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ON A DECOMPOSITION OF NON-NEGATIVE RADON MEASURES

BÉRENGER AKON KPATA

ABSTRACT. We establish a decomposition of non-negative Radon measures on \mathbb{R}^d which extends that obtained by Strichartz [6] in the setting of α -dimensional measures. As consequences, we deduce some well-known properties concerning the density of non-negative Radon measures. Furthermore, some properties of non-negative Radon measures having their Riesz potential in a Lebesgue space are obtained.

1. INTRODUCTION – MAIN RESULTS

Let d be a positive integer. Let $0 < \theta \leq 1$. We denote by dx the Lebesgue measure on \mathbb{R}^d . For any Lebesgue measurable subset E of \mathbb{R}^d , $|E|$ stands for its Lebesgue measure. For $1 \leq p < \infty$, $\|\cdot\|_p$ denotes the usual norm on the classical Lebesgue space $L^p = L^p(\mathbb{R}^d)$. The $d\theta$ -dimensional Hausdorff measure on \mathbb{R}^d is denoted by $\mathcal{H}_{d\theta}$ (see Section 2 for the definition of this measure and some of its basic properties). If μ is a measure on \mathbb{R}^d and $A \subset \mathbb{R}^d$, we denote by $\mu|_A$ the restriction of μ to A .

A Borel measure μ on \mathbb{R}^d is locally uniformly $d\theta$ -dimensional if there exists a constant $C > 0$ such that

$$\mu(B(x, r)) \leq Cr^{d\theta},$$

for every open ball $B(x, r)$ centered at x with radius $r \leq 1$.

This definition easily implies that μ is absolutely continuous with respect to $\mathcal{H}_{d\theta}$, but since $\mathcal{H}_{d\theta}$ is not σ -finite, the Radon-Nikodym theorem does not apply. Instead Strichartz proved in [6] the following substitute.

Proposition 1.1. *If μ is a locally uniformly $d\theta$ -dimensional measure, then there exists a function φ and a measure ν such that $\mu = \varphi d\mathcal{H}_{d\theta} + \nu$, where ν has the property $\mathcal{H}_{d\theta}(A) < \infty$ implies $\nu(A) = 0$ for any Borel subset A of \mathbb{R}^d .*

Next, he gave the following definition motivated by Proposition 1.1.

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Definition 1.2. Let ν and μ be two Borel measures on \mathbb{R}^d . The measure ν is null with respect to μ on \mathbb{R}^d and we will denote this with $\nu \lll \mu$, if for any Borel subset A of \mathbb{R}^d ,

$$\mu(A) < \infty \Rightarrow \nu(A) = 0.$$

In [6] the author established the following result concerning the density of non-negative Radon measures that are null with respect to $\mathcal{H}_{d\theta}$.

Proposition 1.3. *Suppose that ν is a non-negative Radon measure on \mathbb{R}^d . If $\nu \lll \mathcal{H}_{d\theta}$ then*

$$\limsup_{r \rightarrow 0} r^{-d\theta} \nu(B(x, r)) = 0$$

for $\mathcal{H}_{d\theta}$ -almost every x .

A generalization of Proposition 1.1 and Proposition 1.3 was obtained in [4].

In the present note, we establish the following decomposition of non-negative Radon measures.

Proposition 1.4. *Suppose that μ is a non-negative Radon measure on \mathbb{R}^d . Let us consider the following subsets of \mathbb{R}^d :*

$$N_\theta = \{x \in \mathbb{R}^d : \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) = 0\},$$

$$P_\theta = \{x \in \mathbb{R}^d : 0 < \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) < \infty\},$$

$$E_\theta^\infty = \{x \in \mathbb{R}^d : \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) = \infty\}.$$

Then $\mu = \mu \llcorner N_\theta + \mu \llcorner P_\theta + \mu \llcorner E_\theta^\infty$ and

- (i) for any Borel set $F \subset N_\theta$, $\mathcal{H}_{d\theta}(F) < \infty \Rightarrow \mu(F) = 0$,
- (ii) P_θ is $\mathcal{H}_{d\theta}$ σ -finite and for any Borel set $F \subset P_\theta$, $\mathcal{H}_{d\theta}(F) = 0 \Rightarrow \mu(F) = 0$,
- (iii) $\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$.

The remark below shows that in the setting of non-negative Radon measures, Proposition 1.1 derives from Proposition 1.4.

