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A RELATIONAL SEMANTICS FOR THE LOGIC OF BOUNDED LATTICES

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Abstract. This paper aims to propose a complete relational semantics for the so-called logic of bounded lattices, and prove a completeness theorem with regard to a class of two-sorted frames that is dually equivalent (categorically) to the variety of bounded lattices.

Keywords: logic of bounded lattice; polarity; two-sorted frame; relational semantics

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1. Introduction

By the general theory of abstract algebraic logic (see [6], [7], [5]), a unique class of algebras $\mathrm{Alg}(\mathcal{S})$ is canonically associated with each sentential logic S.

In Rebagliato and Verdú [21] a Gentzen system $\mathcal{G}_{\mathbb{L}}$ is associated with the variety $\mathbb{L}$ of lattices, and then Font and Jansana in [6] showed that the class of algebras $\mathrm{Alg}(\mathcal{S}_{\mathcal{G}_{\mathbb{L}}})$ of the sentential logic $\mathcal{S}_{\mathcal{G}_{\mathbb{L}}}$ defined by the Gentzen system $\mathcal{G}_{\mathbb{L}}$ (see Definition 2.3) coincides with $\mathbb{L}$. For this reason, in [6] the logic $\mathcal{S}_{\mathcal{G}_{\mathbb{L}}}$ was called the *logic of lattices*. In this paper, we are interested in the *logic of bounded lattices*, which is defined by a Gentzen system $\mathcal{G}_{\mathbb{BL}}$ that have the same rules as $\mathcal{G}_{\mathbb{L}}$ and two rules more for the True and False connectives.

The purpose of the present paper is to prove a completeness theorem for the logic of bounded lattices using a particular class of two-sorted frames (frames with worlds and co-worlds).

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Relational semantics (possible world semantics or Kripke semantics) is an essential and powerful tool to study and understand intuitionistic and modal logics. Moreover, relational semantics play a fundamental role in making these logics useful. For these reasons, the theory of relational semantics was extended and generalized to several non-classical logics. Recently, for example, a relational semantics was developed for some fragments of several substructural logics (see [4], [8]). Moreover, there is a wide range of papers containing complete relational semantics for non-classical logics, for instance [15], [16], [1], [2], [18], [17], [10], [11], [12].

The frames considered in [8] to get a complete relational semantics for some fragments of substructural logics are formed by a set of "worlds", by a set of "co-worlds" (or "information quanta") and by a binary relation between the worlds and co-worlds. That is, the frames considered in [8] are structures $\langle X, Y, R\rangle$, where X and Y are nonempty sets and R is a binary relation from X to Y. These structures are known in the literature as *polarities* (see [8], [9]) or *contexts* (see [3], Chapters 3 and 7). This consideration of two-sorted frames allows the treatment of problems created by the lack of distributivity of the lattice operations. As it was mentioned in [8], page 253, these two-sorted frames already encode, using an adequate definition of interpretation, a notion of conjunction and disjunction. Thus, we will use this concept of how the lattice connectives are interpreted in two-sorted frames to present a complete relational semantics for the logic of bounded lattices concerning a particular special class of polarities, which are categorically related to the bounded lattices.

There are several papers developing categorical dualities for the variety of (bounded) lattices, see [23], [13], [19], [20]. In [19] a topological duality was established for the variety of bounded lattices. The dual spaces of bounded lattices were called BL-*spaces.* Then in [20], relational structures, categorically equivalent to BL-spaces, were introduced to study quasioperators on bounded lattices. These relational structures, called *mirrored* BL-*spaces*, are polarities $\langle X, Y, R\rangle$ such that X is the dual BL-space of a bounded lattice L and Y is the dual BL-space of the opposite lattice L^{∂} of L.

The main result of this article is to present a complete relational semantics for the logic of bounded lattices [21], [6] through the relational structures (*mirrored* BL-*spaces*) introduced in [19], [20] and by using the definition of interpretation of the lattice connectives presented in [8]. Moreover, it is worth noting that to attain this, we build up the canonical frame taking the dual mirrored BL-space of the corresponding Lindenbaum algebra.

