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Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 3, 621–636

Persistent URL: `http://dml.cz/dmlcz/147781`

Czechoslovak Mathematical Journal, 69 (144) (2019), 621–636

RESOLVING SETS OF DIRECTED CAYLEY GRAPHS FOR THE DIRECT PRODUCT OF CYCLIC GROUPS

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Received March 18, 2017. Published online May 17, 2019.

Abstract. A directed Cayley graph $C(\Gamma, X)$ is specified by a group Γ and an identity-free generating set X for this group. Vertices of $C(\Gamma, X)$ are elements of Γ and there is a directed edge from the vertex u to the vertex v in $C(\Gamma, X)$ if and only if there is a generator $x \in X$ such that $ux = v$. We study graphs $C(\Gamma, X)$ for the direct product $Z_m \times Z_n$ of two cyclic groups Z_m and Z_n, and the generating set $X = \{(0,1), (1,0), (2,0), \dots, (p,0)\}$. We present resolving sets which yield upper bounds on the metric dimension of these graphs for $p = 2$ and 3.

Keywords: metric dimension; resolving set; Cayley graph; direct product; cyclic group

MSC 2010: 05C25, 05C12

1. Introduction

Let G be a directed graph with vertex set $V(G)$. The distance $d(u,v)$ from a vertex $u \in V(G)$ to a vertex $v \in V(G)$ is the length of a shortest directed path from u to v. A vertex w resolves two vertices u and v if $d(u,w) \neq d(v,w)$. For an ordered set of vertices $W = \{w_1, w_2, \dots, w_z\}$, the representation of distances of v with respect to W is the ordered z-tuple

$$r(v|W) = (d(v,w_1), d(v,w_2), \dots, d(v,w_z)).$$

A set $W \subset V(G)$ is a resolving set of G if every two distinct vertices of G have different representations of distances with respect to W (if every two vertices of G are resolved by a vertex in W). The metric dimension of G is the number of vertices in

The work of T. Vetrík has been supported by the National Research Foundation of South Africa; grant numbers: 90793, 112122.

DOI: 10.21136/CMJ.2019.0127-17

a smallest resolving set and it is denoted by $\dim(G)$. The ith coordinate in $r(v|W)$ is 0 if and only if $v = w_i$. Thus, in order to prove that W is a resolving set of G, it suffices to show that $r(u|W) \neq r(v|W)$ for every two different vertices $u, v \in V(G) \setminus W$.

A directed Cayley graph $C(\Gamma, X)$ is specified by a group Γ and an identity-free generating set X for this group. Vertices of $C(\Gamma, X)$ are elements of Γ and there is a directed edge from the vertex u to the vertex v in $C(\Gamma, X)$ if and only if there is a generator $x \in X$ such that $ux = v$.

The concept of metric dimension was introduced by Slater in [11] and investigated independently by Harary and Melter in [4]. Slater referred to a metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of minimum number of loran/sonar detecting devices in a network so that the position of each vertex in the network can be uniquely represented in terms of its distances to the devices in the set.

The metric dimension has extensive applications in robotics, since this invariant can represent the minimum number of landmarks, which uniquely determine the position of a robot moving in a graph space, see [8]. Other applications are for example in pharmaceutical chemistry, see [2], pattern recognition and image processing, see [9].

Finding the metric dimension of a graph is an NP-hard problem. The metric dimension of various families of graphs has been studied for four decades. In [2] it was proved that a connected graph G has $\dim(G) = 1$ if and only if G is a path. Cycles have metric dimension 2. Oellermann, Pawluck and Stokke in [10] presented results on the metric dimension of directed Cayley graphs for the direct product of 3 cyclic groups. Fehr, Gosselin and Oellermann in [3] studied the metric dimension of the directed Cayley graph of dihedral groups with a minimum set of generators and they established upper and lower bounds on the metric dimension for directed Cayley graphs whose vertices are elements of the direct product of t cyclic groups for $t \geqslant 2$. They also found the exact values of the metric dimension of directed Cayley graphs for the direct product $Z_m \times Z_n$ of two cyclic groups of orders m and n, and the generating set $X = \{(0,1),(1,0)\}$. We generalize this problem and study the generating sets $X = \{(0,1),(1,0),(2,0),\ldots,(p,0)\}$ for $p = 2$ and 3.

Let us note that the metric dimension of undirected Cayley graphs of cyclic groups with small number of generators was considered in [6] and [7]. Barycentric subdivision of undirected Cayley graphs was studied in [1] and [5].

2. Directed Cayley graphs with 3 generators

In this section we study directed Cayley graphs $C(\Gamma, X)$ for the group $\Gamma = Z_m \times Z_n$ and the generating set $X = \{(0,1),(1,0),(2,0)\}$. Let us note that the graph $C(\Gamma, X)$ is isomorphic to the graph $C(\Gamma, -X)$, where $-X = \{(0,-1),(-1,0),(-2,0)\}$.

We present Theorems 2.1 and 2.2 for the graph $C(\Gamma, -X)$, because it is easier to express distances from vertices in a graph to vertices in chosen resolving sets when considering $C(\Gamma, -X)$. Let $V(C(\Gamma, -X)) = \{v_{i,j}\colon i = 0, 1, \ldots, m-1; j = 0, 1, \ldots, n-1\}$. Then the graph $C(\Gamma, -X)$ contains directed edges $v_{i,j}v_{i,j-1}$, $v_{i,j}v_{i-1,j}$ and $v_{i,j}v_{i-2,j}$, where $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n-1$. The distance from vertex $v_{i,j}$ to vertex $v_{z,l}$ in $C(\Gamma, -X)$ is

$$d(v_{i,j}, v_{z,l}) = \left\lceil \frac{i-z}{2} \right\rceil + j - l \qquad \text{if } i \geqslant z,\ j \geqslant l, \tag{2.1}$$

$$d(v_{i,j}, v_{z,l}) = \left\lceil \frac{i-z}{2} \right\rceil + n + (j - l) \qquad \text{if } i \geqslant z,\ j < l, \tag{2.2}$$

$$d(v_{i,j}, v_{z,l}) = \left\lceil \frac{m + (i-z)}{2} \right\rceil + j - l \qquad \text{if } i < z,\ j \geqslant l, \tag{2.3}$$

$$d(v_{i,j}, v_{z,l}) = \left\lceil \frac{m + (i-z)}{2} \right\rceil + n + (j - l) \qquad \text{if } i < z,\ j < l. \tag{2.4}$$

