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# ON ROW-SUM MAJORIZATION

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*Abstract.* Let $\mathbb{M}_{n,m}$ be the set of all $n \times m$ real or complex matrices. For $A, B \in \mathbb{M}_{n,m}$, we say that $A$ is row-sum majorized by $B$ (written as $A \prec^{\mathrm{rs}} B$) if $R(A) \prec R(B)$, where $R(A)$ is the row sum vector of $A$ and $\prec$ is the classical majorization on $\mathbb{R}^n$. In the present paper, the structure of all linear operators $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ preserving or strongly preserving row-sum majorization is characterized. Also we consider the concepts of even and circulant majorization on $\mathbb{R}^n$ and then find the linear preservers of row-sum majorization of these relations on $\mathbb{M}_{n,m}$.

*Keywords*: majorization; linear preserver; doubly stochastic matrix

*MSC 2010*: 15A04, 15A21

## 1. Introduction

In the recent years, the concept of majorization has been widely applied to the related research areas of mathematics and statistics. Although this concept is rather old, it is an active field of research and recently many papers in this topic have been published, see for example [4], [5], [6], [7]. The following notation will be fixed throughout the paper: $\mathbb{M}_{n,m}$ is the set of all $n \times m$ matrices with entries in $\mathbb{R}$, $\mathbb{M}_n := \mathbb{M}_{n,n}$. The set of all $n \times 1$ column vectors is denoted by $\mathbb{R}^n$. The symbol $\mathbb{N}_k$ is used for the set $\{1, \ldots, k\}$. The symbol $e_i$ is the row (or column) vector with 1 as the $i$th component and 0 elsewhere. $E_{ij}$ is the $n \times m$ matrix whose $(i,j)$ entry is one and all other entries are zero. The summation of all components of a vector $x$ in $\mathbb{R}^n$ is denoted by $\operatorname{tr}(x)$. The symbol $[x_1 \mid x_2 \mid \ldots \mid x_m]$ is used for the $n \times m$ matrix whose columns are $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$. $A_{[i]} \in \mathbb{M}_{n-1}$ is the principal submatrix of $A \in \mathbb{M}_n$ obtained by deleting the $i$th row and the $i$th column of $A$. The letter $\mathbb{J}$ stands for the (rank-1) square matrix all of whose entries are 1. Let $\sim$ be a relation on $\mathbb{M}_{n,m}$. A linear operator $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ is said to be a *linear preserver* (or strong linear preserver) of $\sim$ if $TX \sim TY$ whenever $X \sim Y$ (or $TX \sim TY$ if and only if $X \sim Y$). For

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an $n \times m$ matrix $A$ we define the row sum vector $R(A) = (r_1(A), r_2(A), \ldots, r_n(A))^t$ where $r_i(A) = \sum_{j=1}^{m} a_{ij}$ for $i = 1, \ldots, n$. The set of all $n \times n$ permutation matrices is denoted by $\mathcal{P}_n$. A nonnegative matrix $R \in \mathbb{M}_{n,m}$ is called row stochastic if $\mathrm{Re} = e$ where $e = (1, 1, \ldots, 1)^t$. An $n \times n$ matrix $D$ is called doubly stochastic if both $D$ and $D^t$ are row stochastic. The collection of all $n \times n$ doubly stochastic matrices is denoted by $\mathbb{D}_n$.

A real matrix $D \in \mathbb{M}_n$ is called *even doubly stochastic* if it is a convex combination of even permutation matrices. Let $v = (v_1, \ldots, v_n)^t$ be a vector in $\mathbb{R}^n$. The *shift operator* $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by $T(v) = (v_n, v_1, \ldots, v_{n-1})^t$. The *circulant matrix associated to* $v$ is the matrix whose $k$th column is given by $T^{k-1}v$ $(k \in \mathbb{N}_n)$, where $T$ is the shift operator. If $v = e_i$ for some $1 \leqslant i \leqslant n$, then the circulant matrix associated to $v$ is a permutation matrix which is called a *circulant permutation matrix.* In the present paper $Q_1$ is the circulant permutation matrix associated to $e_2$ and $Q_k = Q_1^k$ for every $k \in \mathbb{N}_n$. Up to now, various kinds of majorization have been introduced. All types of majorization that we are dealing with in this paper are summarized in the following definition:

**Definition 1.1.** Let $x, y \in \mathbb{R}^n$. We define:

(1) classicalmajorization: $x \prec y$ if $x = Dy$ for some doubly stochastic matrix $D \in \mathbb{D}_n$;

(2) evenmajorization: $x \prec_e y$ if $x = Dy$ for some even doubly stochastic matrix $D \in \mathbb{D}_n$;

(3) circulantmajorization: $x \prec_c y$ if $x = Dy$ for some circulant doubly stochastic matrix $D \in \mathbb{D}_n$.