This paper is organized as follows. In Section 2 we consider some basic concepts of Gentzen systems, and we present the Gentzen system $\mathcal{G}_{\mathbb{BL}}$ associated with the variety $\mathbb{BL}$ of bounded lattices. Then we move to consider some basic facts about the theory of polarities. Section 3 introduces the definition of interpretation on polarities and the satisfaction and "part of" relations. Next, we prove soundness for the sentential logic $\mathcal{S}_{\mathbb{BL}}$ defined by the Gentzen system $\mathcal{G}_{\mathbb{BL}}$. The aim of Section 4 is to prove a completeness theorem for the logic of bounded lattices $\mathcal{S}_{\mathbb{BL}}$ with respect to a particular kind of polarities. These particular polarities will be called BL-*frames* and are defined in [20] (and called *mirrored* BL-*spaces*) using a topological duality for bounded lattices developed by Moshier and Jipsen (see [19]). Thus, in the first part of Section 4 we shall consider a sketch of the topological duality for bounded lattices given in [19] and we introduce the definition of BL-frame. In the second part of the section, we provide the construction of the canonical BL-frame, and we prove two completeness theorems for the logic $\mathcal{S}_{\mathbb{BL}}$.

2. Preliminaries

2.1. The logic of bounded lattices. The main references for the following general concepts are [6], [7], [5].

Let $\mathcal{L}$ be an algebraic language (or set of connectives) and Var a countable set of propositional variables. Let us denote by $\mathrm{Fm}(\mathcal{L})$ the absolutely free algebra of type $\mathcal{L}$ generated by Var. The algebra $\mathrm{Fm}(\mathcal{L})$ is called the *algebra of formulas* of type $\mathcal{L}$ and its elements are called *formulas.*

Definition 2.1. A *sentential logic* (also called *deductive system*) of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathrm{Fm}(\mathcal{L}), \vdash_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is an algebraic language and $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}(\mathcal{L})) \times \mathrm{Fm}(\mathcal{L})$ is a relation satisfying the following properties for all $\Gamma \cup \Delta \cup \{\varphi\} \subseteq \mathrm{Fm}(\mathcal{L})$:

(1) if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$;
(2) if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathcal{S}} \varphi$;
(3) if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Delta \vdash_{\mathcal{S}} \psi$ for every $\psi \in \Gamma$, then $\Delta \vdash_{\mathcal{S}} \varphi$;
(4) if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $h[\Gamma] \vdash_{\mathcal{S}} h(\varphi)$ for every substitution $h \in \mathrm{Hom}(\mathrm{Fm}(\mathcal{L}), \mathrm{Fm}(\mathcal{L}))$.

A sentential logic $\mathcal{S}$ is said to be *finitary* if for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathcal{S}} \varphi$ implies that there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{S}} \varphi$.

A *sequent* of formulas is a formal expression of the form $\Gamma \rhd \varphi$, where Γ is a finite (possible empty) set of formulas and φ is a formula. The set of all sequents of formulas over the language $\mathcal{L}$ is denoted by $\mathrm{Seq}(\mathcal{L})$. Let $\mid\!\sim \; \subseteq \mathcal{P}(\mathrm{Seq}(\mathcal{L})) \times \mathrm{Seq}(\mathcal{L})$ be a binary relation. As usual, the derivation of the sequent $\Gamma \rhd \varphi$ from a finite set of sequents $\{\Gamma_1 \rhd \varphi_1, \dots, \Gamma_n \rhd \varphi_n\}$, that is $\{\Gamma_1 \rhd \varphi_1, \dots, \Gamma_n \rhd \varphi_n\} \mid\!\sim \Gamma \rhd \varphi$, is expressed more graphically by