Theorem 2.1. *Let $\Gamma = Z_m \times Z_n$ and $X = \{(0,1), (1,0), (2,0)\}$, where $n \geqslant 2$ and $m \geqslant 4$. Then*

$$\dim(C(\Gamma, -X)) \leqslant \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

P r o o f. Let $m = 2p + \varepsilon$, where $p \geqslant 2$ and $\varepsilon = 0$ or 1. We show that

$$W = \{v_{0,0}, v_{2,0}, \ldots, v_{2(p-1),0}, v_{m-1,0}\}$$

is a resolving set of the graph $C(\Gamma, -X)$, where $\Gamma = Z_m \times Z_n$ and

$$X = \{(0,1), (1,0), (2,0)\}.$$

Note that $|W| = p + 1 = \lfloor m/2 \rfloor + 1$. Let us present all vertices having the same distance to $v_{0,0} \in W$. For $r = 0, 1, 2, \ldots, m-1$, by (2.1),

$$d(v_{r,k-\lceil r/2 \rceil}, v_{0,0}) = \left\lceil \frac{r}{2} \right\rceil + k - \left\lceil \frac{r}{2} \right\rceil = k,$$

where $0 \leqslant k - \lceil r/2 \rceil \leqslant n-1$. This implies that the only vertices having the distance k to $v_{0,0}$ are the vertices $v_{0,k}, v_{1,k-1}, v_{2,k-1}, v_{3,k-2}, \ldots, v_{2(p-1),k-(p-1)}, v_{2p-1,k-p}$ (and $v_{2p,k-p}$ if $\varepsilon = 1$), where the second indices must be at least 0 and at most $n-1$. We show that these vertices are resolved by W.

For $v_{2,0} \in W$ and $r = 2, 3, \ldots, m-1$, by (2.1) we have

$$d(v_{r,k-\lceil r/2 \rceil}, v_{2,0}) = \left\lceil \frac{r-2}{2} \right\rceil + k - \left\lceil \frac{r}{2} \right\rceil = k - 1.$$

For $r = 0$ and $r = 1$, by (2.4) we obtain $d(v_{0,k}, v_{2,0}) = \lceil (m-2)/2 \rceil + k = k - 1 + \lceil m/2 \rceil$ and $d(v_{1,k-1}, v_{2,0}) = \lceil (m-1)/2 \rceil + k - 1 = k - 1 + \lceil (m-1)/2 \rceil$.

In general, for $v_{i,0} \in W$, where $i = 2, 4, \ldots, 2(p-1)$, and for $r = i, i+1, \ldots, m-1$, by (2.1) we have

$$d(v_{r,k-\lceil r/2 \rceil}, v_{i,0}) = \left\lceil \frac{r-i}{2} \right\rceil + k - \left\lceil \frac{r}{2} \right\rceil = k - \frac{i}{2}.$$

For $r = 0, 1, \ldots, i-1$, by (2.4) we obtain

$$d(v_{r,k-\lceil r/2 \rceil}, v_{i,0}) = \left\lceil \frac{m+r-i}{2} \right\rceil + k - \left\lceil \frac{r}{2} \right\rceil = k - \frac{i}{2} + \left\lceil \frac{m+r}{2} \right\rceil - \left\lceil \frac{r}{2} \right\rceil > k - \frac{i}{2},$$

where $0 \leqslant k - \lceil r/2 \rceil \leqslant n-1$.

This implies that if $\varepsilon = 0$, the only vertices which can have the same representations with respect to $W' = \{v_{0,0}, v_{2,0}, \ldots, v_{2(p-1),0}\} \subset W$ are the vertices $v_{i,k-i/2}$, $v_{i+1,k-i/2-1}$ for $i = 0, 2, 4, \ldots, 2(p-1)$, which are the pairs

$$(v_{0,k}, v_{1,k-1}), (v_{2,k-1}, v_{3,k-2}), \ldots, (v_{2(p-1),k-(p-1)}, v_{2p-1,k-p}).$$

If $\varepsilon = 1$, the representation of the vertices $v_{2(p-1),k-(p-1)}$, $v_{2p-1,k-p}$ with respect to W' is the same as the representation of $v_{2p,k-p}$ (which is $(k, k-1, \ldots, k-p+1)$). Let us note that if $\varepsilon = 1$, the pairs

$$(v_{0,k}, v_{1,k-1}), (v_{2,k-1}, v_{3,k-2}), \ldots, (v_{2(p-2),k-(p-2)}, v_{2p-3,k-(p-1)})$$

are resolved by W', but we present the proof which applies for both cases, $\varepsilon = 0$ and $\varepsilon = 1$.

We use $v_{m-1,0} \in W$ to resolve the vertices $v_{i,k-i/2}$ and $v_{i+1,k-i/2-1}$ for $i = 0, 2, 4, \ldots, 2(p-1)$ (if $\varepsilon = 1$ and $i = 2(p-1)$, we also need to resolve 3 vertices $v_{2(p-1),k-(p-1)}$, $v_{2p-1,k-p}$, $v_{2p,k-p}$).

For $i = 0, 2, 4, \ldots, 2(p-1)$, by (2.4) we obtain

$$d(v_{i,k-i/2}, v_{m-1,0}) = \left\lceil \frac{m+i-(m-1)}{2} \right\rceil + k - \frac{i}{2} = k+1,$$
$$d(v_{i+1,k-i/2-1}, v_{m-1,0}) = \left\lceil \frac{m+i+1-(m-1)}{2} \right\rceil + k - \frac{i}{2} - 1 = k,$$

and for $v_{2p+\varepsilon-1,k-p} = v_{m-1,k-\lceil (m-1)/2 \rceil}$, by (2.1),

$$d(v_{m-1,k-\lceil (m-1)/2 \rceil}, v_{m-1,0}) = k - \left\lceil \frac{m-1}{2} \right\rceil = k - p.$$

All vertices of $C(\Gamma, -X)$ are resolved by W, hence $\dim(C(\Gamma, -X)) \leqslant |W| = \lfloor m/2 \rfloor + 1$. □

Theorem 2.2. *Let $\Gamma = Z_m \times Z_n$ and $X = \{(0,1),(1,0),(2,0)\}$, where $n \geqslant 3$ and $m \geqslant 2n+2$. Then $\dim(C(\Gamma,-X)) \leqslant n$.*

P r o o f. Let us show that

$$W = \{v_{0,0}, v_{0,1}, v_{0,2}, \ldots, v_{0,n-2}, v_{1,n-1}\}$$

is a resolving set of the graph $C(\Gamma,-X)$. We present all vertices having the same distance to $v_{0,0} \in W$. For $r = 0,1,2,\ldots,n-1$, by (2.1), we have

$$d(v_{2k-2r,r}, v_{0,0}) = \left\lceil \frac{2k-2r}{2} \right\rceil + r = k,$$
$$d(v_{2k-2r-1,r}, v_{0,0}) = \left\lceil \frac{2k-2r-1}{2} \right\rceil + r = k,$$

where $2k-2r-1, 2k-2r \in \{0,1,\ldots,m-1\}$. This implies that the only vertices having the distance k to $v_{0,0}$ are the vertices

$$v_{2k,0}, v_{2k-1,0}, v_{2k-2,1}, v_{2k-3,1}, \ldots, v_{2k-2(n-1),n-1}, v_{2k-2(n-1)-1,n-1},$$

where the first indices are at least 0 and at most $m-1$. We show that these vertices are resolved by W.