Now, we introduce the relation of row-sum majorization on $\mathbb{M}_{n,m}$ with respect to $\prec$, $\prec_e$ and $\prec_c$ as follows:

**Definition 1.2.** Let $A, B \in \mathbb{M}_{n,m}$. The matrix $A$ is said to be

(1) *row-sum majorized* by $B$ (denoted by $A \prec^{\mathrm{rs}} B$) if $R(A) \prec R(B)$;

(2) *row-sum evenmajorized* by $B$ (denoted by $A \prec_e^{\mathrm{rs}} B$) if $R(A) \prec_e R(B)$;

(3) *row-sum circulantmajorized* by $B$ (denoted by $A \prec_c^{\mathrm{rs}} B$) if $R(A) \prec_c R(B)$.

The main aim of this paper is to characterize all linear and strong linear preservers of row-sum majorization on $\mathbb{M}_{n,m}$ with respect to $\prec$, $\prec_e$ and $\prec_c$.

## 2. Row-sum majorization on $\mathbb{M}_{n,m}$

In this section we characterize all linear operators that preserve (or strongly preserve) row-sum majorization on $\mathbb{M}_{n,m}$. First, we state the statements that we need to prove the main results of this paper. The following elementary properties of classical majorization are proved in [3].

**Proposition 2.1.** *For all $x, y \in \mathbb{R}^n$ the following assertions are true.*

(1) *If $x \prec y$, then $\min y \leqslant \min x \leqslant \max x \leqslant \max y$, where $\max x$ and $\min x$ denote the maximum and minimum values of the components of a given real vector $x$, respectively.*

(2) *$x \prec y$ if and only if $(x, \alpha) \prec (y, \alpha)$, where $\alpha$ is the maximum value of the components of the vectors $x$ and $y$.*

In [1], Ando characterized all linear preservers of classical majorization on $\mathbb{R}^n$. In fact he proved the following proposition.

**Proposition 2.2** ([2], Corollary 2.7). *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then $T$ preserves classical majorization if and only if one of the following holds:*

(i) *$T(x) = \operatorname{tr}(x)a$ for some $a \in \mathbb{R}^n$.*

(ii) *$T(x) = \alpha Px + \beta \mathbb{J}x$ for some $\alpha, \beta \in \mathbb{R}$, and $P \in \mathcal{P}_n$.*

The following lemma that states an interesting fact about a vector $a \in \mathbb{R}^n$, is used to prove the main results.

**Lemma 2.3.** *Let $a = (a_1, \ldots, a_n)^t \in \mathbb{R}^n$ and $P \in \mathcal{P}_n$. Then $2a \prec a + Pa$ if and only if $Pa = a$.*

P r o o f. The sufficiency is obvious. We prove the necessity by induction on $n$. If $n = 1$, it is clear that $Pa = a$. If $n = 2$, then $P = I$ or $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The case $P = I$ is clear, so assume that $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $2a \prec a + Pa$, $\begin{pmatrix} 2a_1 \\ 2a_2 \end{pmatrix} \prec \begin{pmatrix} a_1+a_2 \\ a_1+a_2 \end{pmatrix}$, and by Proposition 2.1, $a_1 = a_2$ and hence $Pa = a$.

Now, suppose that $n \geqslant 2$, and that for $k = 1, 2, \ldots, n-1$, the assertion has been proved. Let $Pa = (a_{i_1}, a_{i_2}, \ldots, a_{i_n})$ and let $a_t = \max\{a_1, a_2, \ldots, a_n\}$. The maximum value of the vector $a$ is $a_t$, therefore $2a_t \geqslant a_k + a_{i_k}$, for every $k \in \mathbb{N}_n$. On the other hand $2a \prec a + Pa$ and by Proposition 2.1, $\max(2a) \leqslant \max(a + Pa)$. Therefore there exists $l \in \mathbb{N}_n$ such that $2a_t \leqslant a_l + a_{i_l}$. Since $a_t$ is the maximum value of $a$, we have $a_t = a_l = a_{i_l}$. Then without loss of generality we may assume that the $(l, l)$-entry of $P$ is 1. Put $a' = (a_1, a_2, \ldots, a_{l-1}, a_{l+1}, \ldots, a_n)^t \in \mathbb{R}^{n-1}$ and $P' = P_{[l]} \in \mathcal{P}_{n-1}$. Thus, $P'a' = (a_{i_1}, a_{i_2}, \ldots, a_{i_{l-1}}, a_{i_{l+1}}, \ldots, a_{i_n})$. Since $2a \prec a + Pa$, it follows from Proposition 2.1, that $2a' \prec a' + P'a'$, and by the induction hypothesis, $P'a' = a'$. Now, since the $(l, l)$-entry of $P$ is 1, we have $Pa = a$. $\square$

If $X \prec^{\mathrm{rs}} Y$ and $Y \prec^{\mathrm{rs}} X$ (or $x \prec y$ and $y \prec x$), then we write $X \sim_{\mathrm{rs}} Y$ (respectively $x \sim y$). It is well known that $x \sim y$ if and only if there exists a permutation matrix $P$ such that $x = Py$.

