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Δ -WEAK CHARACTER AMENABILITY OF CERTAIN BANACH ALGEBRAS

HAMID SADEGHI

ABSTRACT. In this paper we investigate Δ -weak character amenability of certain Banach algebras such as projective tensor product $A \widehat{\otimes} B$ and Lau product $A \times_{\theta} B$, where A and B are two arbitrary Banach algebras and $\theta \in \Delta(B)$, the character space of B . We also investigate Δ -weak character amenability of abstract Segal algebras and module extension Banach algebras.

1. INTRODUCTION

Let A be a Banach algebra and let $\varphi \in \Delta(A)$, consisting of all nonzero homomorphisms from A into \mathbb{C} . The concept of φ -amenability was first introduced by Kaniuth et al. in [6]. Specifically, A is called φ -amenable if there exist a $m \in A^{**}$ such that

- (i) $m(\varphi) = 1$;
- (ii) $m(f \cdot a) = \varphi(a)m(f)$ ($a \in A, f \in A^*$).

Monfared in [10], introduced and studied the notion of character amenable Banach algebra. A was called character amenable if it has a bounded right approximate identity and it is φ -amenable for all $\varphi \in \Delta(A)$. Many aspects of φ -amenability have been investigated in [3, 6, 9].

Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Following [7], A is called Δ -weak φ -amenable if, there exists a $m \in A^{**}$ such that

- (i) $m(\varphi) = 0$;
- (ii) $m(\psi \cdot a) = \psi(a)$ ($a \in \ker(\varphi), \psi \in \Delta(A)$).

In this paper we use above definition with a slight difference. In fact we say that A is Δ -weak φ -amenable if, there exists a $m \in A^{**}$ such that

- (i) $m(\varphi) = 0$;
- (ii) $m(\psi \cdot a) = \psi(a)$ ($a \in A, \psi \in \Delta(A) \setminus \{\varphi\}$).

The aim of the present work is to study Δ -weak character amenability of certain Banach algebras such as projective tensor product $A \widehat{\otimes} B$, Lau product $A \times_{\theta} B$, where $\theta \in \Delta(B)$, abstract Segal algebras and module extension Banach algebras. Indeed, we show that $A \widehat{\otimes} B$ (resp. $A \times_{\theta} B$) is Δ -weak character amenable if and

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only if both A and B are Δ -weak character amenable. For abstract Segal algebra B with respect to A , we investigate relations between Δ -weak character amenability of A and B . Finally, for a Banach algebra A and A -bimodule X we show that $A \oplus_1 X$ is Δ -weak character amenable if and only if A is Δ -weak character amenable.

2. Δ -WEAK CHARACTER AMENABILITY OF $A \widehat{\otimes} B$

We commence this section with the following definition:

Definition 2.1. Let A be a Banach algebra. The net $(a_\alpha)_\alpha$ in A is called a Δ -weak approximate identity if, $|\varphi(aa_\alpha) - \varphi(a)| \rightarrow 0$, for each $a \in A$ and $\varphi \in \Delta(A)$.

Note that the approximate identity and Δ -weak approximate identity of a Banach algebra can be different. Jones and Lahr proved that if $S = \mathbb{Q}^+$ the semigroup algebra $l^1(S)$ has a bounded Δ -weak approximate identity, but it does not have any bounded or unbounded approximate identity (see [4]).

Definition 2.2. Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. We say that A is Δ -weak φ -amenable if, there exists a $m \in A^{**}$ such that

- (i) $m(\varphi) = 0$;
- (ii) $m(\psi \cdot a) = \psi(a)$ ($a \in A, \psi \in \Delta(A) \setminus \{\varphi\}$).

Definition 2.3. Let A be a Banach algebra. We say that A is Δ -weak character amenable if it is Δ -weak φ -amenable for every $\varphi \in \Delta(A) \cup \{0\}$.