For $v_{0,1} \in W$ and $r = 1,2,\ldots,n-1$, by (2.1) we have

$$d(v_{2k-2r,r}, v_{0,1}) = d(v_{2k-2r-1,r}, v_{0,1}) = (k-r) + r - 1 = k-1$$

(note that $2k-2r-1, 2k-2r \in \{0,1,\ldots,m-1\}$). For $r = 0$, by (2.2) we obtain

$$d(v_{2k,0}, v_{0,1}) = d(v_{2k-1,0}, v_{0,1}) = k+n-1.$$

In general, for $v_{0,i} \in W$, where $i = 1,2,\ldots,n-2$, and for $r = i, i+1, \ldots, n-1$, by (2.1) we have

$$d(v_{2k-2r,r}, v_{0,i}) = \left\lceil \frac{2k-2r}{2} \right\rceil + r - i = k-i,$$
$$d(v_{2k-2r-1,r}, v_{0,i}) = \left\lceil \frac{2k-2r-1}{2} \right\rceil + r - i = k-i.$$

For $r = 0,1,\ldots,i-1$, by (2.2) we obtain

$$d(v_{2k-2r,r}, v_{0,i}) = \left\lceil \frac{2k-2r}{2} \right\rceil + n + r - i = k+n-i,$$
$$d(v_{2k-2r-1,r}, v_{0,i}) = \left\lceil \frac{2k-2r-1}{2} \right\rceil + n + r - i = k+n-i,$$

where $2k-2r-1, 2k-2r \in \{0,1,\ldots,m-1\}$.

Let $W' = \{v_{0,0}, v_{0,1}, v_{0,2}, \ldots, v_{0,n-2}\} \subset W$. It follows that

$$r(v_{2k,0}|W') = r(v_{2k-1,0}|W') = (k, k+n-1, k+n-2, \ldots, k+2),$$
$$r(v_{2k-2,1}|W') = r(v_{2k-3,1}|W') = (k, k-1, k+n-2, \ldots, k+2),$$
$$\vdots$$
$$r(v_{2k-2(n-3),n-3}|W') =$$
$$r(v_{2k-2(n-3)-1,n-3}|W') = (k, k-1, k-2, \ldots, k-n+3, k+2),$$

which means that the vertices $v_{2k-2r,r}$ and $v_{2k-2r-1,r}$ have the same representations with respect to W' for $r = 0, 1, \ldots, n-3$. The representation of the vertices $v_{2k-2(n-2),n-2}$, $v_{2k-2(n-2)-1,n-2}$, $v_{2k-2(n-1),n-1}$ and $v_{2k-2(n-1)-1,n-1}$ with respect to W' is $(k, k-1, k-2, \ldots, k-n+2)$.

For $v_{1,n-1} \in W$ and $r = 0, 1, \ldots, n-3$, by (2.2) we obtain

$$d(v_{2k-2r,r}, v_{1,n-1}) = \left\lceil \frac{2k-2r-1}{2} \right\rceil + n + r - (n-1) = k+1,$$
$$d(v_{2k-2r-1,r}, v_{1,n-1}) = \left\lceil \frac{2k-2r-1-1}{2} \right\rceil + n + r - (n-1) = k,$$

where $2k-2r-1, 2k-2r \in \{1, 2, \ldots, m-1\}$. This implies that the vertices $v_{2k,0}, v_{2k-1,0}, v_{2k-2,1}, v_{2k-3,1}, \ldots, v_{2k-2(n-3),n-3}, v_{2k-2(n-3)-1,n-3}$ are resolved.

It remains to resolve the vertices $v_{2k-2(n-2),n-2}$, $v_{2k-2(n-2)-1,n-2}$, $v_{2k-2(n-1),n-1}$ and $v_{2k-2(n-1)-1,n-1}$. For $v_{1,n-1} \in W$, by (2.2) we have

$$d(v_{2k-2(n-2),n-2}, v_{1,n-1}) = k-(n-2)+n+(n-2)-(n-1) = k+1,$$
$$d(v_{2k-2(n-2)-1,n-2}, v_{1,n-1}) = k-(n-2)-1+n+(n-2)-(n-1) = k,$$

and by (2.1),

$$d(v_{2k-2(n-1),n-1}, v_{1,n-1}) = k-n+1,$$
$$d(v_{2k-2(n-1)-1,n-1}, v_{1,n-1}) = k-(n-1)-1 = k.$$

Hence W is a resolving set of $C(\Gamma, -X)$ which means that $\dim(C(\Gamma, -X)) \leqslant |W|$.

□

It is easy to check that Theorem 2.2 does not hold for $m = 2n$ and $2n+1$. From Theorems 2.1 and 2.2 we obtain the following corollary.

Corollary 2.3. *Let $\Gamma = Z_m \times Z_n$ and $X = \{(0,1), (1,0), (2,0)\}$, where $n \geqslant 3$ and $m \geqslant 4$. Then*

$$\dim(C(\Gamma, -X)) \leqslant \begin{cases} \min\left\{n, \left\lfloor \frac{m}{2} \right\rfloor + 1\right\} & \text{if } m \notin \{2n, 2n+1\}, \\ n+1 & \text{if } m = 2n \text{ or } 2n+1. \end{cases}$$

P r o o f. If $n \geqslant 3$ and $m \geqslant 2n+2$ then by Theorem 2.2 we get $\dim(C(\Gamma, -X)) \leqslant n$. Since $n \leqslant m/2 - 1$, we obtain $n = \min\{n, \lfloor m/2 \rfloor + 1\}$.

If $n \geqslant 3$ and $4 \leqslant m \leqslant 2n-1$, then by Theorem 2.1 we have $\dim(C(\Gamma, -X)) \leqslant \lfloor m/2 \rfloor + 1$. Since $\lfloor m/2 \rfloor + 1 \leqslant \lfloor (2n-1)/2 \rfloor + 1 = n$, we obtain $\lfloor m/2 \rfloor + 1 = \min\{n, \lfloor m/2 \rfloor + 1\}$.