The following remark gives the relation between $R(TE_{ij})$ ($i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$) and $R(TE_{11})$ when $T$ is a linear preserver of row-sum majorization.

**Remark 2.4.** Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear preserver of row-sum majorization. It is clear that for every $i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$, $E_{ij} \sim_{\mathrm{rs}} E_{11}$. So $TE_{ij} \sim_{\mathrm{rs}} TE_{11}$, therefore $R(TE_{ij}) \sim R(TE_{11})$. Thus there exists $P_{ij} \in \mathcal{P}_n$ such that $R(TE_{ij}) = P_{ij}R(TE_{11})$.

Now, we state and prove the main theorem of this section.

**Theorem 2.5.** *Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ preserves row-sum majorization if and only if there exists $A \in \mathbb{M}_n$ such that $R(TX) = AR(X)$ for all $X \in \mathbb{M}_{n,m}$, and $A$ has one of the following forms:*

(i) *$A = (a \mid a \mid \ldots \mid a)$ for some $a \in \mathbb{R}^n$.*

(ii) *$A = \lambda P + \mu \mathbb{J}$ for some $\lambda, \mu \in \mathbb{R}$ and $P \in \mathcal{P}_n$.*

Proof. If there exists $A \in \mathbb{M}_n$ such that $R(TX) = AR(X)$ for all $X \in \mathbb{M}_{n,m}$, and $A$ has one of the forms (i) or (ii), then it is easy to see that $Ax \prec Ay$, whenever $x \prec y$, by Proposition 2.2. Now, let $X \prec^{\mathrm{rs}} Y$. Then $R(X) \prec R(Y)$, hence $AR(X) \prec AR(Y)$. Thus $R(TX) \prec R(TY)$, and therefore $T(X) \prec^{\mathrm{rs}} T(Y)$.

Conversely, assume that $T$ preserves row-sum majorization. Let $\alpha = R(TE_{11})$. Since for every $i$, $j$ ($i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$), $E_{ij} \sim_{\mathrm{rs}} E_{11}$ and $T$ preserves $\prec^{\mathrm{rs}}$, we have $R(TE_{ij}) = P_{ij}\alpha$ for some $P_{ij} \in \mathcal{P}_n$. So $R(TX) = \sum\limits_{i,j} x_{ij}P_{ij}\alpha$ where $P_{ij} \in \mathcal{P}_n$. Now, we show that for every $i \in \mathbb{N}_n$, $P_{ij}\alpha = P_{ik}\alpha$ for all $j,k \in \mathbb{N}_m$. Assume there exists $i \in \mathbb{N}_n$ such that $P_{ij}\alpha \neq P_{ik}\alpha$, for some $j,k \in \mathbb{N}_m$. Therefore $\alpha \neq P_{ij}^t P_{ik}\alpha$. Let $X = 2E_{ij}$ and $Y = E_{ij} + E_{ik}$. It is clear that $X \prec^{\mathrm{rs}} Y$, so we have $TX \prec^{\mathrm{rs}} TY$. It is clear that $R(TX) = 2P_{ij}\alpha$ and $R(TY) = P_{ij}\alpha + P_{ik}\alpha$. Then

$$T(X) \prec^{\mathrm{rs}} T(Y) \Rightarrow R(TX) \prec R(TY) \Rightarrow 2P_{ij}\alpha \prec P_{ij}\alpha + P_{ik}\alpha \Rightarrow 2\alpha \prec \alpha + P_{ij}^t P_{ik}\alpha,$$

and by Lemma 2.3, $P_{ij}^t P_{ik}\alpha = \alpha$, which is a contradiction. Therefore $P_{ij}\alpha = P_{ik}\alpha$ for every $i \in \mathbb{N}_n$ and $j,k \in \mathbb{N}_m$. Set $P_i = P_{i1}$, for every $i \in \mathbb{N}_n$. Therefore $P_i\alpha = P_{i1}\alpha = P_{ij}\alpha$ for every $j \in \mathbb{N}_m$. Thus,

$$R(TX) = \sum_{i,j} x_{ij}P_{ij}\alpha = \sum_{i=1}^{n} r_i(X)P_i\alpha.$$

Put $A = [P_1\alpha \mid P_2\alpha \mid \ldots \mid P_n\alpha]$ and hence $R(TX) = AR(X)$.