Lemma 2.4. *Let A be a Banach algebra such that $0 < |\Delta(A)| \leq 2$. Then A is Δ -weak character amenable.*

Proof. If A has only one character, the proof is easy. Let $\Delta(A) = \{\varphi, \psi\}$, where $\varphi \neq \psi$. Hence, by the proof of Theorem 3.3.14 of [5], there exists a $a_0 \in A$ with $\varphi(a_0) = 0$ and $\psi(a_0) = 1$. Put $m = \widehat{a}_0$. Then $m(\varphi) = \widehat{a}_0(\varphi) = \varphi(a_0) = 0$ and for every $a \in A$, we have

$$m(\psi \cdot a) = \widehat{a}_0(\psi \cdot a) = \psi \cdot a(a_0) = \psi(aa_0) = \psi(a).$$

So, A is Δ -weak φ -amenable. A Similar argument shows that A is Δ -weak ψ -amenable. Therefore A is Δ -weak character amenable. □

The proof of the following theorem is omitted, since it can be proved in the same direction as Theorem 2.2 of [7].

Theorem 2.5. *Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Then A is Δ -weak φ -amenable if and only if there exists a net $(a_\alpha)_\alpha \subseteq \ker(\varphi)$ such that $|\psi(aa_\alpha) - \psi(a)| \rightarrow 0$, for each $a \in A$ and $\psi \in \Delta(A) \setminus \{\varphi\}$.*

Example 2.6. (i) Let A be a Banach algebra with a bounded approximate identity. By Theorem 2.5, A is Δ -weak 0-amenable.

(ii) Let $S = \mathbb{Q}^+$. Then the semigroup algebra $l^1(S)$ has a bounded Δ -weak approximate identity (see [4]). So, Theorem 2.5, implies that $l^1(S)$ is Δ -weak 0-amenable.

Example 2.7. Let X be a Banach space and let $\varphi \in X^* \setminus \{0\}$ with $\|\varphi\| \leq 1$. Define a product on X by $ab = \varphi(a)b$ for all $a, b \in X$. With this product X is a Banach algebra which is denoted by $A_\varphi(X)$ (see [11]). Clearly, $\Delta(A_\varphi(X)) = \{\varphi\}$. Therefore by Lemma 2.4, $A_\varphi(X)$ is Δ -weak φ -amenable.

Example 2.8. Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Suppose that A is a φ -amenable and has a bounded right approximate identity. By Corollary 2.3 of [6], $\ker(\varphi)$ has a bounded right approximate identity. Let $(e_\alpha)_\alpha$ be a bounded right approximate identity for $\ker(\varphi)$. If there exists $a_0 \in A$ with $\varphi(a_0) = 1$ and $\lim_\alpha |\psi(a_0 e_\alpha) - \psi(a_0)| = 0$ for all $\psi \in \Delta(A) \setminus \{\varphi\}$, then A is Δ -weak φ -amenable. For seeing this suppose that m is w^* - $\lim_\alpha(\widehat{e_\alpha})$. Now, we have

$$m(\varphi) = \lim_\alpha \widehat{e_\alpha}(\varphi) = \lim_\alpha \varphi(e_\alpha) = 0,$$

and for every $\psi \in \Delta(A) \setminus \{\varphi\}$ and $a \in \ker(\varphi)$,

$$m(\psi \cdot a) = \lim_\alpha \widehat{e_\alpha}(\psi \cdot a) = \lim_\alpha \psi \cdot a(e_\alpha) = \lim_\alpha \psi(ae_\alpha) = \psi(a).$$

Let $a \in A$. Then $a - \varphi(a)a_0 \in \ker(\varphi)$ and for every $\psi \in \Delta(A) \setminus \{\varphi\}$, we have

$$m(\psi \cdot (a - \varphi(a)a_0)) = \psi(a - \varphi(a)a_0).$$

Therefore $m(\psi \cdot a) = \psi(a)$. So A is Δ -weak φ -amenable.