If $m = 2n$ or $2n+1$, then by Theorem 2.1 we have $\dim(C(\Gamma, -X)) \leqslant \lfloor m/2 \rfloor + 1 = n+1$. □

Since the graphs $C(\Gamma, X)$ and $C(\Gamma, -X)$ are isomorphic, we get an upper bound on the metric dimension of $C(\Gamma, X)$.

Corollary 2.4. *Let* $\Gamma = Z_m \times Z_n$ *and* $X = \{(0,1), (1,0), (2,0)\}$, *where* $n \geqslant 3$ *and* $m \geqslant 4$. *Then*

$$\dim(C(\Gamma, X)) \leqslant \begin{cases} \min\Big\{n, \Big\lfloor \dfrac{m}{2} \Big\rfloor + 1\Big\} & \text{if } m \notin \{2n, 2n+1\}, \\ n+1 & \text{if } m = 2n \text{ or } 2n+1. \end{cases}$$

3. Directed Cayley graphs with 4 generators

Let us consider directed Cayley graphs $C(\Gamma, X)$ for the group $\Gamma = Z_m \times Z_n$ and the generating set $X = \{(0,1), (1,0), (2,0), (3,0)\}$. We present resolving sets which yield upper bounds on the metric dimension of $C(\Gamma, -X)$. The graph $C(\Gamma, -X)$ contains directed edges $v_{i,j}v_{i,j-1}$, $v_{i,j}v_{i-1,j}$, $v_{i,j}v_{i-2,j}$ and $v_{i,j}v_{i-3,j}$, where $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n-1$. The distance from vertex $v_{i,j}$ to vertex $v_{z,l}$ in $C(\Gamma, -X)$ is

$$d(v_{i,j}, v_{z,l}) = \Big\lceil \frac{i-z}{3} \Big\rceil + j - l \qquad \text{if } i \geqslant z,\ j \geqslant l, \tag{3.1}$$

$$d(v_{i,j}, v_{z,l}) = \Big\lceil \frac{i-z}{3} \Big\rceil + n + (j - l) \qquad \text{if } i \geqslant z,\ j < l, \tag{3.2}$$

$$d(v_{i,j}, v_{z,l}) = \Big\lceil \frac{m+(i-z)}{3} \Big\rceil + j - l \qquad \text{if } i < z,\ j \geqslant l, \tag{3.3}$$

$$d(v_{i,j}, v_{z,l}) = \Big\lceil \frac{m+(i-z)}{3} \Big\rceil + n + (j - l) \qquad \text{if } i < z,\ j < l. \tag{3.4}$$

Theorem 3.1. *Let* $\Gamma = Z_m \times Z_n$ *and* $X = \{(0,1), (1,0), (2,0), (3,0)\}$, *where* $n \geqslant 3$ *and* $m \geqslant 5$. *Then*

$$\dim(C(\Gamma, -X)) \leqslant \Big\lfloor \frac{m+1}{3} \Big\rfloor + 1.$$

P r o o f. Let $m = 3p + \varepsilon$, where $p \geqslant 2$ and $\varepsilon \in \{-1, 0, 1\}$. Let us show that

$$W = \{v_{0,0}, v_{3,0}, \ldots, v_{3(p-2),0}, v_{m-4,0}, v_{m-2,0}\}$$

is a resolving set of the graph $C(\Gamma, -X)$, where $\Gamma = Z_m \times Z_n$ and $X = \{(0,1), (1,0), (2,0), (3,0)\}$. Note that

$$|W| = (p-1) + 2 = p + 1 = \frac{m-\varepsilon}{3} + 1 = \left\lfloor \frac{m+1}{3} \right\rfloor + 1.$$

We present all vertices having the same distance to $v_{0,0} \in W$. For $r = 0, 1, 2, \ldots, m-1$, by (3.1),

$$d(v_{r,k-\lceil r/3\rceil}, v_{0,0}) = \left\lceil \frac{r}{3} \right\rceil + k - \left\lceil \frac{r}{3} \right\rceil = k,$$

where $0 \leqslant k - \lceil r/3 \rceil \leqslant n-1$. This implies that the only vertices having the distance k to $v_{0,0}$ are the vertices of the set

$$S_{-1} = \{v_{0,k}, v_{1,k-1}, v_{2,k-1}, v_{3,k-1}, v_{4,k-2}, \ldots, v_{3(p-1),k-(p-1)}, v_{3p-2,k-p}\}$$

if $\varepsilon = -1$ (where the second indices must be at least 0 and at most $n-1$). If $\varepsilon = 0$, we have one extra vertex $v_{3p-1,k-p}$, so for $\varepsilon = 0$ we define $S_0 = S_{-1} \cup \{v_{3p-1,k-p}\}$. If $\varepsilon = 1$, then the vertices $v_{3p-1,k-p}$ and $v_{3p,k-p}$ also have distance k to $v_{0,0}$, so $S_1 = S_{-1} \cup \{v_{3p-1,k-p}, v_{3p,k-p}\}$ is the set of vertices having the distance k to $v_{0,0} \in W$. We show that all vertices in S_ε for $\varepsilon = -1, 0, 1$, are resolved by W.

For $v_{3,0} \in W$ and $r = 3, 4, \ldots, m-1$, by (3.1) we have

$$d(v_{r,k-\lceil r/3\rceil}, v_{3,0}) = \left\lceil \frac{r-3}{3} \right\rceil + k - \left\lceil \frac{r}{3} \right\rceil = k - 1.$$

For $r = 0, 1, 2$, by (3.3) we obtain

$$d(v_{0,k}, v_{3,0}) = \left\lceil \frac{m-3}{3} \right\rceil + k = k - 1 + \left\lceil \frac{m}{3} \right\rceil,$$
$$d(v_{1,k-1}, v_{3,0}) = \left\lceil \frac{m-2}{3} \right\rceil + k - 1 = k - 1 + \left\lceil \frac{m-2}{3} \right\rceil$$

and

$$d(v_{2,k-1}, v_{3,0}) = \left\lceil \frac{m-1}{3} \right\rceil + k - 1 = k - 1 + \left\lceil \frac{m-1}{3} \right\rceil.$$

In general, for $v_{i,0} \in W$, where $i = 3, 6, \ldots, 3(p-2)$, and for $r = i, i+1, \ldots, m-1$, by (3.1) we have

$$d(v_{r,k-\lceil r/3\rceil}, v_{i,0}) = \left\lceil \frac{r-i}{3} \right\rceil + k - \left\lceil \frac{r}{3} \right\rceil = k - \frac{i}{3}.$$

For $r = 0, 1, \ldots, i-1$, by (3.3) we obtain

$$d(v_{r,k-\lceil r/3\rceil}, v_{i,0}) = \left\lceil \frac{m+r-i}{3} \right\rceil + k - \left\lceil \frac{r}{3} \right\rceil = k - \frac{i}{3} + \left\lceil \frac{m+r}{3} \right\rceil - \left\lceil \frac{r}{3} \right\rceil > k - \frac{i}{3},$$

where $0 \leqslant k - \lceil r/3 \rceil \leqslant n-1$.