Now, define $S\colon \mathbb{R}^n \to \mathbb{R}^n$ by $S(x) = Ax$. First, we show that $S$ is a linear preserver of $\prec$. Let $x \prec y$. Then

$$\begin{aligned} x \prec y &\Rightarrow (x \mid 0 \mid 0 \mid \ldots \mid 0) \prec^{\mathrm{rs}} (y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Rightarrow T(x \mid 0 \mid 0 \mid \ldots \mid 0) \prec^{\mathrm{rs}} T(y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Rightarrow R(T(x \mid 0 \mid 0 \mid \ldots \mid 0)) \prec R(T(y \mid 0 \mid 0 \mid \ldots \mid 0)) \end{aligned}$$

$$\Rightarrow AR(x \mid 0 \mid 0 \mid \dots \mid 0) \prec AR(y \mid 0 \mid 0 \mid \dots \mid 0)$$
$$\Rightarrow Ax \prec Ay \Rightarrow S(x) \prec S(y).$$

Thus $S$ is a linear preserver of $\prec$ and hence by Proposition 2.2, $A$ is of the form (i) or (ii). $\square$

**Corollary 2.6.** *Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ preserves row-sum majorization if and only if $R(TE_{ij})$, for every $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$, has one of the following forms:*
(i) *$R(TE_{ij}) = a$ for some $a \in \mathbb{R}^n$.*
(ii) *$R(TE_{ij}) = (\alpha P + \beta \mathbb{J})e_i$ for some $\alpha, \beta \in \mathbb{R}$ and $P \in \mathcal{P}_n$.*

Here, we characterize all linear operators on $\mathbb{M}_{n,m}$ that strongly preserve row-sum majorization. The following example shows that not every linear operator that strongly preserves $\prec^{\mathrm{rs}}$, is necessarily invertible.

**Example 2.7.** Let $m \geqslant 2$ and let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be defined by $T(X) = X\mathbb{J}$. Let $X, Y \in \mathbb{M}_{n,m}$. Therefore

$$T(X) = X\mathbb{J} = \begin{pmatrix} r_1(X) & \dots & r_1(X) \\ r_2(X) & \dots & r_2(X) \\ \vdots & & \vdots \\ r_n(X) & \dots & r_n(X) \end{pmatrix},$$

and hence $R(TX) = nR(X)$. Assume that $X \prec^{\mathrm{rs}} Y$, then

$$X \prec^{\mathrm{rs}} Y \Leftrightarrow R(X) \prec R(Y) \Leftrightarrow nR(X) \prec nR(Y)$$
$$\Leftrightarrow R(TX) \prec R(TY) \Leftrightarrow T(X) \prec^{\mathrm{rs}} T(Y).$$

Then $T$ strongly preserves $\prec^{\mathrm{rs}}$. Since $m \geqslant 2$, $T(X) = 0$ for some $X \neq 0$, which implies that $T$ is not invertible.

In [1], [4], the authors characterized the structure of all strong linear preservers of $\prec$ on $\mathbb{R}^n$ as follows:

**Proposition 2.8.** *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then $T$ strongly preserves classical majorization if and only if there exist $\alpha, \beta \in \mathbb{R}$, and a permutation matrix $P \in \mathcal{P}_n$ such that $T(x) = \alpha Px + \beta \mathbb{J}x$ for all $x \in \mathbb{R}^n$ and $\alpha(\alpha + n\beta) \neq 0$.*

**Theorem 2.9.** *Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ strongly preserves row-sum majorization if and only if there exist $\alpha, \beta \in \mathbb{R}$, $P \in \mathcal{P}_n$ such that $\alpha(\alpha + n\beta) \neq 0$ and $R(TX) = (\alpha P + \beta \mathbb{J})R(X)$ for all $X \in \mathbb{M}_{n,m}$.*