Remark 1.5. If μ is a non-negative locally uniformly $d\theta$ -dimensional measure, then

$$0 \leq \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) < \infty.$$

Therefore, by applying Proposition 1.4, μ has the following decomposition: $\mu = \mu \llcorner N_\theta + \mu \llcorner P_\theta$. In addition, on P_θ , $\mathcal{H}_{d\theta}$ and μ are σ -finite and μ is absolutely continuous with respect to $\mathcal{H}_{d\theta}$. Therefore, according to the Radon-Nikodym theorem there exists a function $\varphi \geq 0$ such that for all Borel sets $E \subset P_\theta$, we have

$$\mu(E) = \int_E \varphi(x) d\mathcal{H}_{d\theta}(x).$$

As immediate consequences of Proposition 1.4, we have Proposition 1.3 and the following result.

Corollary 1.6. *Suppose that $0 < \theta < 1$ and μ is a non-negative Radon measure on \mathbb{R}^d that is absolutely continuous with respect to the Lebesgue measure. Let*

$$A = \left\{ x \in \mathbb{R}^d : \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) > 0 \right\}.$$

Then $\mathcal{H}_{d\theta}(A) = 0$.

In particular, for $u \in L^p$, $1 \leq p < \infty$, if E is defined by

$$E = \left\{ x \in \mathbb{R}^d : \limsup_{r \rightarrow 0} r^{-d\theta} \int_{B(x,r)} |u(x)|^p dx > 0 \right\},$$

then $\mathcal{H}_{d\theta}(E) = 0$.

Let us stress that Corollary 1.6 was already established in [8] in order to investigate the Lebesgue points for Sobolev functions.

For $0 < \gamma < 1$, we define the Riesz potential operator I_γ by

$$I_\gamma \mu(x) = \int_{\mathbb{R}^d} |x - y|^{d(\gamma-1)} d\mu(y), \quad x \in \mathbb{R}^d,$$

for any suitable Radon measure μ on \mathbb{R}^d .

The next results that give some properties of non-negative Radon measures having their Riesz potential in a classical Lebesgue space also arise from Proposition 1.4.

Proposition 1.7. *Suppose that $0 < \gamma < 1$ and $1 < p < \infty$. Then for any non-negative Radon measure μ on \mathbb{R}^d satisfying $I_\gamma \mu \in L^p$, we have*

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^d} \left(r^{d(\gamma-1)} \mu(B(x, r)) \right)^p dx = 0.$$

Proposition 1.8. *Suppose that $\frac{d}{d-1} < p < \infty$ and μ is a non-negative Radon measure. Then we have*

$$I_{\frac{1}{d}} \mu \in L^p \Rightarrow \lim_{r \rightarrow 0} r^{p'-d} \mu(B(x, r)) = 0 \quad \mu\text{-almost everywhere,}$$

where $p' = \frac{p}{p-1}$.

Notice that Proposition 1.7 and Proposition 1.8 are related to the solvability in $L^p(\mathbb{R}^d, \mathbb{R}^d)$ of the equation

$$(1) \quad \operatorname{div} F = \mu$$

with measure data μ .

Indeed, Phuc and Torrès have obtained the following criterion.

Proposition 1.9 ([5]). *Suppose that μ is a non-negative Radon measure on \mathbb{R}^d and $\frac{d}{d-1} < p < \infty$. Then the following conditions are equivalent:*

- (i) Equation (1) has a solution in $L^p(\mathbb{R}^d, \mathbb{R}^d)$.
- (ii) $I_{\frac{1}{d}} \mu$ belongs to $L^p(\mathbb{R}^d, \mathbb{R})$.

The remainder of this paper is organized as follows. In Section 2 we prove Proposition 1.4, Proposition 1.3 and Corollary 1.6. Section 3 is devoted to the proof of Proposition 1.7. In Section 4 we establish the proof of Proposition 1.8.

2. PROOFS OF PROPOSITION 1.4, PROPOSITION 1.3 AND COROLLARY 1.6

In the sequel, we shall use the following notation.

Notation 2.1. For any non-empty subset B of \mathbb{R}^d , we denote by $\text{diam } B$ its diameter.

Let us recall the definition of the ξ -dimensional Hausdorff measure \mathcal{H}_ξ in \mathbb{R}^d , where $0 < \xi \leq d$ (see [3] for a detailed exposition on this measure). Let A be a subset of \mathbb{R}^d . For any $\delta > 0$,

$$\mathcal{H}_\xi^\delta(A) = \inf \left\{ \sum_{i \in I} (\text{diam } U_i)^\xi : A \subset \bigcup_{i \in I} U_i, I \text{ countable and } \text{diam } U_i < \delta \text{ for } i \in I \right\}$$

and

$$\mathcal{H}_\xi(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\xi^\delta(A).$$

Remark 2.2. (i) If $0 < t < r$, then for any subset E of \mathbb{R}^d we have

$$\mathcal{H}_t(E) < \infty \Rightarrow \mathcal{H}_r(E) = 0.$$

(ii) There exists a positive constant $C(d)$ such that for any Lebesgue measurable subset E of \mathbb{R}^d ,

$$\mathcal{H}_d(E) = C(d)|E|.$$

The following result (see [8]) will be useful in the proof of Proposition 1.4.

