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BIGRAPHIC PAIRS WITH A REALIZATION CONTAINING A SPLIT BIPARTITE-GRAPH

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Abstract. Let $K_{s,t}$ be the complete bipartite graph with partite sets $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$. A split bipartite-graph on $(s+s')+(t+t')$ vertices, denoted by $\mathrm{SB}_{s+s',t+t'}$, is the graph obtained from $K_{s,t}$ by adding $s'+t'$ new vertices $x_{s+1}, \dots, x_{s+s'}$, $y_{t+1}, \dots, y_{t+t'}$ such that each of $x_{s+1}, \dots, x_{s+s'}$ is adjacent to each of $y_1, \dots, y_t$ and each of $y_{t+1}, \dots, y_{t+t'}$ is adjacent to each of $x_1, \dots, x_s$. Let A and B be nonincreasing lists of nonnegative integers, having lengths m and n, respectively. The pair $(A; B)$ is potentially $\mathrm{SB}_{s+s',t+t'}$-bigraphic if there is a simple bipartite graph containing $\mathrm{SB}_{s+s',t+t'}$ (with $s+s'$ vertices $x_1, \dots, x_{s+s'}$ in the part of size m and $t+t'$ vertices $y_1, \dots, y_{t+t'}$ in the part of size n) such that the lists of vertex degrees in the two partite sets are A and B. In this paper, we give a characterization for $(A; B)$ to be potentially $\mathrm{SB}_{s+s',t+t'}$-bigraphic. A simplification of this characterization is also presented.

Keywords: degree sequence; bigraphic pair; potentially $\mathrm{SB}_{s+s',t+t'}$-bigraphic pair

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1. Introduction

All graphs considered here are simple, that is, contain neither loops nor multiple edges. A sequence $\pi = (d_1, d_2, \dots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is referred to as a *realization* of π. The following well-known result due to Erdős and Gallai in [1] gives a characterization for π to be graphic.

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Theorem 1.1 ([1]). *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers, where $\sum\limits_{i=1}^{n} d_i$ is even. Then π is graphic if and only if*

$$\sum_{i=1}^{t} d_i \leqslant t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\} \tag{1}$$

for all t with $1 \leqslant t \leqslant n$.

Nash-Williams in [6] further showed that Theorem 1.1 remains valid if condition (1) is assumed only for those t for which $d_t > d_{t+1}$. Recently, Tripathi et al. in [9] gave a short constructive proof of Theorem 1.1.

For a given graph H, a graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be *potentially H-graphic* if there is a realization of π containing H as a subgraph. Rao in [7] gave a characterization of π that is potentially K_{r+1}-graphic. This is an extension of Theorem 1.1 ($r = 0$).

Theorem 1.2 ([7]). *Let $n \geqslant r+1$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers, where $d_{r+1} \geqslant r$ and $\sum\limits_{i=1}^{n} d_i$ is even. Then π is potentially K_{r+1}-graphic if and only if*

$$\begin{aligned}\sum_{i=1}^{p}(d_i - r) + \sum_{i=r+2}^{r+1+q} d_i \leqslant{}& (p+q)(p+q-1) - p(p-1)\\ &+ \sum_{i=p+1}^{r+1} \min\{q, d_i - r\} + \sum_{i=r+q+2}^{n} \min\{p+q, d_i\}\end{aligned} \tag{2}$$

for all p and q with $0 \leqslant p \leqslant r+1$ and $0 \leqslant q \leqslant n-r-1$.

Rao in [7] also further showed that Theorem 1.2 remains valid if condition (2) is assumed only for those p and q for which $d_p > d_{p+1}$ or $p = 0$ or $p = r+1$ and $d_{r+1+q} > d_{r+2+q}$ or $q = 0$ or $q = n-m-1$. In [7], Rao gave a lengthy induction proof of Theorem 1.2 via linear algebraic techniques that remains unpublished, but Kézdy and Lehel in [5] have given another proof using network flows. Recently, Yin in [11] obtained a short constructive proof of Theorem 1.2.