P r o o f. If $R(TX) = (\alpha P + \beta \mathbb{J})R(X)$ for some permutation matrix $P \in \mathcal{P}_n$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha(\alpha + n\beta) \neq 0$, then it is clear that $T$ is a strong linear preserver of row-sum majorization by Proposition 2.8. Conversely, let $T$ strongly preserve row-sum majorization. By Theorem 2.5, there exists $A \in \mathbb{M}_n$ such that $R(TX) = AR(X)$. Now, define $S\colon \mathbb{R}^n \to \mathbb{R}^n$ by $S(x) = Ax$. First we show that $S$ is a strong linear preserver of $\prec$. Let $x \prec y$. Then we have

$$\begin{aligned} x \prec y &\Leftrightarrow (x \mid 0 \mid 0 \mid \ldots \mid 0) \prec^{\rm rs} (y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Leftrightarrow T(x \mid 0 \mid 0 \mid \ldots \mid 0) \prec^{\rm rs} T(y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Leftrightarrow R(T(x \mid 0 \mid 0 \mid \ldots \mid 0)) \prec R(T(y \mid 0 \mid 0 \mid \ldots \mid 0)) \\ &\Leftrightarrow AR(x \mid 0 \mid 0 \mid \ldots \mid 0) \prec AR(y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Leftrightarrow Ax \prec Ay \Leftrightarrow S(x) \prec S(y). \end{aligned}$$

Thus $S$ is a strong linear preserver of $\prec$ and by Proposition 2.8, $A = \alpha P + \beta \mathbb{J}$ for some $P \in \mathcal{P}_n$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha(\alpha + n\beta) \neq 0$. □

## 3. Row-sum even and circulant majorization

In this section we consider the concepts of even and circulant majorization on $\mathbb{R}^n$ and then we characterize the linear and strong linear preservers of row-sum majorization of these relations on $\mathbb{M}_{n,m}$. Here, we give some necessary prerequisites. The following lemma gives the relation between two vectors $x, y \in \mathbb{R}^n$ when $x \sim_e y$. This lemma is used to find the structure of linear preservers of $\prec_e^{\rm rs}$.

**Lemma 3.1.** *Let $x, y \in \mathbb{R}^n$ and $x \sim_e y$. Then there exists an even permutation matrix $P \in \mathcal{P}_n$ such that $x = Py$.*

P r o o f. Let $x, y \in \mathbb{R}^n$ and $x \sim_e y$. It is clear that even majorization implies multivariate majorization, so we have $x \sim y$. Therefore there exists a permutation matrix $P \in \mathcal{P}_n$ such that $x = Py$. On the other hand $x \prec_e y$, and by the definition of even majorization there exists an even doubly stochastic matrix $D$ such that $x = Dy$. Since $D$ is even doubly stochastic, there exist even permutations $P_{i_1}, \ldots, P_{i_k} \in \mathcal{P}_n$ and scalars $c_1, \ldots, c_k \in \mathbb{R}^+$ such that $\sum\limits_{i=1}^{k} c_j = 1$ and $D = \sum\limits_{i=1}^{k} c_j P_{i_j}$. Thus, $x = \sum\limits_{i=1}^{k} c_j P_{i_j} y$. Without loss of generality assume that $c_1 \neq 0$. If $c_1 = 1$, then $x = P_{i_1} y$ and the assertion holds trivially. If $c_1 \neq 1$, put

$$z = \frac{1}{1 - c_1} \sum_{j=2}^{k} c_j P_{i_j} y$$

and hence $Py = c_1 P_{i_1} y + (1 - c_1)z$. By the triangle inequality we have

$$\|y\| = \|Py\| \leqslant c_1\|y\| + (1 - c_1)\|z\| \leqslant \|y\|.$$

Since equality occures in the triangle inequality, there is a scalar $\alpha$ such that $z = \alpha P_{i_1} y$. Thus, $Py = (c_1 + \alpha(1 - c_1))P_{i_1} y$ and this implies that $\alpha = 1$. Therefore $x = P_{i_1} y$, as desired. $\square$

We use the following matrices throughout this section:

$$P_1 = I, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_1' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$P_2' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_3' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here, we mention the structure of all linear preservers of $\prec_e$ on $\mathbb{R}^n$.

**Proposition 3.2** ([2], Theorem 2.2)**.** *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and let $n \geqslant 4$. Then $T$ preserves even majorization if and only if one of the following holds:*
(i) *$T(x) = \mathrm{tr}(x)a$ for some $a \in \mathbb{R}^n$.*
(ii) *$T(x) = \alpha Px + \beta \mathbb{J}x$ for some $\alpha, \beta \in \mathbb{R}$, and a permutation matrix $P \in \mathcal{P}_n$.*

**Proposition 3.3** ([2], Proposition 2.4)**.** *Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator. Then $T$ preserves even majorization if and only if one of the following holds:*
(i) *$T(x) = \mathrm{tr}(x)a$ for some $a \in \mathbb{R}^3$.*
(ii) *$T(x) = \alpha P_1 x + \beta P_2 x + \gamma P_3 x$ for some $\alpha, \beta, \gamma \in \mathbb{R}$.*
(iii) *$T(x) = \alpha P_1' x + \beta P_2' x + \gamma P_3' x$ for some $\alpha, \beta, \gamma \in \mathbb{R}$.*

The next theorem characterizes the structure of all linear operators $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ preserving $\prec_e^{\mathrm{rs}}$.