For $f \in A^*$ and $g \in B^*$, let $f \otimes g$ denote the element of $(A \widehat{\otimes} B)^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ for all $a \in A$ and $b \in B$. Then, with this notion,

$$\Delta(A \widehat{\otimes} B) = \{\varphi \otimes \psi : \varphi \in \Delta(A), \psi \in \Delta(B)\}.$$

Theorem 2.9. *Let A and B be Banach algebras and let $\varphi \in \Delta(A) \cup \{0\}$ and $\psi \in \Delta(B) \cup \{0\}$. Then $A \widehat{\otimes} B$ is Δ -weak $(\varphi \otimes \psi)$ -amenable if and only if A is Δ -weak φ -amenable and B is Δ -weak ψ -amenable.*

Proof. Suppose that $A \widehat{\otimes} B$ is Δ -weak $(\varphi \otimes \psi)$ -amenable. So, there exists $m \in (A \widehat{\otimes} B)^{**}$ such that

$$m(\varphi \otimes \psi) = 0, \quad m((\varphi' \otimes \psi') \cdot (a \otimes b)) = (\varphi' \otimes \psi')(a \otimes b),$$

for all $a \otimes b \in A \widehat{\otimes} B$ and $(\varphi' \otimes \psi') \in \Delta(A \widehat{\otimes} B) \setminus \{\varphi \otimes \psi\}$. Choose $b_0 \in B$ such that $\psi(b_0) = 1$, and define $m_\psi \in A^{**}$ by $m_\psi(f) = m(f \otimes \psi)$ ($f \in A^*$). Then $m_\psi(\varphi) = m(\varphi \otimes \psi) = 0$ and for every $a \in A$ and $\varphi' \in \Delta(A) \setminus \{\varphi\}$, we have

$$\begin{aligned} m_\psi(\varphi' \cdot a) &= m(\varphi' \cdot a \otimes \psi) = m(\varphi' \cdot a \otimes \psi \cdot b_0) \\ &= m((\varphi' \otimes \psi) \cdot (a \otimes b_0)) = (\varphi' \otimes \psi)(a \otimes b_0) \\ &= \varphi'(a). \end{aligned}$$

Thus A is Δ -weak φ -amenable. By a similar argument one can prove that B is Δ -weak ψ -amenable.

Conversely, assume that A is Δ -weak φ -amenable and B is Δ -weak ψ -amenable. By Theorem 2.5, there are bounded nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\ker(\varphi)$ and $\ker(\psi)$, respectively, such that $|\varphi'(aa_\alpha) - \varphi'(a)| \rightarrow 0$ and $|\psi'(bb_\beta) - \psi'(b)| \rightarrow 0$ for all $a \in A, b \in B, \varphi' \in \Delta(A) \setminus \{\varphi\}$ and $\psi' \in \Delta(B) \setminus \{\psi\}$. Consider the bounded net

$((a_\alpha \otimes b_\beta))_{(\alpha, \beta)}$ in $A \widehat{\otimes} B$. Let $\|a_\alpha\| \leq M_1$, $\|b_\beta\| \leq M_2$ and let $F = \sum_{i=1}^N c_i \otimes d_i \in A \widehat{\otimes} B$. For every $\varphi' \in \Delta(A) \setminus \{0\}$ and $\psi' \in \Delta(B) \setminus \{0\}$, we have

$$\begin{aligned} & \left| \varphi' \otimes \psi'(F \cdot (a_\alpha \otimes b_\beta)) - \varphi' \otimes \psi'(F) \right| \\ &= \left| \sum_{i=1}^N \left[(\varphi'(c_i a_\alpha) - \varphi'(c_i)) \psi'(d_i b_\beta) + \varphi'(c_i) (\psi'(d_i b_\beta) - \psi'(d_i)) \right] \right| \\ &\leq \sum_{i=1}^N M_2 \|d_i\| \|\psi'\| \left| \varphi'(c_i a_\alpha) - \varphi'(c_i) \right| + \sum_{i=1}^N \|\varphi'\| \|c_i\| \|\psi'(d_i b_\beta) - \psi'(d_i)\| \\ &\longrightarrow 0. \end{aligned}$$