Let K_r be the complete graph with vertex set $\{v_1, \ldots, v_r\}$. A *complete split graph* on $r+s$ vertices, denoted by $S_{r,s}$, is the graph obtained from K_r by adding s new vertices $v_{r+1}, \ldots, v_{r+s}$ such that each of $v_{r+1}, \ldots, v_{r+s}$ is adjacent to each of $v_1, \ldots, v_r$. Clearly, $S_{r,1} = K_{r+1}$. Therefore, $S_{r,s}$ is an extension of K_{r+1}. Yin in [10] established a Rao-type characterization of π that is potentially $S_{r,s}$-graphic. This is an extension of Theorem 1.2 ($s = 1$).

Theorem 1.3 ([10]). *Let $n \geqslant r+s$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers, where $d_r \geqslant r+s-1$, $d_{r+s} \geqslant r$ and $\sum_{i=1}^{n} d_i$ is even. Then π is potentially $S_{r,s}$-graphic if and only if*

$$
\begin{aligned}
(3)\quad \sum_{i=1}^{p}(d_i - r - s + 1) + \sum_{i=r+1}^{r+p'}(d_i - r) + \sum_{i=r+s+1}^{r+s+q} d_i \\
\leqslant (p+p'+q)(p+p'+q-1) - p(p-1) - 2pp' \\
+ \sum_{i=p+1}^{r} \min\{q, d_i - r - s + 1\} + \sum_{i=r+p'+1}^{r+s} \min\{p'+q, d_i - r\} \\
+ \sum_{i=r+s+q+1}^{n} \min\{p+p'+q, d_i\}
\end{aligned}
$$

for all p, p' and q with $0 \leqslant p \leqslant r$, $0 \leqslant p' \leqslant s$ and $0 \leqslant q \leqslant n-r-s$.

Yin in [10] also further showed that Theorem 1.3 remains valid if condition (3) is assumed only for those p, p' and q for which $d_p > d_{p+1}$ or $p = 0$ or $p = r$, $d_{r+p'} > d_{r+p'+1}$ or $p' = 0$ or $p' = s$ and $d_{r+s+q} > d_{r+s+q+1}$ or $q = 0$ or $q = n-r-s$.

Let A be an m-tuple and B an n-tuple of nonnegative integers; we take $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$, indexed so that each list is nonincreasing. If there is a simple bipartite graph G such that A and B are the lists of vertex degrees for the two partite sets, then the pair $(A; B)$ is *bigraphic* and G is a *realization* of the pair $(A; B)$. Let $K_{s,t}$ be the complete bipartite graph with partite sets $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$. We say that the pair $(A; B)$ is *potentially $K_{s,t}$-bigraphic* if some realization of $(A; B)$ contains $K_{s,t}$ (with s vertices $x_1, \ldots, x_s$ in the part of size m and t vertices $y_1, \ldots, y_t$ in the part of size n). The following Theorem 1.4 is the well-known Gale-Ryser characterization of bigraphic pairs.

Theorem 1.4 ([3], [8]). *The pair $(A; B)$ is bigraphic if and only if $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$ and*

$$
(4)\quad \sum_{i=1}^{k} a_i \leqslant \sum_{j=1}^{n} \min\{k, b_j\}
$$

for all k with $1 \leqslant k \leqslant m-1$.

Recently, Garg et al. in [4] presented a short constructive proof of Theorem 1.4. Yin and Huang in [13] gave a Gale-Ryser type characterization of potentially $K_{s,t}$-bigraphic pairs.

Theorem 1.5 ([13]). *The pair* $(A;B)$ *is potentially* $K_{s,t}$*-bigraphic if and only if* $a_s \geqslant t$, $b_t \geqslant s$, $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$ *and*

$$(5) \qquad \sum_{i=1}^{p} a_i + \sum_{i=1}^{q} a_{s+i} \leqslant pt + \sum_{j=1}^{t} \min\{q, b_j - s\} + \sum_{j=t+1}^{n} \min\{p+q, b_j\}$$

for all p *and* q *with* $0 \leqslant p \leqslant s$ *and* $0 \leqslant q \leqslant m - s$.