**Theorem 3.4.** *Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ preserves row-sum even majorization if and only if there exists $A \in \mathbb{M}_n$ such that $R(TX) = AR(X)$ for all $X \in \mathbb{M}_{n,m}$, and one of the following statements holds:*
(a) *$n \geqslant 4$, and $A$ has one of the following forms:*
  (i) *$A = (a \mid a \mid \ldots \mid a)$ for some $a \in \mathbb{R}^n$,*
  (ii) *$A = \lambda P + \mu J$ for some $\lambda, \mu \in \mathbb{R}$ and $P \in \mathcal{P}_n$,*
(b) *$n = 3$, and $A$ has one of the following forms:*
  (i) *$A = (a \mid a \mid \ldots \mid a)$ for some $a \in \mathbb{R}^n$,*
  (ii) *$A = \alpha P_1 + \beta P_2 + \gamma P_3$ for some $\alpha, \beta, \gamma \in \mathbb{R}$,*
  (iii) *$A = \alpha P_1' x + \beta P_2' x + \gamma P_3'$ for some $\alpha, \beta, \gamma \in \mathbb{R}$.*

P r o o f. First, assume that $n \geqslant 4$ (or $n = 3$), and there exists $A \in \mathbb{M}_n$ such that $R(TX) = AR(X)$ for all $X \in \mathbb{M}_{n,m}$ and $A$ has one of the forms (i) or (ii) of the case (a) (respectively (i) or (ii) or (iii) of the case (b)). Now, let $X \prec_e^{\mathrm{rs}} Y$. Then $R(X) \prec_e R(Y)$, and it is clear that $AR(X) \prec_e AR(Y)$ by Proposition 3.2 (or Proposition 3.3). Thus $R(TX) \prec_e R(TY)$ and therefore $T(X) \prec_e^{\mathrm{rs}} T(Y)$.

Conversely, assume that $T$ preserves row-sum even majorization and let $\alpha = R(TE_{11})$. It is clear that for every $i$, $j$ ($i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$), $R(E_{ij}) \sim_e R(E_{11})$, so $R(TE_{ij}) = P_{ij}\alpha$, for some even permutation $P_{ij} \in \mathcal{P}_n$, by Lemma 3.1. Thus $R(TX) = \sum\limits_{i,j} x_{ij}P_{ij}\alpha$. We claim that for every $i \in \mathbb{N}_n$, $P_{ij}\alpha = P_{ik}\alpha$ for all $j, k \in \mathbb{N}_m$. Assume there exists $i \in \mathbb{N}_n$ such that $P_{ij}\alpha \neq P_{ik}\alpha$, for some $j, k \in \mathbb{N}_m$. Therefore $\alpha \neq P_{ij}^t P_{ik}\alpha$. Let $X = 2E_{ij}$ and $Y = E_{ij} + E_{ik}$. It is clear that $X \prec_e^{\mathrm{rs}} Y$, so we have $TX \prec_e^{\mathrm{rs}} TY$. Then

$$\begin{aligned} TX \prec_e^{\mathrm{rs}} TY &\Rightarrow R(TX) \prec_e R(TY) \Rightarrow 2P_{ij}\alpha \prec_e P_{ij}\alpha + P_{ik}\alpha \\ &\Rightarrow 2\alpha \prec_e \alpha + P_{ij}^t P_{ik}\alpha \Rightarrow 2\alpha \prec \alpha + P_{ij}^t P_{ik}\alpha, \end{aligned}$$

and by Lemma 2.3, $P_{ij}^t P_{ik}\alpha = \alpha$, which is a contradiction. So $P_{ij}\alpha = P_{ik}\alpha$ for every $i \in \mathbb{N}_n$ and $j, k \in \mathbb{N}_m$. Set $P_i = P_{i1}$ for every $i \in \mathbb{N}_n$. So $P_i\alpha = P_{i1}\alpha = P_{ij}\alpha$ for every $j \in \mathbb{N}_m$ and hence,

$$R(TX) = \sum_{i,j} x_{ij}P_{ij}\alpha = \sum_{i=1}^{n} r_i(X)P_i\alpha.$$

Put $A = [P_1\alpha \mid P_2\alpha \mid \ldots \mid P_n\alpha]$, therefore $R(TX) = AR(X)$. Now, define $S\colon \mathbb{R}^n \to \mathbb{R}^n$ by $S(x) = Ax$. Now we show that $S$ is a linear preserver of $\prec_e$. Let $x \prec_e y$. Then

