_a^b f_{n_0} \, \mathrm{d}g \right\|_Z < \varepsilon \tag{3.15}$$

holds for each δ_ε-fine partition P of $[a, b]$.

Similarly, in the case of the Dushnik integral, we will choose a division $\boldsymbol{\alpha}_\varepsilon$ of $[a, b]$ such that (3.15) holds for each partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of $[a, b]$ such that $\boldsymbol{\alpha} \supset \boldsymbol{\alpha}_\varepsilon$ and $\xi_j \in (\alpha_{j-1}, \alpha_j)$ for all $j \in \{1, \ldots, \nu(P)\}$.

Finally, in the case of the Young integral, we can choose a division $\boldsymbol{\alpha}_\varepsilon$ of $[a, b]$ such that

$$\left\| S_Y(f_{n_0}, \mathrm{d}g, P) - \int_a^b f_{n_0} \, \mathrm{d}g \right\|_Z < \varepsilon$$

is true whenever $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ where $\boldsymbol{\alpha} \supset \boldsymbol{\alpha}_\varepsilon$ and $\xi_j \in (\alpha_{j-1}, \alpha_j)$ for all $j \in \{1, \ldots, \nu(P)\}$.

To summarize, for the Kurzweil integral the relations

$$\begin{aligned}&\|S(f, \mathrm{d}g, P) - I\|_Z\\ &\qquad \leqslant \|S(f - f_{n_0}, \mathrm{d}g, P)\|_Z + \|S(f - f_{n_0}, \mathrm{d}g, P) - I_{n_0}\|_Z + \|I_{n_0} - I\|_Z < 3\varepsilon\end{aligned}$$

are true for each δ_ε-fine partition P of $[a, b]$, in case of the Dushnik integral the inequality

$$\|S(f, \mathrm{d}g, P) - I\|_Z < 3\varepsilon$$

holds for each partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ such that $\boldsymbol{\alpha} \supset \boldsymbol{\alpha}_\varepsilon$ and $\xi_j \in (\alpha_{j-1}, \alpha_j)$ for all $j \in \{1, \ldots, \nu(P)\}$, and in the case of the Young integral we can see that the inequality

$$\|S_Y(f, \mathrm{d}g, P) - I\|_Z < 3\varepsilon$$

is true for each partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ such that $\boldsymbol{\alpha} \supset \boldsymbol{\alpha}_\varepsilon$ and $\xi_j \in (\alpha_{j-1}, \alpha_j)$ for all $j \in \{1, \ldots, \nu(P)\}$. Thus, $\int_a^b f \, \mathrm{d}g = I$ holds in any of the considered cases. The proof is complete. $\square$

The next assertions are complementary to Theorem 3.4. Their proofs are quite analogous to that of Theorem 3.4 and we leave them to the reader.

Theorem 3.5. *Let $f\colon [a, b] \to X$ and $g, g_n\colon [a, b] \to Y$ for $n \in \mathbb{N}$ be such that the integrals $\int_a^b f \, \mathrm{d}g_n$ exist for all $n \in \mathbb{N}$. Suppose that at least one of the following conditions is satisfied:*

(i) *$f \in \mathrm{BV}([a, b], X)$ and the sequence $\{g_n\}$ converges on $[a, b]$ uniformly to g.*
(ii) *f is bounded on $[a, b]$ and $\lim\limits_{n\to\infty} \operatorname{var}_a^b(g_n - g) = 0$.*

Then the integral $\int_a^b f \, \mathrm{d}g$ exists as well, and

$$\lim_{n\to\infty} \int_a^b f \, \mathrm{d}g_n = \int_a^b f \, \mathrm{d}g.$$

Theorem 3.6. *Let $f\colon [a, b] \to X$ and $g, g_n\colon [a, b] \to Y$ for $n \in \mathbb{N}$ be such that the integrals $\int_a^b \mathrm{d}f \, g_n$ exist for all $n \in \mathbb{N}$. Suppose that at least one of the following conditions is satisfied:*

(i) *$f \in \mathrm{BV}([a, b], Y)$ and the sequence g_n converges on $[a, b]$ uniformly to g.*
(ii) *f is bounded on $[a, b]$ and $\lim\limits_{n\to\infty} \|g_n - g\|_{\mathrm{BV}} = 0$.*