This implies that the only vertices which can have the same representations with respect to $W' = \{v_{0,0}, v_{3,0}, \ldots, v_{3(p-2),0}\} \subset W$ are the vertices $v_{i,k-i/3}$, $v_{i+1,k-i/3-1}$, $v_{i+2,k-i/3-1}$ for $i = 0, 3, 6, \ldots, 3(p-3)$, which are the triples $(v_{0,k}, v_{1,k-1}, v_{2,k-1})$, $(v_{3,k-1}, v_{4,k-2}, v_{5,k-2}), \ldots, (v_{3(p-3),k-(p-3)}, v_{3p-8,k-(p-2)}, v_{3p-7,k-(p-2)})$ and the last $6+\varepsilon$ vertices of S_ε (which are the vertices $v_{3(p-2),k-(p-2)}$, $v_{3p-5,k-(p-1)}, \ldots$) have the representation $(k, k-1, \ldots, k-(p-2))$ with respect to W'.

Let us use $v_{m-4,0}, v_{m-2,0} \in W$ to resolve the vertices $v_{i,k-i/3}$, $v_{i+1,k-i/3-1}$, $v_{i+2,k-i/3-1}$ for $i = 0, 3, 6, \ldots, 3(p-3)$. For $v_{m-4,0} \in W$, by (3.3) we obtain

$$\begin{aligned}
d(v_{i,k-i/3}, v_{m-4,0}) &= \left\lceil \frac{m+i-(m-4)}{3} \right\rceil + k - \frac{i}{3} = k+2, \\
d(v_{i+1,k-i/3-1}, v_{m-4,0}) &= \left\lceil \frac{m+i+1-(m-4)}{3} \right\rceil + k - \frac{i}{3} - 1 = k+1, \\
d(v_{i+2,k-i/3-1}, v_{m-4,0}) &= \left\lceil \frac{m+i+2-(m-4)}{3} \right\rceil + k - \frac{i}{3} - 1 = k+1,
\end{aligned}$$

and for $v_{m-2,0} \in W$,

$$\begin{aligned}
d(v_{i,k-i/3}, v_{m-2,0}) &= \left\lceil \frac{m+i-(m-2)}{3} \right\rceil + k - \frac{i}{3} = k+1, \\
d(v_{i+1,k-i/3-1}, v_{m-2,0}) &= \left\lceil \frac{m+i+1-(m-2)}{3} \right\rceil + k - \frac{i}{3} - 1 = k, \\
d(v_{i+2,k-i/3-1}, v_{m-2,0}) &= \left\lceil \frac{m+i+2-(m-2)}{3} \right\rceil + k - \frac{i}{3} - 1 = k+1,
\end{aligned}$$

so the vertices $v_{i,k-i/3}$, $v_{i+1,k-i/3-1}$, $v_{i+2,k-i/3-1}$ are resolved. It remains to resolve the last $6+\varepsilon$ vertices of S_ε.

For $\varepsilon = -1$ we need to resolve the 5 vertices $v_{3p-6,k-(p-2)}$, $v_{3p-5,k-(p-1)}$, $v_{3p-4,k-(p-1)}$, $v_{3p-3,k-(p-1)}$, $v_{3p-2,k-p}$. Since $m = 3p-1$, we have $v_{m-4,0} = v_{3p-5,0}$ and $v_{m-2,0} = v_{3p-3,0}$. Then for $v_{3p-5,0} \in W$,

$$\begin{aligned}
d(v_{3p-6,k-(p-2)}, v_{3p-5,0}) &= \left\lceil \frac{m-1}{3} \right\rceil + k - (p-2) = k+2, \\
d(v_{3p-5,k-(p-1)}, v_{3p-5,0}) &= k-p+1, \\
d(v_{3p-4,k-(p-1)}, v_{3p-5,0}) &= \left\lceil \frac{1}{3} \right\rceil + k - (p-1) = k-p+2, \\
d(v_{3p-3,k-(p-1)}, v_{3p-5,0}) &= \left\lceil \frac{2}{3} \right\rceil + k - (p-1) = k-p+2, \\
d(v_{3p-2,k-p}, v_{3p-5,0}) &= \left\lceil \frac{3}{3} \right\rceil + k - p = k-p+1.
\end{aligned}$$

We resolve the pairs $(v_{3p-5,k-(p-1)}, v_{3p-2,k-p})$ and $(v_{3p-4,k-(p-1)}, v_{3p-3,k-(p-1)})$ by $v_{3p-3,0} \in W$. We get

$$\begin{aligned}
d(v_{3p-5,k-(p-1)}, v_{3p-3,0}) &= \left\lceil \frac{m-2}{3} \right\rceil + k - (p-1) = k, \\
d(v_{3p-4,k-(p-1)}, v_{3p-3,0}) &= \left\lceil \frac{m-1}{3} \right\rceil + k - (p-1) = k+1, \\
d(v_{3p-3,k-(p-1)}, v_{3p-3,0}) &= k - p + 1, \\
d(v_{3p-2,k-p}, v_{3p-3,0}) &= \left\lceil \frac{1}{3} \right\rceil + k - p = k - p + 1,
\end{aligned}$$

thus all vertices of $V(C(\Gamma, -X))$ are resolved if $\varepsilon = -1$.