$$\frac{\Gamma_1 \rhd \varphi_1, \dots, \Gamma_n \rhd \varphi_n}{\Gamma \rhd \varphi} \tag{2.1}$$

and this expression is called a *Gentzen-style rule.* The relation $\mid\!\sim$ is said to be *substitution-invariant* if for all $\Sigma \cup \{\Gamma \rhd \varphi\} \subseteq \mathrm{Seq}(\mathcal{L})$,

$$\Sigma \mid\!\sim \Gamma \rhd \varphi \ \text{ implies } \ h[\Sigma] \mid\!\sim h[\Gamma] \rhd h(\varphi)$$

for all substitutions $h \in \mathrm{Hom}(\mathrm{Fm}, \mathrm{Fm})$, where $h[\Sigma] = \{h[\Delta] \rhd h(\psi) \colon \ \Delta \rhd \psi \in \Sigma\}$.

Definition 2.2 ([6])**.** A *Gentzen system* is a pair $\mathcal{G} = \langle \mathrm{Fm}(\mathcal{L}), \mid\!\sim_{\mathcal{G}} \rangle$, where $\mid\!\sim_{\mathcal{G}}$ is a finitary closure operator on the set $\mathrm{Seq}(\mathcal{L})$ that is substitution-invariant and has the following *structural rules*: for every formulas φ and ψ and every finite subset $\Gamma \subseteq \mathrm{Fm}(\mathcal{L})$,

$$\text{(Axiom)}\ \frac{\emptyset}{\varphi \rhd \varphi} \qquad \text{(Cut)}\ \frac{\Gamma \rhd \varphi \quad \Gamma, \varphi \rhd \psi}{\Gamma \rhd \psi} \qquad \text{(Weakening)}\ \frac{\Gamma \rhd \varphi}{\Gamma, \psi \rhd \varphi}.$$

We say that a sequent $\Gamma \rhd \varphi$ is *derivable in* (or is a *derivable sequent of*) $\mathcal{G}$ if $\emptyset \mid\!\sim_{\mathcal{G}} \Gamma \rhd \varphi$.

Definition 2.3 ([6])**.** Let $\mathcal{G}$ be a Gentzen system. The *sentential logic* defined by $\mathcal{G}$ is the sentential logic $\langle \mathrm{Fm}(\mathcal{L}), \vdash_{\mathcal{G}} \rangle$, where the consequence relation $\vdash_{\mathcal{G}}$ is defined as follows: for all $\Gamma \subseteq \mathrm{Fm}(\mathcal{L})$, $\varphi \in \mathrm{Fm}(\mathcal{L})$,

$$\Gamma \vdash_{\mathcal{G}} \varphi \Leftrightarrow \text{there is a finite } \Delta \subseteq \Gamma \text{ such that } \emptyset \mid\!\sim_{\mathcal{G}} \Delta \rhd \varphi.$$

A *Gentzen calculus* is a set of Gentzen-style rules. Every Gentzen calculus $\mathbf{G}$ containing the structural rules defines in a standard way a Gentzen system $\mathcal{G}_{\mathbf{G}} = \langle \mathrm{Fm}, \mid\!\sim_{\mathbf{G}} \rangle$, see for instance [22], [21].

It should be noticed that the previous concepts are given in their finite versions, for example, finite Gentzen-style rules, finitary Gentzen systems, finitary sentential logics. Now we introduce the Gentzen system that will define the sentential logic that concerns us.