Theorem 1.5 reduces to Theorem 1.4 when $s = t = 0$. Recently, Yin in [12] presented a simplification of Theorem 1.5, that is, Theorem 1.5 remains valid if condition (5) is assumed only for those p and q for which $a_p > a_{p+1}$ or $p = 0$ or $p = s$ and $a_{s+q} > a_{s+q+1}$ or $q = 0$ or $q = m - s$.

A *split bipartite-graph* on $(s+s') + (t+t')$ vertices, denoted by $\mathrm{SB}_{s+s',t+t'}$, is the graph obtained from $K_{s,t}$ by adding $s' + t'$ new vertices $x_{s+1}, \ldots, x_{s+s'}$, $y_{t+1}, \ldots, y_{t+t'}$ such that each of $x_{s+1}, \ldots, x_{s+s'}$ is adjacent to each of $y_1, \ldots, y_t$ and each of $y_{t+1}, \ldots, y_{t+t'}$ is adjacent to each of $x_1, \ldots, x_s$. The pair $(A;B)$ is *potentially* $\mathrm{SB}_{s+s',t+t'}$*-bigraphic* if some realization of $(A;B)$ contains $\mathrm{SB}_{s+s',t+t'}$ (with $s+s'$ vertices $x_1, \ldots, x_{s+s'}$ in the part of size m and $t+t'$ vertices $y_1, \ldots, y_{t+t'}$ in the part of size n). Clearly, if $s' = t' = 0$, then $\mathrm{SB}_{s,t} = K_{s,t}$. Therefore $\mathrm{SB}_{s+s',t+t'}$ is an extension of $K_{s,t}$. The purpose of this paper is to establish a characterization of the pairs $(A;B)$ that are potentially $\mathrm{SB}_{s+s',t+t'}$-bigraphic. That is the following Theorem 1.6.

Theorem 1.6. *The pair* $(A;B)$ *is potentially* $\mathrm{SB}_{s+s',t+t'}$*-bigraphic if and only if* $a_s \geqslant t+t'$, $b_t \geqslant s+s'$, $a_{s+s'} \geqslant t$, $b_{t+t'} \geqslant s$, $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$ *and*

$$(6) \qquad \sum_{i=1}^{p} a_i + \sum_{i=1}^{q} a_{s+i} + \sum_{i=1}^{r} a_{s+s'+i}$$
$$\leqslant (p+q)t + pt' + \sum_{j=1}^{t} \min\{r, b_j - s - s'\} + \sum_{j=t+1}^{t+t'} \min\{q+r, b_j - s\}$$
$$+ \sum_{j=t+t'+1}^{n} \min\{p+q+r, b_j\}$$

for all p, q *and* r *with* $0 \leqslant p \leqslant s$, $0 \leqslant q \leqslant s'$ *and* $0 \leqslant r \leqslant m - s - s'$.

Theorem 1.6 reduces to Theorem 1.5 when $s' = t' = 0$. We also present a simplification of Theorem 1.6.

Theorem 1.7. *Theorem* 1.6 *remains valid if condition* (6) *is assumed only for those* p, q *and* r *for which* $a_p > a_{p+1}$ *or* $p = 0$ *or* $p = s$, $a_{s+q} > a_{s+q+1}$ *or* $q = 0$ *or* $q = s'$ *and* $a_{s+s'+r} > a_{s+s'+r+1}$ *or* $r = 0$ *or* $r = m - s - s'$.

2. Proofs of theorems 1.6–1.7

The following useful lemma is due to Ferrara et al. in [2].

Lemma 2.1 ([2]). *Let G be a realization of the pair $(A;B)$ with partite sets X and Y. If H is a subgraph of G whose vertex set consists of X' in X and Y' in Y, then $(A;B)$ has a realization G' containing H such that the vertices of H are the highest-degree vertices both in X and in Y.*

The necessity of Theorem 1.6 relies on the following lemma. For a graph G and a vertex u in G, $N_G(u)$ denotes the set of neighbors of u in G.