For $\varepsilon = 0$ we resolve the 6 vertices $v_{3p-6,k-(p-2)}, v_{3p-5,k-(p-1)}, \ldots, v_{3p-1,k-p}$ by $v_{m-4,0} = v_{3p-4,0}$ and $v_{m-2,0} = v_{3p-2,0}$. For $v_{3p-4,0} \in W$,

$$\begin{aligned}
d(v_{3p-6,k-(p-2)}, v_{3p-4,0}) &= \left\lceil \frac{m-2}{3} \right\rceil + k - (p-2) = k+2, \\
d(v_{3p-5,k-(p-1)}, v_{3p-4,0}) &= \left\lceil \frac{m-1}{3} \right\rceil + k - (p-1) = k+1, \\
d(v_{3p-4,k-(p-1)}, v_{3p-4,0}) &= k - p + 1, \\
d(v_{3p-3,k-(p-1)}, v_{3p-4,0}) &= \left\lceil \frac{1}{3} \right\rceil + k - (p-1) = k - p + 2, \\
d(v_{3p-2,k-p}, v_{3p-4,0}) &= \left\lceil \frac{2}{3} \right\rceil + k - p = k - p + 1, \\
d(v_{3p-1,k-p}, v_{3p-4,0}) &= \left\lceil \frac{3}{3} \right\rceil + k - p = k - p + 1.
\end{aligned}$$

It remains to resolve the vertices $v_{3p-4,k-(p-1)}$, $v_{3p-2,k-p}$, $v_{3p-1,k-p}$. For $v_{3p-2,0} \in W$ we obtain

$$\begin{aligned}
d(v_{3p-4,k-(p-1)}, v_{3p-2,0}) &= \left\lceil \frac{m-2}{3} \right\rceil + k - (p-1) = k+1, \\
d(v_{3p-2,k-p}, v_{3p-2,0}) &= k - p, \\
d(v_{3p-1,k-p}, v_{3p-2,0}) &= \left\lceil \frac{1}{3} \right\rceil + k - p = k - p + 1,
\end{aligned}$$

so if $\varepsilon = 1$, all vertices are resolved.

For $\varepsilon = 1$ we resolve the 7 vertices $v_{3p-6,k-(p-2)}, v_{3p-5,k-(p-1)}, \ldots, v_{3p,k-p}$ by $v_{m-4,0} = v_{3p-3,0}$ and $v_{m-2,0} = v_{3p-1,0}$. For $v_{3p-3,0} \in W$,

$$\begin{aligned}
d(v_{3p-6,k-(p-2)}, v_{3p-3,0}) &= \left\lceil \frac{m-3}{3} \right\rceil + k - (p-2) = k+2, \\
d(v_{3p-5,k-(p-1)}, v_{3p-3,0}) &= \left\lceil \frac{m-2}{3} \right\rceil + k - (p-1) = k+1, \\
d(v_{3p-4,k-(p-1)}, v_{3p-3,0}) &= \left\lceil \frac{m-1}{3} \right\rceil + k - (p-1) = k+1, \\
d(v_{3p-3,k-(p-1)}, v_{3p-3,0}) &= k - p + 1,
\end{aligned}$$

$$\begin{aligned}
d(v_{3p-2,k-p}, v_{3p-3,0}) &= \left\lceil\frac{1}{3}\right\rceil + k - p = k - p + 1,\\
d(v_{3p-1,k-p}, v_{3p-3,0}) &= \left\lceil\frac{2}{3}\right\rceil + k - p = k - p + 1,\\
d(v_{3p,k-p}, v_{3p-3,0}) &= \left\lceil\frac{3}{3}\right\rceil + k - p = k - p + 1.
\end{aligned}$$

It remains to resolve the vertices $v_{3p-5,k-(p-1)}$, $v_{3p-4,k-(p-1)}$ and the vertices $v_{3p-3,k-(p-1)}$, $v_{3p-2,k-p}$, $v_{3p-1,k-p}$, $v_{3p,k-p}$. For $v_{3p-1,0} \in W$ we obtain

$$\begin{aligned}
d(v_{3p-5,k-(p-1)}, v_{3p-1,0}) &= \left\lceil\frac{m-4}{3}\right\rceil + k - (p-1) = k,\\
d(v_{3p-4,k-(p-1)}, v_{3p-1,0}) &= \left\lceil\frac{m-3}{3}\right\rceil + k - (p-1) = k + 1,\\
d(v_{3p-3,k-(p-1)}, v_{3p-1,0}) &= \left\lceil\frac{m-2}{3}\right\rceil + k - (p-1) = k + 1,\\
d(v_{3p-2,k-p}, v_{3p-1,0}) &= \left\lceil\frac{m-1}{3}\right\rceil + k - p = k,\\
d(v_{3p-1,k-p}, v_{3p-1,0}) &= k - p,\\
d(v_{3p,k-p}, v_{3p-1,0}) &= \left\lceil\frac{1}{3}\right\rceil + k - p = k - p + 1.
\end{aligned}$$

All vertices are resolved by W, therefore $\dim(C(\Gamma, -X)) \leqslant |W| = \lfloor (m+1)/3 \rfloor + 1$. □

Theorem 3.2. *Let* $\Gamma = Z_m \times Z_n$ *and* $X = \{(0,1), (1,0), (2,0), (3,0)\}$, *where* $n \geqslant 3$ *and* $m \geqslant 3n + 2$. *Then* $\dim(C(\Gamma, -X)) \leqslant n$.

P r o o f. We show that

$$W = \{v_{0,0}, v_{0,1}, v_{0,2}, \ldots, v_{0,n-3}, v_{1,n-2}, v_{2,n-1}\}$$

is a resolving set of the graph $C(\Gamma, -X)$. Let us present all vertices having the same distance to $v_{0,0} \in W$. For $r = 0, 1, 2, \ldots, n-1$, by (3.1) we have

$$\begin{aligned}
d(v_{3k-3r,r}, v_{0,0}) &= \left\lceil\frac{3k-3r}{3}\right\rceil + r = k,\\
d(v_{3k-3r-1,r}, v_{0,0}) &= \left\lceil\frac{3k-3r-1}{3}\right\rceil + r = k,\\
d(v_{3k-3r-2,r}, v_{0,0}) &= \left\lceil\frac{3k-3r-2}{3}\right\rceil + r = k,
\end{aligned}$$

where $3k-3r-2, 3k-3r-1, 3k-3r \in \{0, 1, \ldots, m-1\}$. This implies that the only vertices having the distance k to $v_{0,0}$ are the vertices $v_{3k,0}, v_{3k-1,0}, v_{3k-2,0}, v_{3k-3,1}$,

$v_{3k-4,1}, v_{3k-5,1}, \ldots, v_{3k-3(n-1),n-1}, v_{3k-3(n-1)-1,n-1}, v_{3k-3(n-1)-2,n-1}$, where the first indices are at least 0 and at most $m-1$. We show that these vertices are resolved by W.