Definition 2.4. Let $\mathcal{L}_{\mathrm{b}} = \{\wedge, \vee, \bot, \top\}$ be an algebraic language, where $\{\wedge, \vee\}$ are binary connectives and $\{\bot, \top\}$ are constants. Let $\mathcal{G}_{\mathbb{BL}} = \langle \mathrm{Fm}(\mathcal{L}_{\mathrm{b}}), \mid\!\sim_{\mathcal{G}_{\mathbb{BL}}} \rangle$ be the Gentzen system defined by the Gentzen calculus that contains the structural rules and the following rules:

$$(\wedge \rhd)\colon\ \frac{\Gamma, \varphi, \psi \rhd \chi}{\Gamma, \varphi \wedge \psi \rhd \chi}, \quad (\vee \rhd)\colon\ \frac{\varphi \rhd \chi \quad \psi \rhd \chi}{\varphi \vee \psi \rhd \chi}, \quad (\top)\colon\ \frac{\emptyset}{\emptyset \rhd \top},$$
$$(\rhd \wedge)\colon\ \frac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi}, \quad (\rhd \vee)\colon\ \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi} \frac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi}, \quad (\bot)\colon\ \frac{\emptyset}{\bot \rhd \varphi}.$$

The *logic of bounded lattices* is the sentential logic $\mathcal{S}_{\mathbb{BL}} = \langle \mathrm{Fm}(\mathcal{L}_{\mathrm{b}}), \vdash_{\mathbb{BL}} \rangle$ defined by the Gentzen system $\mathcal{G}_{\mathbb{BL}}$.

2.2. Polarities. Definitions and properties about polarities can be found, for instance, in [3], Chapters 3 and 7 and in [8], [9].

Definition 2.5. A *polarity* is a triple $P = \langle X, Y, R\rangle$, where X and Y are non-empty sets and $R \subseteq X \times Y$ is a binary relation from X to Y.

For every polarity $P = \langle X, Y, R\rangle$, the Galois connection (Φ_R, Ψ_R) is defined as:

$$\begin{aligned}
\Phi_R \colon\ & \mathcal{P}(X) \to \mathcal{P}(Y) \\
& \qquad A \mapsto \Phi_R(A) = \{y \in Y \colon\ \forall x \in X \colon\ x \in A \Rightarrow x\, R\, y\}, \\
\Psi_R \colon\ & \mathcal{P}(Y) \to \mathcal{P}(X) \\
& \qquad B \mapsto \Psi_R(B) = \{x \in X \colon\ \forall y \in Y \colon\ y \in B \Rightarrow x\, R\, y\}.
\end{aligned}$$

So, we have the lattice of *Galois closed subsets* of X

$$\mathcal{C}(P) = \{A \in \mathcal{P}(X) \colon\ (\Psi_R \circ \Phi_R)(A) = A\}$$

and the lattice of *dual Galois closed subsets* of Y

$$\mathcal{C}^{\mathrm{d}}(P) = \{B \in \mathcal{P}(Y) \colon\ (\Phi_R \circ \Psi_R)(B) = B\}.$$

It is well known that $\mathcal{C}(P)$ and $\mathcal{C}^{\mathrm{d}}(P)$ are complete lattices, see for instance [3]. For instance, in $\mathcal{C}(P)$ the meet is the set-theoretic intersection and the join of a family $\mathcal{A} \subseteq \mathcal{C}(P)$ is $\bigvee \mathcal{A} = (\Psi_R \circ \Phi_R)\big(\bigcup \mathcal{A}\big)$. The following properties are known and easy to check.

Proposition 2.6. *Let $P = \langle X, Y, R\rangle$ be a polarity. For every $A \in \mathcal{P}(X)$ and $B \in \mathcal{P}(Y)$ we have:*

(1) $\Phi_R(A) = \bigcap_{x \in A} R[x]$ *and* $\Psi_R(B) = \bigcap_{y \in B} R^{-1}[y]$*;*
(2) $y \in \Phi_R(A) \Leftrightarrow A \subseteq R^{-1}[y]$*;*
(3) $x \in \Psi_R(B) \Leftrightarrow B \subseteq R[x]$*;*
(4) $\mathcal{C}(P) = \{\Psi_R(B) \colon\ B \subseteq Y\}$*;*
(5) $\mathcal{C}^{\mathrm{d}}(P) = \{\Phi_R(A) \colon\ A \subseteq X\}$*;*
(6) $\Phi_R \colon \mathcal{C}(P) \to \mathcal{C}^{\mathrm{d}}(P)$ *is a dual order-isomorphism whose inverse is the map* $\Psi_R \colon \mathcal{C}^{\mathrm{d}}(P) \to \mathcal{C}(P)$*. Thus, we will identify the opposite lattice of* $\mathcal{C}(P)$ *with* $\mathcal{C}^{\mathrm{d}}(P)$*.*