Now let $G \in A \widehat{\otimes} B$, so there exist sequences $(c_i)_i \subseteq A$ and $(d_i)_i \subseteq B$ such that $G = \sum_{i=1}^\infty c_i \otimes d_i$ with $\sum_{i=1}^\infty \|c_i\| \|d_i\| < \infty$. Let $\varepsilon > 0$ be given, we choose $N \in \mathbb{N}$ such that $\sum_{i=N+1}^\infty \|c_i\| \|d_i\| < \varepsilon/4M_1M_2\|\varphi'\|\|\psi'\|$. Put $F = \sum_{i=1}^N c_i \otimes d_i$. Since $|\varphi' \otimes \psi'(F \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)| \longrightarrow 0$, it follows that there exists (α_0, β_0) such that $|\varphi' \otimes \psi'(F \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)| < \varepsilon/2$ for all $(\alpha, \beta) \geq (\alpha_0, \beta_0)$. Now for such a (α, β) , we have

$$\begin{aligned} & \left| \varphi' \otimes \psi'(G \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G) \right| \\ &= \left| \varphi' \otimes \psi'(F \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F) + \sum_{i=1+N}^\infty (\varphi'(c_i a_\alpha) \psi'(d_i b_\beta) - \varphi'(c_i) \psi'(d_i)) \right| \\ &\leq \varepsilon/2 + 2M_1M_2\|\varphi'\|\|\psi'\| \sum_{i=N+1}^\infty \|c_i\| \|d_i\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $|\varphi' \otimes \psi'(G \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)| \longrightarrow 0$. Also, clearly $|\varphi' \otimes \psi'(G \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)| \longrightarrow 0$ for $\varphi' = 0$ and $\psi' = 0$ and it is easy to see that $((a_\alpha \otimes b_\beta))_{(\alpha, \beta)} \subseteq \ker(\varphi \otimes \psi)$. Therefore $A \widehat{\otimes} B$ is Δ -weak $(\varphi \otimes \psi)$ -amenable, again by Theorem 2.5. \square

Corollary 2.10. *Let A and B be Banach algebras. Then $A \widehat{\otimes} B$ is Δ -weak character amenable if and only if both A and B are Δ -weak character amenable.*

By using above corollary and Theorem 2.9, we can proof following proposition.

Proposition 2.11. *Let A and B be Banach algebras. Then $A \widehat{\otimes} B$ is Δ -weak character amenable if and only if $B \widehat{\otimes} A$ is Δ -weak character amenable.*

3. Δ -WEAK CHARACTER AMENABILITY OF $A \times_\theta B$

Let A and B be Banach algebras with $\Delta(B) \neq \emptyset$ and $\theta \in \Delta(B)$. Then the set $A \times B$ equipped with the multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 + \theta(b_2) a_1 + \theta(b_1) a_2, b_1 b_2) \quad (a_1, a_2 \in A, b_1, b_2 \in B),$$

and the norm $\|(a, b)\| = \|a\| + \|b\|$ ($a \in A, b \in B$), is a Banach algebra which is called the θ -Lau product of A and B and is denoted by $A \times_\theta B$. Lau product was introduced by Lau [8] for certain class of Banach algebras and was extended by Monfared [9] for the general case.

We note that the dual space $(A \times_{\theta} B)^*$ can be identified with $A^* \times B^*$, via

$$\langle (f, g), (a, b) \rangle = \langle a, f \rangle + \langle b, g \rangle \quad (a \in A, f \in A^*, b \in B, g \in B^*).$$

Moreover, $(A \times_{\theta} B)^*$ is a $(A \times_{\theta} B)$ -bimodule with the module operations given by

$$(3.1) \quad (f, g) \cdot (a, b) = (f \cdot a + \theta(b)f, f(a)\theta + g \cdot b),$$

and

$$(3.2) \quad (a, b) \cdot (f, g) = (a \cdot f + \theta(b)f, f(a)\theta + b \cdot g),$$

for all $a \in A, b \in B, f \in A^*$ and $g \in B^*$.