For $v_{0,1} \in W$ and $r = 1, 2, \ldots, n-1$, by (3.1) we have

$$d(v_{3k-3r,r}, v_{0,1}) = d(v_{3k-3r-1,r}, v_{0,1}) = d(v_{3k-3r-2,r}, v_{0,1}) = (k-r) + r - 1 = k - 1$$

(note that $3k-3r-2, 3k-3r-1, 3k-3r \in \{0, 1, \ldots, m-1\}$). For $r = 0$, by (3.2) we obtain

$$d(v_{3k,0}, v_{0,1}) = d(v_{3k-1,0}, v_{0,1}) = d(v_{3k-2,0}, v_{0,1}) = k + n - 1.$$

In general, for $v_{0,i} \in W$, where $i = 1, 2, \ldots, n-3$, and for $r = i, i+1, \ldots, n-1$, by (3.1) we have

$$\begin{aligned} d(v_{3k-3r,r}, v_{0,i}) &= \left\lceil \frac{3k-3r}{3} \right\rceil + r - i = k - i, \\ d(v_{3k-3r-1,r}, v_{0,i}) &= \left\lceil \frac{3k-3r-1}{3} \right\rceil + r - i = k - i, \\ d(v_{3k-3r-2,r}, v_{0,i}) &= \left\lceil \frac{3k-3r-2}{3} \right\rceil + r - i = k - i. \end{aligned}$$

For $r = 0, 1, \ldots, i-1$, by (3.2) we obtain

$$\begin{aligned} d(v_{3k-3r,r}, v_{0,i}) &= \left\lceil \frac{3k-3r}{3} \right\rceil + n + r - i = k + n - i, \\ d(v_{3k-3r-1,r}, v_{0,i}) &= \left\lceil \frac{3k-3r-1}{3} \right\rceil + n + r - i = k + n - i, \\ d(v_{3k-3r-2,r}, v_{0,i}) &= \left\lceil \frac{3k-3r-2}{3} \right\rceil + n + r - i = k + n - i, \end{aligned}$$

where $3k-3r-2, 3k-3r-1, 3k-3r \in \{0, 1, \ldots, m-1\}$.

Let $W' = \{v_{0,0}, v_{0,1}, v_{0,2}, \ldots, v_{0,n-3}\} \subset W$. It follows that

$$\begin{aligned} r(v_{3k,0}|W') &= r(v_{3k-1,0}|W') = r(v_{3k-2,0}|W') \\ &= (k, k+n-1, k+n-2, \ldots, k+3), \\ r(v_{3k-3,1}|W') &= r(v_{3k-4,1}|W') = r(v_{3k-5,1}|W') \\ &= (k, k-1, k+n-2, \ldots, k+3), \\ &\vdots \\ r(v_{3k-3(n-4),n-4}|W') &= r(v_{3k-3(n-4)-1,n-4}|W') = r(v_{3k-3(n-4)-2,n-4}|W') \\ &= (k, k-1, k-2, \ldots, k-n+4, k+3), \end{aligned}$$

which means that the vertices $v_{3k-3r,r}$, $v_{3k-3r-1,r}$, $v_{3k-3r-2,r}$ have the same representations with respect to W' for $r = 0, 1, \ldots, n-4$. The representation of the vertices

$$v_{3k-3(n-3),n-3}, v_{3k-3(n-3)-1,n-3}, v_{3k-3(n-3)-2,n-3}, v_{3k-3(n-2),n-2}, v_{3k-3(n-2)-1,n-2},$$
$$v_{3k-3(n-2)-2,n-2}, v_{3k-3(n-1),n-1}, v_{3k-3(n-1)-1,n-1}, v_{3k-3(n-1)-2,n-1}$$

with respect to W' is $(k, k-1, k-2, \ldots, k-n+3)$.

For $v_{1,n-2} \in W$ and $r = 0, 1, \ldots, n-4$, by (3.2) we obtain

$$d(v_{3k-3r,r}, v_{1,n-2}) = \left\lceil \frac{3k-3r-1}{3} \right\rceil + n + r - (n-2) = k+2,$$
$$d(v_{3k-3r-1,r}, v_{1,n-2}) = \left\lceil \frac{3k-3r-1-1}{3} \right\rceil + n + r - (n-2) = k+2,$$
$$d(v_{3k-3r-2,r}, v_{1,n-2}) = \left\lceil \frac{3k-3r-2-1}{3} \right\rceil + n + r - (n-2) = k+1,$$

where $3k-3r-2, 3k-3r-1, 3k-3r \in \{1, 2, \ldots, m-1\}$.

For $v_{2,n-1} \in W$ and $r = 0, 1, \ldots, n-4$, by (3.2) we obtain

$$d(v_{3k-3r,r}, v_{2,n-1}) = \left\lceil \frac{3k-3r-2}{3} \right\rceil + n + r - (n-1) = k+1,$$
$$d(v_{3k-3r-1,r}, v_{2,n-1}) = \left\lceil \frac{3k-3r-1-2}{3} \right\rceil + n + r - (n-1) = k,$$
$$d(v_{3k-3r-2,r}, v_{2,n-1}) = \left\lceil \frac{3k-3r-2-2}{3} \right\rceil + n + r - (n-1) = k,$$

where $3k-3r-2, 3k-3r-1, 3k-3r \in \{2, 3, \ldots, m-1\}$. It follows that the vertices $v_{3k,0}$, $v_{3k-1,0}$, $v_{3k-2,0}$, $v_{3k-3,1}$, $v_{3k-4,1}$, $v_{3k-5,1}$, $v_{3k-3(n-4),n-4}$, $v_{3k-3(n-4)-1}$, $v_{3k-3(n-4)-2,n-4}$ are resolved.

It remains to resolve vertices $v_{3k-3(n-3),n-3}$, $v_{3k-3(n-3)-1,n-3}$, $v_{3k-3(n-3)-2,n-3}$, $v_{3k-3(n-2),n-2}$, $v_{3k-3(n-2)-1,n-2}$, $v_{3k-3(n-2)-2,n-2}$, $v_{3k-3(n-1),n-1}$, $v_{3k-3(n-1)-1,n-1}$ and $v_{3k-3(n-1)-2,n-1}$. For $v_{1,n-2} \in W$ we have

$$d(v_{3k-3(n-3),n-3}, v_{1,n-2}) = d(v_{3k-3(n-3)-1,n-3}, v_{1,n-2}) = k+2,$$
$$d(v_{3k-3(n-3)-2,n-3}, v_{1,n-2}) = k+1,$$
$$d(v_{3k-3(n-2),n-2}, v_{1,n-2}) = d(v_{3k-3(n-2)-1,n-2}, v_{1,n-2}) = k-n+2,$$
$$d(v_{3k-3(n-2)-2,n-2}, v_{1,n-2}) = k-n+1,$$
$$d(v_{3k-3(n-1),n-1}, v_{1,n-2}) = d(v_{3k-3(n-1)-1,n-1}, v_{1,n-2}) = k-n+2,$$
$$d(v_{3k-3(n-1)-2,n-1}, v_{1,n-2}) = k-n+1.$$