Lemma 2.2. *If $(A;B)$ is potentially $\mathrm{SB}_{s+s',t+t'}$-bigraphic, then $(A;B)$ has a realization G with partite sets $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_n\}$ such that $d_G(x_i)=a_i$ for $1\leqslant i\leqslant m$, $d_G(y_i)=b_i$ for $1\leqslant i\leqslant n$, each of $x_1,\dots,x_s$ is adjacent to each of $y_1,\dots,y_{t+t'}$ and each of $y_1,\dots,y_t$ is adjacent to each of $x_1,\dots,x_{s+s'}$.*

Proof. By Lemma 2.1, we may assume that G is a realization of $(A;B)$ with partite sets $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_n\}$ such that $d_G(x_i)=a_i$ for $1\leqslant i\leqslant m$, $d_G(y_i)=b_i$ for $1\leqslant i\leqslant n$ and G contains $\mathrm{SB}_{s+s',t+t'}$ on $x_1,\dots,x_{s+s'},y_1,\dots,y_{t+t'}$. If there is a $u\in\{x_1,\dots,x_s\}$ such that u is not adjacent to each of $y_1,\dots,y_{t+t'}$, then there is a $u'\in\{x_{s+1},\dots,x_{s+s'}\}$ such that u' is adjacent to each of $y_1,\dots,y_{t+t'}$. Denote $A_1=\{y_1,\dots,y_{t+t'}\}\setminus N_G(u)$, $B_1=N_G(u)\setminus\{y_1,\dots,y_{t+t'}\}$, $B_2=N_G(u')\setminus\{y_1,\dots,y_{t+t'}\}$ and $C=B_1\setminus B_2$. Since $d_G(u)\geqslant d_G(u')$, we have $t+t'-|A_1|+|B_1|\geqslant t+t'+|B_2|$, i.e., $|B_1|\geqslant|A_1|+|B_2|$, implying that $|C|=|B_1|-|B_1\cap B_2|\geqslant|B_1|-|B_2|\geqslant|A_1|$. Choose any subset $C'\subseteq C$ having $|C'|=|A_1|$. Now form a new realization G' of $(A;B)$ by interchanging the edges of the star centered at u with endvertices in C' with the non-edges of the star centered at u with endvertices in A_1, and interchanging the edges of the star centered at u' with endvertices in A_1 with the non-edges of the star centered at u' with endvertices in C'. Then u is adjacent to each of $y_1,\dots,y_{t+t'}$ in G'. Repeat this process until each of $x_1,\dots,x_s$ is adjacent to each of $y_1,\dots,y_{t+t'}$. In a similar way we can achieve that each of $y_1,\dots,y_t$ is adjacent to each of $x_1,\dots,x_{s+s'}$. □

Proof of Theorem 1.6. To prove the necessity, by Lemma 2.2, we may let G be a realization of $(A;B)$ with partite sets $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_n\}$ such that $d_G(x_i)=a_i$ for $1\leqslant i\leqslant m$, $d_G(y_i)=b_i$ for $1\leqslant i\leqslant n$, each of $x_1,\dots,x_s$ is adjacent to each of $y_1,\dots,y_{t+t'}$ and each of $y_1,\dots,y_t$ is adjacent to each of $x_1,\dots,x_{s+s'}$. This requires $a_s\geqslant t+t'$, $b_t\geqslant s+s'$, $a_{s+s'}\geqslant t$ and $b_{t+t'}\geqslant s$. Moreover, $\sum\limits_{i=1}^{p}a_i+\sum\limits_{i=1}^{q}a_{s+i}+\sum\limits_{i=1}^{r}a_{s+s'+i}$ is the sum of the number of edges from y_h to

$\{x_1,\dots,x_p,x_{s+1},\dots,x_{s+q},x_{s+s'+1},\dots,x_{s+s'+r}\}$, the summation being taken over $h=1,2,\dots,n$. Now the contribution of y_h to this sum is at most $\min\{p+q+r, b_j-(s-p)-(s'-q)\}$ if $h\in\{1,\dots,t\}$, at most $\min\{p+q+r,b_j-(s-p)\}$ if $h\in\{t+1,\dots,t+t'\}$ and at most $\min\{p+q+r,b_j\}$ if $h\in\{t+t'+1,\dots,n\}$. This gives, after easy algebraic manipulations, the right side and the necessity is proved.