$$\begin{aligned} x \prec_e y &\Rightarrow (x \mid 0 \mid 0 \mid \ldots \mid 0) \prec_e^{\mathrm{rs}} (y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Rightarrow T(x \mid 0 \mid 0 \mid \ldots \mid 0) \prec_e^{\mathrm{rs}} T(y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Rightarrow R(T(x \mid 0 \mid 0 \mid \ldots \mid 0)) \prec_e R(T(y \mid 0 \mid 0 \mid \ldots \mid 0)) \\ &\Rightarrow AR(x \mid 0 \mid 0 \mid \ldots \mid 0) \prec_e AR(y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Rightarrow Ax \prec_e Ay \Rightarrow S(x) \prec_e S(y). \end{aligned}$$

Thus $S$ is a linear preserver of $\prec_e$ and by Proposition 3.2 (or Proposition 3.3) for $n \geqslant 4$ (or $n = 3$), the case (a) (or (b)) holds and $A$ is of the form (i) or (ii) (respectively (i) or (ii) or (iii)), as desired. □

**Corollary 3.5.** *Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ preserves row-sum even majorization if and only if for every $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$, $R(T(E_{ij}))$ has one of the following forms:*

(a) $n \geqslant 4$, *and one of the following holds:*

  (i) $R(TE_{ij}) = a$ *for some* $a \in \mathbb{R}^n$,

  (ii) $R(TE_{ij}) = (\alpha P + \beta \mathbb{J})e_i$ *for some* $\alpha, \beta \in \mathbb{R}$ *and* $P \in \mathcal{P}_n$,

(b) $n = 3$, *and one of the following holds:*

  (i) $R(TE_{ij}) = a$ *for some* $a \in \mathbb{R}^n$,

  (ii) $R(TE_{ij}) = \alpha P_1 + \beta P_2 + \gamma P_3$ *for some* $\alpha, \beta, \gamma \in \mathbb{R}$,

  (iii) $R(TE_{ij}) = \alpha P_1' x + \beta P_2' x + \gamma P_3'$ *for some* $\alpha, \beta, \gamma \in \mathbb{R}$.

The linear operators strongly preserving $\prec_e$ on $\mathbb{R}^n$ have been characterized as follows:

**Proposition 3.6** ([2], Theorem 2.4). *Let* $T\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a linear operator and let* $n \geqslant 4$. *Then* $T$ *strongly preserves even majorization if and only if* $T$ *has the form* $T(x) = \alpha P x + \beta \mathbb{J} x$, *where* $P \in \mathcal{P}_n$ *and* $\alpha, \beta \in \mathbb{R}$ *are such that* $\alpha(\alpha + n\beta) \neq 0$.

**Proposition 3.7** ([2], Theorem 2.4). *Let* $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ *be a linear operator. Then* $T$ *strongly preserves even majorization if and only one of the following holds:*

(i) $T(x) = \alpha P_1 x + \beta P_2 x + \gamma P_3 x$,

(ii) $T(x) = \alpha P_1' x + \beta P_2' x + \gamma P_3' x$,

*where* $\alpha, \beta, \gamma \in \mathbb{R}$ *with* $\alpha^3 + \beta^3 + \gamma^3 \neq 3\alpha\beta\gamma$.

The following example shows that in general a linear operator that strongly preserves $\prec_e^{\mathrm{rs}}$ is not invertible.

**Example 3.8.** Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be defined by $T(X) = X\mathbb{J}$. Then $T$ strongly preserves row-sum even majorization but is not invertible.

**Theorem 3.9.** *Let* $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ *be a linear operator. Then* $T$ *strongly preserves row-sum even majorization if and only if there exists* $A \in \mathbb{M}_n$ *such that* $R(TX) = AR(X)$ *for all* $X \in \mathbb{M}_{n,m}$, *and one of the following holds:*

(i) $n \geqslant 4$, *and* $A = \alpha P + \beta \mathbb{J}$ *for some permutation matrix* $P \in \mathcal{P}_n$ *and* $\alpha, \beta \in \mathbb{R}$;

(ii) $n = 3$, *and* $A = \alpha P_1 + \beta P_2 + \gamma P_3$ *or* $A = \alpha P_1' + \beta P_2' + \gamma P_3'$ *for some* $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $\alpha^3 + \beta^3 + \gamma^3 \neq 3\alpha\beta\gamma$.