The two-sorted frames will be the base for the complete relational semantics for the logic of bounded lattices $\mathcal{S}_{\mathbb{BL}}$, and polarities will be underlying two-sorted structures of this semantics. Thus, since the polarities attempt to be a generalization of the classical frames, we want that the sets of worlds and co-worlds of polarity $P = \langle X, Y, R\rangle$ are represented in the potential interpretants $\mathcal{C}(P)$. Hence, we restrict ourselves to the following polarities.

Definition 2.7. A polarity $P = \langle X, Y, R\rangle$ is said to be a *separating frame* (*S-frame* for short) when the following two conditions hold:

▷ $\forall x_1, x_2 \in X\colon\ x_1 \neq x_2 \Rightarrow R[x_1] \neq R[x_2]$,
▷ $\forall y_1, y_2 \in Y\colon\ y_1 \neq y_2 \Rightarrow R^{-1}[y_1] \neq R^{-1}[y_2]$.

Proposition 2.8 ([8])**.** *Let $P = \langle X, Y, R\rangle$ be an S-frame. Then the maps $\Theta\colon X \to \mathcal{C}(P)$ and $\Upsilon\colon Y \to \mathcal{C}(P)$ defined by: $\Theta(x) = (\Psi_R \circ \Phi_R)(\{x\})$ and $\Upsilon(y) = R^{-1}[y]$ for every $x \in X$ and $y \in Y$, are injective.*

Thus, by the previous proposition, we can see that for every S-frame $P = \langle X, Y, R\rangle$, the sets X and Y are represented in the lattice $\mathcal{C}(P)$. Moreover, since we will deal with bounded lattices, we need to restrict ourselves to a particular class of S-frames, namely to *bounded* S-frames. The following definition will be clear when we consider the particular type of S-frames, called BL-*frames* (see Remark 4.9), used to obtain the completeness theorem for the logic $\mathcal{S}_{\mathbb{BL}}$.

Definition 2.9. A polarity $P = \langle X, Y, R\rangle$ is said to be *bounded* if there exists $x \in X$ such that $R[x] = Y$ and there exists $y \in Y$ such that $R^{-1}[y] = X$.

From Definitions 2.7 and 2.9, it should be noted that for every bounded S-frame $P = \langle X, Y, R\rangle$ there exists a unique $x \in X$ such that $R[x] = Y$ and there exists a unique $y \in Y$ such that $R^{-1}[y] = X$. Thus, we denote these elements by 1_X and 1_Y, respectively. It should be kept in mind that $1_X \in A$ for all $A \in \mathcal{C}(P)$ and $1_Y \in B$ for all $B \in \mathcal{C}^{\mathrm{d}}(P)$.

3. Generalized relational semantics

We consider the algebraic language $\mathcal{L}_{\mathrm{b}} = \{\wedge, \vee, \bot, \top\}$ of type $(2, 2, 0, 0)$. Given a set of propositional variables Var, we recall that $\mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ denotes the algebra of formulas of type $\mathcal{L}_{\mathrm{b}}$ generated by Var. Now we consider the notion of interpretation and the satisfaction relation introduced by Gehrke in [8].

Definition 3.1. Let $P = \langle X, Y, R\rangle$ be a bounded S-frame. An *interpretation* (or *valuation*) of Var in P is a map $v\colon \mathrm{Var} \to \mathcal{C}(P)$. We say that the pair $\langle P, v\rangle$ is a *model*. For every model $M = \langle P, v\rangle$ we define the following relations: for $p \in \mathrm{Var}$ and for $x \in X$ and $y \in Y$,

▷ $M, x \Vdash p \Leftrightarrow x \in v(p)$,
▷ $M, y \succ p \Leftrightarrow v(p) \subseteq R^{-1}[y]$.