For the sufficiency, we let a *subrealization* of $(A;B)$ be a bipartite graph with partite sets $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_n\}$ such that $d(x_i)\leqslant a_i$ for each i and $d(y_j)\leqslant b_j$ for each j. We will construct a realization of $(A;B)$ through successive subrealizations. In the initial subrealization, each of $x_1,\dots,x_s$ is adjacent to each of $y_1,\dots,y_{t+t'}$ and each of $y_1,\dots,y_t$ is adjacent to each of $x_1,\dots,x_{s+s'}$. This subrealization contains $\mathrm{SB}_{s+s',t+t'}$ in the desired location and has no other edges.

A subrealization has three critical indices. Let p be the largest index such that $d(x_i)=a_i$ for $1\leqslant i<p\leqslant s$, let q be the largest index such that $d(x_{s+i})=a_{s+i}$ for $1\leqslant i<q\leqslant s'$ and let r be the largest index such that $d(x_{s+s'+i})=a_{s+s'+i}$ for $1\leqslant i<r\leqslant m-s-s'$. The *critical deficiency* is $(a_p-d(x_p))+(a_{s+q}-d(x_{s+q}))+(a_{s+s'+r}-d(x_{s+s'+r}))$. While $p\leqslant s$ or $q\leqslant s'$ or $r\leqslant m-s-s'$, we obtain a new subrealization having the same degrees of $x_1,\dots,x_{p-1},x_{s+1},\dots,x_{s+q-1}$ and $x_{s+s'+1},\dots,x_{s+s'+q-1}$ but smaller critical deficiency or larger critical indices. The new subrealization need not contain the previous subrealization, but it contains the initial subrealization and hence contains $\mathrm{SB}_{s+s',t+t'}$. The process can only stop when the subrealization is a realization of $(A;B)$ containing $\mathrm{SB}_{s+s',t+t'}$.

Let $X_1=\{x_{p+1},\dots,x_s\}$, $X_2=\{x_{s+q+1},\dots,x_{s+s'}\}$ and $X_3=\{x_{s+s'+r+1},\dots,x_m\}$. We maintain the conditions that each of $x_1,\dots,x_s$ is adjacent to each of $y_1,\dots,y_{t+t'}$ and each of $y_1,\dots,y_t$ is adjacent to each of $x_1,\dots,x_{s+s'}$, there is no edge joining $\{y_1,\dots,y_t\}$ and X_3, there is no edge joining $\{y_{t+1},\dots,y_{t+t'}\}$ and $X_2\cup X_3$, and there is no edge joining $\{y_{t+t'+1},\dots,y_n\}$ and $X_1\cup X_2\cup X_3$, which certainly hold initially.

Case 0: $x_py_i\notin E(G)$ for some vertex y_i such that $d(y_i)<b_i$. Add the edge x_py_i.

Case 1: $x_{s+q}y_j\notin E(G)$ for some vertex y_j such that $d(y_j)<b_j$. Add the edge $x_{s+q}y_j$.

Case 2: $x_{s+s'+r}y_j\notin E(G)$ for some vertex y_j such that $d(y_j)<b_j$. Add the edge $x_{s+s'+r}y_j$.