For $v_{2,n-1} \in W$ we have

$$\begin{aligned} d(v_{3k-3(n-3),n-3}, v_{2,n-1}) &= k+1, \\ d(v_{3k-3(n-3)-1,n-3}, v_{2,n-1}) = d(v_{3k-3(n-3)-2,n-3}, v_{2,n-1}) &= k, \\ d(v_{3k-3(n-2),n-2}, v_{2,n-1}) &= k+1, \\ d(v_{3k-3(n-2)-1,n-2}, v_{2,n-1}) = d(v_{3k-3(n-2)-2,n-2}, v_{2,n-1}) &= k, \\ d(v_{3k-3(n-1),n-1}, v_{2,n-1}) &= k-n+1, \\ d(v_{3k-3(n-1)-1,n-1}, v_{2,n-1}) = d(v_{3k-3(n-1)-2,n-1}, v_{2,n-1}) &= k-n. \end{aligned}$$

Hence W is a resolving set of $C(\Gamma, -X)$. The proof is complete. □

Let us note that Theorem 3.2 does not hold for $m = 3n-1, 3n$ and $3n+1$. From Theorems 3.1 and 3.2 we get the following upper bound on the metric dimension of $C(\Gamma, -X)$.

Corollary 3.3. *Let* $\Gamma = Z_m \times Z_n$ *and* $X = \{(0,1), (1,0), (2,0), (3,0)\}$, *where* $n \geqslant 3$ *and* $m \geqslant 5$. *Then*

$$\dim(C(\Gamma, -X)) \leqslant \begin{cases} \min\Big\{n, \Big\lfloor \dfrac{m+1}{3} \Big\rfloor + 1\Big\} & \text{if } m \notin \{3n-1, 3n, 3n+1\}, \\ n+1 & \text{if } m = 3n-1,\ 3n \text{ or } 3n+1. \end{cases}$$

P r o o f. If $n \geqslant 3$ and $m \geqslant 3n+2$ then by Theorem 3.2 we have $\dim C(\Gamma, -X) \leqslant n$. Since $n \leqslant (m-2)/3$, we obtain $n = \min\{n, \lfloor (m+1)/3 \rfloor + 1\}$.

If $n \geqslant 3$ and $5 \leqslant m \leqslant 3n-2$, then by Theorem 3.1 we get

$$\dim C(\Gamma, -X) \leqslant \Big\lfloor \frac{m+1}{3} \Big\rfloor + 1.$$

Since

$$\Big\lfloor \frac{m+1}{3} \Big\rfloor + 1 \leqslant \Big\lfloor \frac{3n-1}{3} \Big\rfloor + 1 = n,$$

we obtain

$$\Big\lfloor \frac{m+1}{3} \Big\rfloor + 1 = \min\Big\{n, \Big\lfloor \frac{m+1}{3} \Big\rfloor + 1\Big\}.$$

If $m = 3n-1, 3n$ or $3n+1$, then by Theorem 3.1 we have

$$\dim C(\Gamma, -X) \leqslant \Big\lfloor \frac{(m+1)}{3} \Big\rfloor + 1 = n+1.$$

□

The graphs $C(\Gamma, X)$ and $C(\Gamma, -X)$ are isomorphic, so we obtain Corollary 3.4.

Corollary 3.4. *Let* $\Gamma = Z_m \times Z_n$ *and* $X = \{(0,1),(1,0),(2,0),(3,0)\}$, *where* $n \geqslant 3$ *and* $m \geqslant 5$. *Then*

$$\dim C(\Gamma, X) \leqslant \begin{cases} \min\Big\{n, \Big\lfloor \dfrac{m+1}{3} \Big\rfloor + 1\Big\} & \text{if } m \notin \{3n-1, 3n, 3n+1\}, \\ n+1 & \text{if } m = 3n-1,\ 3n \text{ or } 3n+1. \end{cases}$$

4. Conclusion

In this paper we obtained strong upper bounds on the metric dimension of directed Cayley graphs $C(\Gamma, X)$ for $\Gamma = Z_m \times Z_n$ and $X = \{(0,1),(1,0),(2,0),\dots,(p,0)\}$ where $p = 2$ and 3. Note that the directed Cayley graph $C(Z_n, Y)$ for $Y = \{1, 2, \dots, p\}$ is isomorphic to the circulant graph $C_n(1, 2, \dots, p)$, thus the graph $C(\Gamma, X)$ for the group $\Gamma = Z_m \times Z_n$ and $X = \{(0,1),(1,0),(2,0)\}$ is isomorphic to the Cartesian product $C_m(1,2) \times C_n$, where C_n is the cycle of order n. Similarly, the Cayley graph $C(\Gamma, X)$ for $\Gamma = Z_m \times Z_n$ and $X = \{(0,1),(1,0),(2,0),(3,0)\}$ is isomorphic to the graph $C_m(1,2,3) \times C_n$.

For positive integers n, p and $a_1, a_2, \dots, a_p$ such that $1 \leqslant a_1 < a_2 < \dots < a_p \leqslant n-1$, the directed circulant graph $C_n(a_1, a_2, \dots, a_p)$ consists of the vertices $v_0, v_1, \dots, v_{n-1}$ and directed edges $v_i v_{i+a_j}$ for every $i = 0, 1, \dots, n-1$ and $j = 1, 2, \dots, p$ indices are taken modulo n.

Hence, from Corollaries 2.4 and 3.4 we obtain bounds on the metric dimension for the Cartesian product of a circulant graph and a cycle.

Corollary 4.1. *Let* $n \geqslant 3$ *and* $m \geqslant 4$. *Then*

$$\dim(C_m(1,2) \times C_n) \leqslant \begin{cases} \min\Big\{n, \Big\lfloor \dfrac{m}{2} \Big\rfloor + 1\Big\} & \text{if } m \notin \{2n, 2n+1\}, \\ n+1 & \text{if } m = 2n \text{ or } 2n+1. \end{cases}$$

Corollary 4.2. *Let* $n \geqslant 3$ *and* $m \geqslant 5$. *Then*

$$\dim(C_m(1,2,3) \times C_n) \leqslant \begin{cases} \min\Big\{n, \Big\lfloor \dfrac{m+1}{3} \Big\rfloor + 1\Big\} & \text{if } m \notin \{3n-1, 3n, 3n+1\}, \\ n+1 & \text{if } m = 3n-1,\ 3n \text{ or } 3n+1. \end{cases}$$

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