Lemma 2.3. Let μ be a non-negative Radon measure on \mathbb{R}^d . Let $0 < \lambda < \infty$. Suppose that F is a Borel subset of \mathbb{R}^d such that

$$\limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) > \lambda,$$

for each $x \in F$. Then there exists a constant $C = C(d, \theta)$ such that

$$\mathcal{H}_{d\theta}(F) \leq \frac{C}{\lambda} \mu(F).$$

Proof of Proposition 1.4. a) Let F be a Borel subset of \mathbb{R}^d . Let $0 < \lambda < \infty$ and $0 < \delta < \infty$. Let us set $F_\delta^\lambda = \{x \in F : \sup_{0 < r \leq \delta} r^{-d\theta} \mu(B(x, r)) < \lambda\}$ and

$$F^\lambda = \{x \in F : \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) < \lambda\}.$$

For any countable covering $\{U_i : i \in I\}$ of F such that $\text{diam } U_i < \frac{\delta}{2}$ for all $i \in I$, we have

$$U_i \cap F_\delta^\lambda \neq \emptyset \Rightarrow \exists x \in U_i \cap F_\delta^\lambda \Rightarrow \exists x \in F_\delta^\lambda : U_i \subset B(x, 2 \text{ diam } U_i)$$

$$U_i \cap F_\delta^\lambda \neq \emptyset \Rightarrow \exists x \in F_\delta^\lambda : \mu(U_i) \leq \mu(B(x, 2 \text{ diam } U_i)) < \lambda (2 \text{ diam } U_i)^{d\theta}.$$

It follows that

$$\mu(F_\delta^\lambda) \leq \sum_{\substack{i \in I \\ U_i \cap F_\delta^\lambda \neq \emptyset}} \mu(U_i) \leq \lambda 2^{d\theta} \sum_{i \in I} (\text{diam } U_i)^{d\theta}.$$

Hence,

$$\mu(F_\delta^\lambda) \leq \lambda 2^{d\theta} \mathcal{H}_{d\theta}^\delta(F) \leq \lambda 2^{d\theta} \mathcal{H}_{d\theta}(F), \quad \lambda > 0, \delta > 0.$$

Notice that for any $\lambda > 0$, $(F_{\frac{\lambda}{k}}^\lambda)_{k \geq 1}$ is an increasing sequence which converges to F^λ . So we have

$$(2) \quad \mu(F^\lambda) \leq \lambda 2^{d\theta} \mathcal{H}_{d\theta}(F), \quad \lambda > 0.$$

b) Suppose that F is a Borel set such that $F \subset N_\theta$ and $\mathcal{H}_{d\theta}(F) < \infty$. Then, for any $\lambda > 0$, we have $F = F^\lambda = \{x \in F : \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) < \lambda\}$. So by (2),

we have $\mu(F) = 0$.

c) Suppose that F is a Borel set such that $F \subset P_\theta$ and $\mathcal{H}_{d\theta}(F) = 0$. It follows from (2) that for any $\lambda > 0$, $\mu(F^\lambda) = 0$. Since the increasing sequence $(F^k)_{k \geq 1}$ converges to F , we obtain $\mu(F) = 0$.

d) Let us set

$$A_\theta^{m,k} = \{x \in B(0, m) : \frac{1}{k} < \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) < \infty\}, \quad k \in \mathbb{N}^*, m \in \mathbb{N}^*$$

and

$$B_\theta^{m,k} = \{x \in B(0, m) : k < \limsup_{r \rightarrow 0} r^{-d\theta} \mu(B(x, r)) < \infty\}, \quad k \in \mathbb{N}^*, m \in \mathbb{N}^*.$$

By Lemma 2.3, there exists a real constant $C = C(d, \theta)$ such that for any positive integers k and m

$$(3) \quad \mathcal{H}_{d\theta}(A_\theta^{m,k}) \leq Ck\mu(A_\theta^{m,k}) \leq Ck\mu(B(0, m)) < \infty$$

and

$$(4) \quad \mathcal{H}_{d\theta}(B_\theta^{m,k}) \leq \frac{C}{k}\mu(B(0, m)) < \infty.$$

Since

$$P_\theta = \bigcup_{(m,k) \in \mathbb{N}^* \times \mathbb{N}^*} A_\theta^{m,k},$$

we deduce from (3) that P_θ is $\mathcal{H}_{d\theta}$ σ -finite.