Proposition 3.1. *Let A be a unital Banach algebra and B be a Banach algebra and $\theta \in \Delta(B)$. Then $A \times_{\theta} B$ has a Δ -weak approximate identity if and only if B has a Δ -weak approximate identity.*

Proof. Let $((a_{\alpha}, b_{\alpha}))_{\alpha}$ be a Δ -weak approximate identity for $A \times_{\theta} B$. For every $\psi \in \Delta(B)$ and $b \in B$ we have,

$$|\psi(bb_{\alpha}) - \psi(b)| = |(0, \psi)((0, b)(a_{\alpha}, b_{\alpha})) - (0, \psi)(0, b)| \longrightarrow 0.$$

Then $(b_{\alpha})_{\alpha}$ is a Δ -weak approximate identity for B .

Conversely, let e_A be the identity of A and $(b_{\beta})_{\beta}$ be a Δ -weak approximate identity for B . We claim that $((e_A - \theta(b_{\beta})e_A, b_{\beta}))_{\beta}$ is a Δ -weak approximate identity for $A \times_{\theta} B$. In fact for every $a \in A, b \in B$ and $\varphi \in \Delta(A)$, we have

$$\begin{aligned} & |(\varphi, \theta)((a, b)(e_A - \theta(b_{\beta})e_A, b_{\beta})) - (\varphi, \theta)(a, b)| \\ &= |(\varphi, \theta)(a + \theta(b)e_A - \theta(bb_{\beta})e_A, bb_{\beta}) - (\varphi, \theta)(a, b)| \\ &= 0. \end{aligned}$$

Also for every $a \in A, b \in B$ and $\psi \in \Delta(B)$, we have

$$|(0, \psi)((a, b)(e_A - \theta(b_{\beta})e_A, b_{\beta})) - (0, \psi)(a, b)| = |\psi(bb_{\beta}) - \psi(b)| \longrightarrow 0.$$

Therefore $((e_A - \theta(b_{\beta})e_A, b_{\beta}))_{\beta}$ is a Δ -weak approximate identity for $A \times_{\theta} B$. \square

Theorem 3.2. *Let A be a unital Banach algebra and B be a Banach algebra and $\theta \in \Delta(B)$. Then $A \times_{\theta} B$ is Δ -weak character amenable if and only if both A and B are Δ -weak character amenable.*

Proof. Suppose that $A \times_{\theta} B$ is Δ -weak character amenable. Let $\varphi \in \Delta(A) \cup \{0\}$. Then there exists $m \in (A \times_{\theta} B)^{**}$ such that $m(\varphi, \theta) = 0$ and $m(h \cdot (a, b)) = h(a, b)$ for all $(a, b) \in A \times_{\theta} B$ and $h \in \Delta(A \times_{\theta} B)$, where $h \neq (\varphi, \theta)$. Let e_A be the identity of A and define $m_{\psi} \in A^{**}$ by $m_{\psi}(f) = m(f, f(e_A)\theta)$ ($f \in A^*$). For every $a \in A$ and $\varphi' \in \Delta(A) \setminus \{\varphi\}$, we have

$$\begin{aligned} m_{\psi}(\varphi' \cdot a) &= m(\varphi' \cdot a, (\varphi' \cdot a)(e_A)\theta) \\ &= m(\varphi' \cdot a, \varphi'(a)\theta) \\ &= m((\varphi', \theta) \cdot (a, 0)) \\ &= (\varphi', \theta)(a, 0) \\ &= \varphi'(a). \end{aligned}$$

Also $m_\psi(\varphi) = m(\varphi, \theta) = 0$. Thus A is a Δ -weak φ -amenable. Therefore A is Δ -weak character amenable.