Then the integral $\int_a^b f \, \mathrm{d}g$ exists as well, and

$$\lim_{n\to\infty} \int_a^b \mathrm{d}f \, g_n = \int_a^b \mathrm{d}f \, g.$$

Theorem 3.7. *Let $f, f_n\colon [a,b] \to X$ for $n \in \mathbb{N}$ and $g\colon [a,b] \to Y$ be such that the integrals $\int_a^b \mathrm{d}f_n g$ exist for all $n \in \mathbb{N}$. Suppose that at least one of the following conditions is satisfied:*

(i) *The sequence $\{f_n\}$ converges on $[a,b]$ uniformly to f and g has a bounded variation on $[a,b]$.*

(ii) $\lim\limits_{n\to\infty} \operatorname{var}_a^b(f_n - f) = 0$ *and g is bounded on $[a,b]$.*

Then the integral $\int_a^b f\,\mathrm{d}g$ exists as well, and

$$\lim_{n\to\infty} \int_a^b \mathrm{d}f_n\, g = \int_a^b \mathrm{d}f\, g.$$

Now we are able to prove Theorem 3.1.

Proof of Theorem 3.1.

a) First, assume that $f \in G([a,b],X)$ and $g \in \mathrm{BV}([a,b],Y)$. Choose a sequence $\{f_n\}$ of finite step functions tending uniformly to f on $[a,b]$. Then by Lemma 3.2 we have

$$(\mathrm{K})\int_a^b f_n\,\mathrm{d}g = (\mathrm{Y})\int_a^b f_n\,\mathrm{d}g \quad \text{for all } n \in \mathbb{N}$$

and, further, by Theorem 3.4,

$$(\mathrm{K})\int_a^b f\,\mathrm{d}g = \lim_{n\to\infty}(\mathrm{K})\int_a^b f_n\,\mathrm{d}g = \lim_{n\to\infty}(\mathrm{Y})\int_a^b f_n\,\mathrm{d}g = (\mathrm{Y})\int_a^b f\,\mathrm{d}g.$$

For the Dushnik integral we have by Lemma 3.2 and Theorem 3.4

$$\begin{aligned}(\mathrm{D})\int_a^b f\,\mathrm{d}g &= \lim_{n\to\infty}(\mathrm{D})\int_a^b f_n\,\mathrm{d}g \\ &= \lim_{n\to\infty}\Big(f_n(b)g(b) - f_n(a)g(a) - (\mathrm{K})\int_a^b f_n\,\mathrm{d}g\Big) \\ &= f(b)g(b) - f(a)g(a) - (\mathrm{K})\int_a^b f\,\mathrm{d}g.\end{aligned}$$

Hence (3.1) is true.

b) Now, let $f \in \mathrm{BV}([a,b],X)$ and $g \in G([a,b],Y)$. Choose a sequence $\{g_n\}$ of finite step functions which converges uniformly on $[a,b]$ to g. By Lemma 3.2 we have

$$(\mathrm{K})\int_a^b f\,\mathrm{d}g_n = (\mathrm{Y})\int_a^b f\,\mathrm{d}g_n \quad \text{for all } n \in \mathbb{N}$$

and, further, by Theorem 3.5,

$$(\mathrm{K})\int_a^b f\,\mathrm{d}g=\lim_{n\to\infty}(\mathrm{K})\int_a^b f\,\mathrm{d}g_n=\lim_{n\to\infty}(\mathrm{Y})\int_a^b f\,\mathrm{d}g_n=(\mathrm{Y})\int_a^b f\,\mathrm{d}g.$$

Moreover, Lemma 3.2 and Theorem 3.5 also imply that the relations

$$\begin{aligned}(\mathrm{D})\int_a^b f\,\mathrm{d}g&=\lim_{n\to\infty}(\mathrm{D})\int_a^b f\,\mathrm{d}g_n\\&=\lim_{n\to\infty}\Big(f(b)g_n(b)-f(a)g_n(a)-(\mathrm{K})\int_a^b f\,\mathrm{d}g_n\Big)\\&=f(b)g(b)-f(a)g(a)-(\mathrm{K})\int_a^b f\,\mathrm{d}g\end{aligned}$$

are true. This completes the proof of (3.1).

c) Relation (3.2) can be proved in a similar way using Lemma 3.2 and Theorems 3.6 and 3.7. □

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