From (4) we have

$$\lim_{k \rightarrow \infty} \mathcal{H}_{d\theta}(B_\theta^{k,m}) = 0, \quad m \in \mathbb{N}^*.$$

In addition, for any positive integer m , $(B_\theta^{k,m})_{k \geq 1}$ is a decreasing sequence which converges to $E_\theta^\infty \cap B(0, m)$. So, by (4),

$$\mathcal{H}_{d\theta}(E_\theta^\infty \cap B(0, m)) = 0, \quad m \in \mathbb{N}^*$$

and therefore $\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$. □

Proof of Proposition 1.3. Since ν is a non-negative Radon measure on \mathbb{R}^d then Proposition 1.4 holds with μ replaced by ν . Let P_θ and E_θ^∞ be as in Proposition 1.4. Since by Proposition 1.4 $\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$ then it is enough to prove that $\mathcal{H}_{d\theta}(P_\theta) = 0$. Let (m, k) be an element of $\mathbb{N}^* \times \mathbb{N}^*$. Let $A_\theta^{m,k}$ be as in the proof of Proposition 1.4. By (3), $\mathcal{H}_{d\theta}(A_\theta^{m,k}) < \infty$. Therefore $\nu(A_\theta^{m,k}) = 0$ by the hypothesis on ν . Using again (3), we obtain $\mathcal{H}_{d\theta}(A_\theta^{m,k}) = 0$. But

$$P_\theta = \bigcup_{(m,k) \in \mathbb{N}^* \times \mathbb{N}^*} A_\theta^{m,k},$$

so $\mathcal{H}_{d\theta}(P_\theta) = 0$. □

Proof of Corollary 1.6. a) Let P_θ and E_θ^∞ be as in Proposition 1.4. Let us notice that $A = P_\theta \cup E_\theta^\infty$. Since Proposition 1.4 ensures that $\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$ then it is enough to prove that $\mathcal{H}_{d\theta}(P_\theta) = 0$ to establish the general case.

Let (m, k) be an element of $\mathbb{N}^* \times \mathbb{N}^*$. Let $A_\theta^{m,k}$ be as in the proof of Proposition 1.4. By (3), $\mathcal{H}_{d\theta}(A_\theta^{m,k}) < \infty$. Since $0 < d\theta < d$, it follows from Remark 2.2 that $\mathcal{H}_d(A_\theta^{m,k}) = 0$ and $|A_\theta^{m,k}| = 0$. The absolute continuity of μ with respect to the Lebesgue measure implies $\mu(A_\theta^{m,k}) = 0$ and consequently $\mathcal{H}_{d\theta}(A_\theta^{m,k}) = 0$ thanks to (3). But

$$P_\theta = \bigcup_{(m,k) \in \mathbb{N}^* \times \mathbb{N}^*} A_\theta^{m,k},$$

so $\mathcal{H}_{d\theta}(P_\theta) = 0$.

b) The particular case follows from the general case by defining a measure μ as $d\mu(x) = |u(x)|^p dx$. □

3. PROOF OF PROPOSITION 1.7

Let us introduce the fractional maximal operator m_β , $1 < \beta < \infty$, defined by

$$m_\beta \mu(x) = \sup_{r>0} |B(x, r)|^{\frac{1}{\beta}-1} \mu(B(x, r)), \quad x \in \mathbb{R}^d,$$

for any non-negative Radon measure μ on \mathbb{R}^d .

We have the following result.

Proposition 3.1. *Suppose that $1 < \beta < \infty$ and $1 \leq p < \infty$. Let μ be a non-negative Radon measure on \mathbb{R}^d such that $m_\beta \mu \in L^p$. Then, we have*

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^d} \left(r^{d(\frac{1}{\beta}-1)} \mu(B(x, r)) \right)^p dx = 0.$$

Proof. By hypothesis, $0 < 1 - \frac{1}{\beta} < 1$.

Define $N_{1-\frac{1}{\beta}}$, $P_{1-\frac{1}{\beta}}$ and $E_{1-\frac{1}{\beta}}^\infty$ as in Proposition 1.4. Then $\mathcal{H}_{d(1-\frac{1}{\beta})}(E_{1-\frac{1}{\beta}}^\infty) = 0$ and there exists a countable family $\{A_i : i \in I\}$ of subsets of \mathbb{R}^d satisfying $P_{1-\frac{1}{\beta}} = \bigcup_{i \in I} A_i$ and $\mathcal{H}_{d(1-\frac{1}{\beta})}(A_i) < \infty$ for all $i \in I$.