Let $\psi \in \Delta(B) \cup \{0\}$. From the Δ -weak character amenability of $A \times_\theta B$ it follows that there exists a $m \in (A \times_\theta B)^{**}$ such that $m(0, \psi) = 0$ and $m(h \cdot (a, b)) = h(a, b)$ for all $(a, b) \in A \times_\theta B$ and $h \in \Delta(A \times_\theta B)$, where $h \neq (0, \psi)$. Define $m_\varphi \in B^{**}$ by $m_\varphi(g) = m(0, g)$. So $m_\varphi(\psi) = m(0, \psi) = 0$ and

$$m_\varphi(\psi' \cdot b) = m(0, \psi' \cdot b) = m((0, \psi') \cdot (0, b)) = (0, \psi')(0, b) = \psi'(b),$$

for all $b \in B$ and $\psi' \in \Delta(B) \setminus \{\psi\}$. Therefore B is Δ -weak character amenable.

Conversely, let A and B be Δ -weak character amenable. We show that for every $h \in \Delta(A \times_\theta B)$, $A \times_\theta B$ is Δ -weak h -amenable. To see this we first assume that $h = (0, \psi)$, where $\psi \in \Delta(B)$. Since B is Δ -weak character amenable, by Theorem 2.5 there exists a net $(b_\beta)_\beta \subseteq \ker \psi$ such that $|\psi'(bb_\beta) - \psi'(b)| \rightarrow 0$, for all $b \in B$ and $\psi' \in \Delta(B)$, where $\psi' \neq \psi$. Consider the bounded net $((e_A - \theta(b_\beta)e_A, b_\beta))_\beta \subseteq A \times_\theta B$. A similar argument as in the proof of Proposition 3.1, shows that

$$|(\varphi, \theta)((a, b)(e_A - \theta(b_\beta)e_A, b_\beta)) - (\varphi, \theta)(a, b)| \rightarrow 0,$$

and

$$|(0, \psi)((a, b)(e_A - \theta(b_\beta)e_A, b_\beta)) - (0, \psi)(a, b)| \rightarrow 0,$$

for all $\varphi \in \Delta(A)$, $\psi \in \Delta(B)$ and $a \in A, b \in B$. Also one can easily check that $((e_A - \theta(b_\beta)e_A, b_\beta))_\beta \subseteq \ker h$. So, by Theorem 2.5, $A \times_\theta B$ is Δ -weak $(0, \psi)$ -amenable.

Now let $h = (\varphi, \theta)$, where $\varphi \in \Delta(A)$. Since A is Δ -weak φ -amenable, by Theorem 2.5 there exists a net $(a_\alpha)_\alpha \subseteq \ker \varphi$ such that $|\varphi'(aa_\alpha) - \varphi'(a)| \rightarrow 0$, for all $a \in A$ and $\varphi' \in \Delta(A)$, where $\varphi' \neq \varphi$. Also since B is Δ -weak θ -amenable again by Theorem 2.5, there exists a net $(b_\beta)_\beta \subseteq \ker(\theta)$ such that $|\psi'(bb_\beta) - \psi'(b)| \rightarrow 0$, for all $b \in B$ and $\psi' \in \Delta(B)$, where $\psi' \neq \theta$. Consider the bounded net $((a_\alpha, b_\beta))_{(\alpha, \beta)} \subseteq A \times_\theta B$. It is easy to see that $((a_\alpha, b_\beta))_{(\alpha, \beta)} \subseteq \ker(\varphi, \theta)$. For every $a \in A, b \in B$ and $\psi' \in \Delta(B)$, we have

$$|(0, \psi')((a, b)(a_\alpha, b_\beta)) - (0, \psi')(a, b)| = |\psi'(bb_\beta) - \psi'(b)| \rightarrow 0,$$