Case 3: $d(y_k)\neq\min\{p+q+r,b_k\}$ for a k with $k\geqslant t+t'+1$. In a subrealization, $d(y_k)\leqslant b_k$. Since there is no edge joining $\{y_{t+t'+1},\dots,y_n\}$ and $X_1\cup X_2\cup X_2$, $d(y_k)\leqslant p+q+r$. Hence $d(y_k)<\min\{p+q+r,b_k\}$. Case 0, Case 1 and Case 2 apply unless $x_py_k,x_{s+q}y_k,x_{s+s'+r}y_k\in E(G)$. Since $d(y_k)<p+q+r$, there exists i with $i\in\{1,\dots,p-1,s+1,\dots,s+q-1,s+s'+1,\dots,s+s'+r-1\}$ such that $x_iy_k\notin E(G)$. If $i\in\{1,\dots,p-1\}$, then since $p\leqslant s$ and $d(x_i)=a_i\geqslant a_p>d(x_p)$, there exists $u\in N(x_i)\setminus N(x_p)$; in this case, replace ux_i with $\{x_iy_k,ux_p\}$. If $i\in\{s+1,\dots,s+q-1\}$, then since $d(x_i)>d(x_{s+q})$, there exists $u\in N(x_i)\setminus N(x_{s+q})$; in this case, replace ux_i

with $\{x_iy_k, ux_{s+q}\}$. If $i \in \{s+s'+1, \dots, s+s'+r-1\}$, then since $d(x_i) > d(x_{s+s'+r})$, there exists $u \in N(x_i) \setminus N(x_{s+s'+r})$; in this case, replace ux_i with $\{x_iy_k, ux_{s+s'+r}\}$.

Case 4: $d(y_k)-s \neq \min\{q+r, b_k-s\}$ for a k with $t < k \leqslant t+t'$. In a subrealization, $d(y_k) - s \leqslant b_k - s$. Since there is no edge joining $\{y_{t+1}, \dots, y_{t+t'}\}$ and $X_2 \cup X_3$, $d(y_k)-s \leqslant q+r$. Hence $d(y_k)-s < \min\{q+r, b_k-s\}$. Case 1 and Case 2 apply unless $x_{s+q}y_k, x_{s+s'+r}y_k \in E(G)$. Since $d(y_k) - s < q+r$ and $x_1y_k, \dots, x_sy_k \in E(G)$, there exists i with $i \in \{s+1, \dots, s+q-1, s+s'+1, \dots, s+s'+r-1\}$ such that $x_iy_k \notin E(G)$. If $i \in \{s+1, \dots, s+q-1\}$, then by $d(x_i) > d(x_{s+q})$, there exists $u \in N(x_i) \setminus N(x_{s+q})$; replace ux_i with $\{x_iy_k, ux_{s+q}\}$. If $i \in \{s+s'+1, \dots, s+s'+r-1\}$, then by $d(x_i) > d(x_{s+s'+r})$, there exists $u \in N(x_i) \setminus N(x_{s+s'+r})$; replace ux_i with $\{x_iy_k, ux_{s+s'+r}\}$.

Case 5: $d(y_k)-s-s' \neq \min\{r, b_k-s-s'\}$ for a k with $1 \leqslant k \leqslant t$. In a subrealization, $d(y_k) - s - s' \leqslant b_k - s - s'$. Since there is no edge joining $\{y_1, \dots, y_t\}$ and X_3, $d(y_k) - s - s' \leqslant r$. Hence $d(y_k) - s - s' < \min\{r, b_k - s - s'\}$. Case 2 applies unless $x_{s+s'+r}y_k \in E(G)$. Since $d(y_k) - s - s' < r$ and $x_1y_k, \dots, x_{s+s'}y_k \in E(G)$, there exists i with $i \in \{s+s'+1, \dots, s+s'+r-1\}$ such that $x_iy_k \notin E(G)$. By $d(x_i) > d(x_{s+s'+r})$, there exists $u \in N(x_i) \setminus N(x_{s+s'+r})$; replace ux_i with $\{x_iy_k, ux_{s+s'+r}\}$.

It is easy to check that the above conditions are preserved by the replacements of edges in all Cases 0–5. If none of these cases applies, then $d(y_k) = \min\{p+q+r, b_k\}$ for $k \geqslant t+t'+1$, $d(y_k) - s = \min\{q+r, b_k-s\}$ for $t < k \leqslant t+t'$ and $d(y_k) - s - s' = \min\{r, b_k - s - s'\}$ for $1 \leqslant k \leqslant t$. Since each of $x_1, \dots, x_s$ is adjacent to each of $y_1, \dots, y_{t+t'}$ and each of $y_1, \dots, y_t$ is adjacent to each of $x_1, \dots, x_{s+s'}$, there is no edge joining $\{y_1, \dots, y_t\}$ and X_3, there is no edge joining $\{y_{t+1}, \dots, y_{t+t'}\}$ and $X_2 \cup X_3$ and there is no edge joining $\{y_{t+t'+1}, \dots, y_n\}$ and $X_1 \cup X_2 \cup X_3$, we have that