So, according to Remark 2.2 we have $|P_{1-\frac{1}{\beta}} \cup E_{1-\frac{1}{\beta}}^\infty| = 0$.

Recall that

$$\lim_{r \rightarrow 0} r^{d(\frac{1}{\beta}-1)} \mu(B(x, r)) = 0, \quad x \in N_{1-\frac{1}{\beta}} = \mathbb{R}^d \setminus (P_{1-\frac{1}{\beta}} \cup E_{1-\frac{1}{\beta}}^\infty).$$

In addition,

$$\omega_d^{\frac{1}{\beta}-1} r^{d(\frac{1}{\beta}-1)} \mu(B(x, r)) \leq m_\beta \mu(x), \quad x \in \mathbb{R}^d,$$

where ω_d is the volume of the unit ball in \mathbb{R}^d and $m_\beta \mu \in L^p$.

An application of the dominated convergence theorem ends the proof. □

For the proof of Proposition 1.7 we need the following well-known connexions between the fractional maximal operator m_β and the Riesz potential operator $I_{\frac{1}{\beta}}$.

Proposition 3.2 ([1]). *Suppose that $1 < \beta < \infty$. Let μ be a non-negative Radon measure on \mathbb{R}^d . Then,*

- (i) $m_\beta \mu \leq I_{\frac{1}{\beta}} \mu$,
- (ii) if $1 - \frac{1}{\beta} > \frac{1}{p} > 0$, there is a real constant $C > 0$ not depending on μ such that

$$C^{-1} \|m_\beta \mu\|_p \leq \|I_{\frac{1}{\beta}} \mu\|_p \leq C \|m_\beta \mu\|_p.$$

It follows that Proposition 1.7 is a consequence of Proposition 3.1.

4. PROOF OF PROPOSITION 1.8

In the sequel, for $1 < p < \infty$, we shall denote by p' the conjugate of p : $p' = \frac{p}{p-1}$. For the proof of Proposition 1.8, we need some basic properties of the Bessel capacity of order (t, p) ($t > 0, p > 1$) denoted by $C_{t,p}$. So we refer the reader to [1], [2] or [7] for a detailed exposition on this capacity.

To prove the sufficiency part of Proposition 1.9, Phuc and Torrès remarked that if $I_{\frac{1}{d}} \mu \in L^p$ then the non-negative Radon measure μ belongs to the dual space of the Sobolev space $W^{1, p'}(\mathbb{R}^d)$. Therefore such a measure is absolutely continuous with respect to the Bessel capacity $C_{1, p'}$ (see Section 2 in [2]). Thus we may state the following result.

Proposition 4.1. *Suppose that $\frac{d}{d-1} < p < \infty$ and μ is a non-negative Radon measure such that $I_{\frac{1}{d}} \mu \in L^p$. Then for any Borel subset E of \mathbb{R}^d we have*

$$C_{1, p'}(E) = 0 \Rightarrow \mu(E) = 0.$$

Another useful result is the following well-known relation between the Hausdorff measure and the Bessel capacity (see [1] for a proof).

Proposition 4.2. *Suppose that $1 < p \leq d$. Then for any subset E of \mathbb{R}^d we have*

$$\mathcal{H}_{d-p}(E) < \infty \Rightarrow C_{1,p}(E) = 0.$$

We may now prove Proposition 1.8.

Proof of Proposition 1.8. Let μ be a non-negative Radon measure on \mathbb{R}^d such that $I_{\frac{1}{d}}\mu \in L^p$ and let $\frac{d}{d-1} < p < \infty$. Applying Proposition 1.4 to μ with $\theta = 1 - \frac{p'}{d}$, we get that $P_{1-\frac{p'}{d}}$ is $\mathcal{H}_{d-p'}$ σ -finite and $\mathcal{H}_{d-p'}\left(E_{1-\frac{p'}{d}}^\infty\right) = 0$. We then deduce from Proposition 4.2 that $C_{1,p'}\left(P_{1-\frac{p'}{d}} \cup E_{1-\frac{p'}{d}}^\infty\right) = 0$ and so $\mu\left(P_{1-\frac{p'}{d}} \cup E_{1-\frac{p'}{d}}^\infty\right) = 0$ by Proposition 4.1. We conclude that $\lim_{r \rightarrow 0} r^{p'-d}\mu(B(x, r)) = 0$ μ -almost everywhere. \square

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