and for every $\varphi' \in \Delta(A)$,

$$\begin{aligned} & |(\varphi', \theta)((a, b)(a_\alpha, b_\beta)) - (\varphi', \theta)((a, b))| \\ &= |\varphi'(aa_\alpha) + \theta(b_\beta)\varphi'(a) + \theta(b)\varphi'(a_\alpha) + \theta(bb_\beta) - \varphi'(a) - \theta(b)| \\ &= |\varphi'(aa_\alpha) + \theta(b)\varphi'(a_\alpha) - \varphi'(a) - \theta(b)| \\ &\leq |\varphi'(aa_\alpha) - \varphi'(a)| + |\theta(b)| |\varphi'(a_\alpha e_A) - \varphi'(e_A)| \rightarrow 0. \end{aligned}$$

So, Theorem 2.5, yields that $A \times_\theta B$ is Δ -weak (φ, θ) -amenable. Therefore $A \times_\theta B$ is Δ -weak character amenable. □

4. Δ -WEAK CHARACTER AMENABILITY OF ABSTRACT SEGAL ALGEBRAS

We start this section with the basic definition of abstract Segal algebra; see [2] for more details. Let $(A, \|\cdot\|_A)$ be a Banach algebra. A Banach algebra $(B, \|\cdot\|_B)$ is an abstract Segal algebra with respect to A if:

- (i) B is a dense left ideal in A ;

- (ii) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for all $b \in B$;
- (iii) there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for all $a, b \in B$.

Several authors have studied various notions of amenability for abstract Segal algebras; see, for example, [1, 12].

To prove our next result we need to quote the following lemma from [1].

Lemma 4.1. *Let A be a Banach algebra and let B be an abstract Segal algebra with respect to A . Then $\Delta(B) = \{\varphi|_B : \varphi \in \Delta(A)\}$.*

Theorem 4.2. *Let A be a Banach algebra and let B be an abstract Segal algebra with respect to A . If B is Δ -weak character amenable, then so is A . In the case that B^2 is dense in B and B has a bounded approximate identity the converse is also valid.*

Proof. Let $\varphi \in \Delta(A)$. Since B is Δ -weak character amenable, by Lemma 4.1 B is Δ -weak $\varphi|_B$ -amenable. Now from the Theorem 2.5, it follows that there exists a bounded net $(b_\alpha)_\alpha$ in $\ker(\varphi|_B)$ such that

$$|\psi|_B(bb_\alpha) - \psi|_B(b)| \longrightarrow 0,$$

for all $b \in B$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi|_B$. Let $\psi \in \Delta(A)$ and $a \in A$. From the density of B in A it follows that there exists a net $(b_i)_i \subseteq B$ such that $\lim_i b_i = a$. So

$$|\psi(ab_\alpha) - \psi(a)| = \lim_i |\psi|_B(b_i b_\alpha) - \psi|_B(b_i)| \longrightarrow 0.$$

Then Theorem 2.5 implies that A is Δ -weak φ -amenable. Therefore A is Δ -weak character amenable.

Conversely, suppose that A is Δ -weak character amenable. Let $\varphi|_B \in \Delta(B)$. By Theorem 2.5, there exists a bounded net $(a_\alpha)_\alpha$ in $\ker(\varphi)$ such that $|\psi(aa_\alpha) - \psi(a)| \longrightarrow 0$, for all $a \in A$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi$. Let $(e_i)_i$ be a bounded approximate identity for B with bound $M > 0$. Set $b_\alpha = \lim_i (e_i a_\alpha e_i)$, for all α . From the fact that B^2 is dense in B and the continuity of φ , we infer that $b_\alpha \subseteq \ker(\varphi|_B)$. Moreover, for every $b \in B$ and $\psi|_B \in \Delta(B)$, with $\psi \neq \varphi$, we have

$$\begin{aligned} |\psi|_B(bb_\alpha) - \psi|_B(b)| &= \lim_i |\psi|_B(b e_i a_\alpha e_i) - \psi|_B(b)| \\ &= \lim_i |\psi|_B(b e_i^2 a_\alpha) - \psi|_B(b)| \\ &= |\psi|_B(b a_\alpha) - \psi|_B(b)| \longrightarrow 0. \end{aligned}$$