$$\begin{aligned}
&\sum_{i=1}^{p} d(x_i) + \sum_{i=1}^{q} d(x_{s+i}) + \sum_{i=1}^{r} d(x_{s+s'+i}) \\
&\qquad = p(t+t') + qt + \sum_{j=1}^{t} (d(y_j) - s - s') + \sum_{j=t+1}^{t+t'} (d(y_j) - s) + \sum_{j=t+t'+1}^{n} d(y_j) \\
&\qquad = (p+q)t + pt' + \sum_{j=1}^{t} \min\{r, b_j - s - s'\} + \sum_{j=t+1}^{t+t'} \min\{q+r, b_j - s\} \\
&\qquad\quad + \sum_{j=t+t'+1}^{n} \min\{p+q+r, b_j\}.
\end{aligned}$$

By (6) and since $d(x_i) \leqslant a_i$, we get that

$$\sum_{i=1}^{p} a_i + \sum_{i=1}^{q} a_{s+i} + \sum_{i=1}^{r} a_{s+s'+i} = \sum_{i=1}^{p} d(x_i) + \sum_{i=1}^{q} d(x_{s+i}) + \sum_{i=1}^{r} d(x_{s+s'+i}),$$

implying that $d(x_p) = a_p$, $d(x_{s+q}) = a_{s+q}$ and $d(x_{s+s'+r}) = a_{s+s'+r}$. Increase p by 1, q by 1 and r by 1, and continue.

Finally, a subrealization containing $\mathrm{SB}_{s+s',t+t'}$ is obtained so that $d(x_i) = a_i$ for $1 \leqslant i \leqslant m$. By $d(y_i) \leqslant b_i$ for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^{n} d(y_i) = \sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$, we have that $d(y_i) = b_i$ for $1 \leqslant i \leqslant n$. Hence we have constructed a realization of $(A; B)$. □

Proof of Theorem 1.7. We only need to show that if condition (6) holds for those p, q and r for which $a_p > a_{p+1}$ or $p = 0$ or $p = s$, $a_{s+q} > a_{s+q+1}$ or $q = 0$ or $q = s'$ and $a_{s+s'+r} > a_{s+s'+r+1}$ or $r = 0$ or $r = m - s - s'$, then (6) holds for all p, q and r with $0 \leqslant p \leqslant s$, $0 \leqslant q \leqslant s'$ and $0 \leqslant r \leqslant m - s - s'$. Suppose not. Let p, q, r be such that

$$\begin{aligned}(7)\qquad &\sum_{i=1}^{p} a_i + \sum_{i=1}^{q} a_{s+i} + \sum_{i=1}^{r} a_{s+s'+i}\\ &\quad > (p+q)t + pt' + \sum_{j=1}^{t} \min\{r, b_j - s - s'\}\\ &\qquad + \sum_{j=t+1}^{t+t'} \min\{q+r, b_j - s\} + \sum_{j=t+t'+1}^{n} \min\{p+q+r, b_j\}\end{aligned}$$

and $p + q + r$ is as small as possible. Let

$$u = \max\{i\colon\ a_i = a_p \text{ and } i \leqslant s\}, \quad v = \max\{i\colon\ a_{s+i} = a_{s+q} \text{ and } i \leqslant s'\}$$

and

$$w = \max\{i\colon\ a_{s+s'+i} = a_{s+s'+r} \text{ and } i \leqslant m - s - s'\}.$$