Proof. The sufficiency is clear by Propositions 3.6 and 3.7. So, we only prove the necessity. Let $T$ strongly preserve row-sum even majorization. By Theorem 3.4, there exist $A \in \mathbb{M}_n$, such that $R(TX) = AR(X)$. Now, define $S\colon \mathbb{R}^n \to \mathbb{R}^n$ by $S(x) = Ax$. Here, we show that $S$ is a strong linear preserver of $\prec_e$. Let $x \prec_e y$. Then

$$\begin{aligned} x \prec_e y &\Leftrightarrow (x \mid 0 \mid 0 \mid \ldots \mid 0) \prec_e^{\mathrm{rs}} (y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Leftrightarrow T(x \mid 0 \mid 0 \mid \ldots \mid 0) \prec_e^{\mathrm{rs}} T(y \mid 0 \mid 0 \mid \ldots \mid 0) \\ &\Leftrightarrow R(T(x \mid 0 \mid 0 \mid \ldots \mid 0)) \prec_e R(T(y \mid 0 \mid 0 \mid \ldots \mid 0)) \end{aligned}$$

$$\Leftrightarrow AR(x \mid 0 \mid 0 \mid \ldots \mid 0) \prec_e AR(y \mid 0 \mid 0 \mid \ldots \mid 0)$$
$$\Leftrightarrow Ax \prec_e Ay \Leftrightarrow S(x) \prec_e S(y).$$

Thus $S$ is a strong linear preserver of $\prec_e$. If $n \geqslant 4$, by Proposition 3.6, $A$ is of the form (i). If $n = 3$, by Proposition 3.7, $A$ is of the form (ii), as desired. □

In the remainder of the paper we consider the circulant majorization and we find the linear operator preserving row-sum circulant majorization on $\mathbb{M}_{n,m}$. The following proposition characterizes the linear operators preserving circulant majorization on $\mathbb{R}^n$.

**Proposition 3.10** ([2], Theorem 2.6)**.** *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then $T$ preserves circulant majorization if and only if there exist a vector $a \in \mathbb{R}^n$ and an integer $k \in \mathbb{N}_n$, such that $Tx = (a \mid Q_k a \mid \ldots \mid Q_k^{n-1} a)x$ for all $x \in \mathbb{R}^n$.*

**Lemma 3.11** ([2], Lemma 2.1)**.** *Let $x, y \in \mathbb{R}^n$ and $x \sim_c y$. Then there exists a circulant permutation matrix $P \in \mathcal{P}_n$ such that $x = Py$.*

By using Proposition 3.10, Lemma 3.11 and arguments similar to those used in the proofs of Theorems 3.4 and 3.9 we can prove the following results.

**Theorem 3.12.** *A linear operator $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ preserves row-sum circulant majorization if and only if there exist a vector $a \in \mathbb{R}^n$ and an integer $k \in \mathbb{N}_n$, such that for all $X \in \mathbb{M}_{n,m}$,*

$$R(TX) = (a \mid Q_k a \mid \ldots \mid Q_k^{n-1} a)R(X).$$

**Corollary 3.13.** *Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ preserves row-sum circulant majorization if and only if for every $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$, there exist a vector $a \in \mathbb{R}^n$ and an integer $k \in \mathbb{N}_n$, such that $R(T(E_{ij})) = (a \mid Q_k a \mid \ldots \mid Q_k^{n-1} a)e_i$.*

In the following theorems, $(\alpha_0, \ldots, \alpha_{n-1})^t$ is the first column of the matrix $A$ and $\varepsilon = \mathrm{e}^{2\pi \mathrm{i}/n}$. The strong linear preservers of $\prec_c$ and $\prec_c^{\mathrm{rs}}$ are as follows:

**Proposition 3.14** ([2], Theorem 2.8)**.** *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then $T$ strongly preserves circulant majorization if and only if there exists a matrix $A \in \mathbb{M}_n$ such that $Tx = Ax$ for all $x \in \mathbb{R}^n$, and $AQ_1 = Q_k A$ for some $k \in \mathbb{N}_n$ and $\prod_{l=0}^{n-1} \Bigl(\sum_{k=0}^{n-1} \varepsilon^{kl} \alpha_k\Bigr) \neq 0$ .*

By using Proposition 3.14 and applying the arguments used in the previous results we can obtain the following theorem.

**Theorem 3.15.** *Let $T\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ strongly preserves row-sum circulant majorization if and only if there exist a matrix $A \in \mathbb{M}_n$ and an integer $k \in \mathbb{N}_n$ such that $AQ_1 = Q_k A$ and $\prod_{l=0}^{n-1}\Bigl(\sum_{k=0}^{n-1} \varepsilon^{kl}\alpha_k\Bigr) \neq 0$ and $R(TX) = AR(X)$ for all $X \in \mathbb{M}_{n,m}$.*

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