Hence, B is Δ -weak $\varphi|_B$ -amenable by Theorem 2.5. Therefore B is Δ -weak character amenable. \square

5. Δ -WEAK CHARACTER AMENABILITY OF MODULE EXTENSION BANACH ALGEBRAS

Let A be a Banach algebra and X be a Banach A -bimodule. The l^1 -direct sum of A and X , denoted by $A \oplus_1 X$, with the product defined by

$$(a, x)(a', x') = (aa', a \cdot x' + x \cdot a') \quad (a, a' \in A, x, x' \in X),$$

is a Banach algebra that is called the module extension Banach algebra of A and X .

Using the fact that the element $(0, x)$ is nilpotent in $A \oplus_1 X$ for all $x \in X$, it is easy to verify that

$$\Delta(A \oplus_1 X) = \{\tilde{\varphi} : \varphi \in \Delta(A)\},$$

where $\tilde{\varphi}(a, x) = \varphi(a)$ for all $a \in A$ and $x \in X$.

Theorem 5.1. *Let A be a Banach algebra and X be a Banach A -bimodule. Then $A \oplus_1 X$ is Δ -weak character amenable if and only if A is Δ -weak character amenable.*

Proof. Suppose that A is Δ -weak character amenable. Let $\tilde{\varphi} \in \Delta(A \oplus_1 X)$. By Theorem 2.5, there exists a bounded net $(a_\alpha)_\alpha$ in $\ker(\varphi)$ such that $|\psi(aa_\alpha) - \psi(a)| \rightarrow 0$, for all $a \in A$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi$. Choose a bounded net $(a_\alpha, 0)_\alpha$ in $A \oplus_1 X$. Clearly, $(a_\alpha, 0)_\alpha \subseteq \ker(\tilde{\varphi})$. For every $a \in A$, $x \in X$ and $\tilde{\psi} \in \Delta(A \oplus_1 X)$, we have

$$\begin{aligned} |\tilde{\psi}((a, x)(a_\alpha, 0)) - \tilde{\psi}(a, x)| &= |\tilde{\psi}(aa_\alpha, x \cdot a_\alpha) - \tilde{\psi}(a, x)| \\ &= |\psi(aa_\alpha) - \psi(a)| \rightarrow 0. \end{aligned}$$

So, Theorem 2.5 implies that $A \oplus_1 X$ is Δ -weak $\tilde{\varphi}$ -amenable. Therefore $A \oplus_1 X$ is Δ -weak character amenable.

For the converse, let $\varphi \in \Delta(A)$. Again by Theorem 2.5 there exists a bounded net $(a_\alpha, x_\alpha)_\alpha$ in $\ker(\tilde{\varphi})$ such that $|\tilde{\psi}((a, x)(a_\alpha, x_\alpha)) - \tilde{\psi}(a, x)| \rightarrow 0$, for all $a \in A$, $x \in X$ and $\tilde{\psi} \in \Delta(A \oplus_1 X)$, with $\tilde{\psi} \neq \tilde{\varphi}$. So,

$$\begin{aligned} |\psi(aa_\alpha) - \psi(a)| &= |\tilde{\psi}(aa_\alpha, a \cdot x_\alpha + x \cdot a_\alpha) - \tilde{\psi}(a, x)| \\ &= |\tilde{\psi}((a, x)(a_\alpha, x_\alpha)) - \tilde{\psi}(a, x)| \rightarrow 0, \end{aligned}$$

for all $a \in A$ and $\psi \in \Delta(A)$. Moreover, $\varphi(a_\alpha) = \tilde{\varphi}(a_\alpha, x_\alpha) = 0$, for all α . Thus $(a_\alpha)_\alpha \subseteq \ker(\varphi)$. By Theorem 2.5, A is Δ -weak φ -amenable. Therefore A is Δ -weak character amenable. \square

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