If $p = 0$, we define u to be 0; if $q = 0$, then $v = 0$; if $r = 0$, then $w = 0$. By the hypothesis, we have that

$$\begin{aligned}(8)\qquad &\sum_{i=1}^{u} a_i + \sum_{i=1}^{v} a_{s+i} + \sum_{i=1}^{w} a_{s+s'+i}\\ &\quad \leqslant (u+v)t + ut' + \sum_{j=1}^{t} \min\{w, b_j - s - s'\}\\ &\qquad + \sum_{j=t+1}^{t+t'} \min\{v+w, b_j - s\} + \sum_{j=t+t'+1}^{n} \min\{u+v+w, b_j\}.\end{aligned}$$

By the choice of p, q and r, we also have that if $p \geqslant 1$ then

$$(9)\qquad \sum_{i=1}^{p-1} a_i + \sum_{i=1}^{q} a_{s+i} + \sum_{i=1}^{r} a_{s+s'+i} \leqslant (p+q-1)t + (p-1)t' + \sum_{j=1}^{t} \min\{r, b_j - s - s'\} + \sum_{j=t+1}^{t+t'} \min\{q+r, b_j - s\} + \sum_{j=t+t'+1}^{n} \min\{p+q+r-1, b_j\};$$

if $q \geqslant 1$ then

$$(10)\qquad \sum_{i=1}^{p} a_i + \sum_{i=1}^{q-1} a_{s+i} + \sum_{i=1}^{r} a_{s+s'+i} \leqslant (p+q-1)t + pt' + \sum_{j=1}^{t} \min\{r, b_j - s - s'\} + \sum_{j=t+1}^{t+t'} \min\{q+r-1, b_j - s\} + \sum_{j=t+t'+1}^{n} \min\{p+q+r-1, b_j\};$$

if $r \geqslant 1$ then

$$(11)\qquad \sum_{i=1}^{p} a_i + \sum_{i=1}^{q} a_{s+i} + \sum_{i=1}^{r-1} a_{s+s'+i} \leqslant (p+q)t + pt' + \sum_{j=1}^{t} \min\{r-1, b_j - s - s'\} + \sum_{j=t+1}^{t+t'} \min\{q+r-1, b_j - s\} + \sum_{j=t+t'+1}^{n} \min\{p+q+r-1, b_j\}.$$

Let α be the number of values of j for which $b_j \geqslant s+s'+r$ and $1 \leqslant j \leqslant t$, let β be the number of values of j for which $b_j \geqslant s+q+r$ and $t+1 \leqslant j \leqslant t+t'$, and let γ be the number of values of j for which $b_j \geqslant p+q+r$ and $t+t'+1 \leqslant j \leqslant n$. From (7) and (9)–(11) we get that

$$(12)\qquad a_p > t + t' + \gamma \quad \text{if } p \geqslant 1,$$

$$(13)\qquad a_{s+q} > t + \beta + \gamma \quad \text{if } q \geqslant 1,$$

$$(14)\qquad a_{s+s'+r} > \alpha + \beta + \gamma \quad \text{if } r \geqslant 1.$$

Now from (7), (8) and (12)–(14) we get that

$$\begin{aligned}
&(u-p)(t+t'+\gamma)+(v-q)(t+\beta+\gamma)+(w-r)(\alpha+\beta+\gamma)\\
&\quad< (u-p)a_p+(v-q)a_{s+q}+(w-r)a_{s+s'+r}\\
&\quad< (u-p+v-q)t+(u-p)t'+\sum_{j=1}^{t}(\min\{w,b_j-s-s'\}-\min\{r,b_j-s-s'\})\\
&\qquad+\sum_{j=t+1}^{t+t'}(\min\{v+w,b_j-s\}-\min\{q+r,b_j-s\})\\
&\qquad+\sum_{j=t+t'+1}^{n}(\min\{u+v+w,b_j\}-\min\{p+q+r,b_j\})\\
&\quad\leqslant (u-p+v-q)t+(u-p)t'+(w-r)\alpha+(v+w-q-r)\beta\\
&\qquad+(u+v+w-p-q-r)\gamma\\
&\quad=(u-p)(t+t'+\gamma)+(v-q)(t+\beta+\gamma)+(w-r)(\alpha+\beta+\gamma),
\end{aligned}$$

a contradiction. □

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