When $M, x \Vdash p$ holds, we say that p holds at x in M and when $M, y \succ p$ holds, we say that y is part of p in M.

Now we extend the relations $\Vdash$ and $\succ$ to $\mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$. Let $M = \langle P, v\rangle$ be a model, i.e. $P = \langle X, Y, R\rangle$ is a bounded S-frame and $v\colon \mathrm{Var} \to \mathcal{C}(P)$ is an interpretantion. Let $\varphi, \psi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ be such that $M, x \Vdash \varphi$; $M, x \Vdash \psi$; $M, y \succ \varphi$ and $M, y \succ \psi$ have or have not already been determined for each $x \in X$ and $y \in Y$. Then we define for $x \in X$ and $y \in Y$:

- $\triangleright$ $M, x \Vdash \varphi \wedge \psi \Leftrightarrow M, x \Vdash \varphi$ and $M, x \Vdash \psi$,
- $\triangleright$ $M, y \succ \varphi \wedge \psi \Leftrightarrow \forall\, x' \in X\colon\ M, x' \Vdash \varphi \wedge \psi$ implies $x'\ R\ y$,
- $\triangleright$ $M, y \succ \varphi \vee \psi \Leftrightarrow M, y \succ \varphi$ and $M, y \succ \psi$,
- $\triangleright$ $M, x \Vdash \varphi \vee \psi \Leftrightarrow \forall\, y' \in Y\colon\ M, y' \succ \varphi \vee \psi$ implies $x\ R\ y'$.
- $\triangleright$ $M, x \nVdash \bot$ if $x \neq 1_X$,
- $\triangleright$ $M, y \succ \bot$,
- $\triangleright$ $M, x \Vdash \top$,
- $\triangleright$ $M, y \not\succ \top$ if $y \neq 1_Y$.

Definitions of $\Vdash$ and $\succ$ for the logical constants $\top$ and $\bot$ considered here, instead of the standard ones (see [8], page 253), are in correspondence with the definition of boundedness of S-frames. So they are also influenced by the class of two-sorted frames (BL-frames) considered in Section 4. It should also be noted that $M, 1_X \Vdash \varphi$ for every $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$.

Given a polarity $P = \langle X, Y, R\rangle$, recall that $\mathcal{C}(P)$ is a bounded (complete) lattice.

Definition 3.2. Let $M = \langle P, v\rangle$ be a model. We denote by $v^{\Vdash}_M$ the unique extension of v such that $v^{\Vdash}_M$ is a homomorphism from $\mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ to $\mathcal{C}(P)$.

The following proposition can be proved inductively using definitions of the relations $\Vdash$ and $\succ$.

Proposition 3.3. *Let $M = \langle P, v\rangle$ be a model. Then for every formula φ, $v^{\Vdash}_M(\varphi) = \{x \in X\colon\ M, x \Vdash \varphi\}$.*

Moreover, in a similar way to Proposition 3.3, it can be proved that the map $v^{\succ}_M\colon\ \mathrm{Fm} \to \mathcal{C}^{\mathrm{d}}(P)$ defined by $v^{\succ}_M(\varphi) := \{y \in Y\colon\ M, y \succ \varphi\}$ is a dual homomorphism, that is, $v^{\succ}_M(\varphi\wedge\psi) = v^{\succ}_M(\varphi)\vee v^{\succ}_M(\psi)$ and $v^{\succ}_M(\varphi\vee\psi) = v^{\succ}_M(\varphi)\cap v^{\succ}_M(\psi)$. For every model $M = \langle P, v\rangle$ it should be noted that $v^{\Vdash}_M(\top) = X \in \mathcal{C}(P)$ and $v^{\Vdash}_M(\bot) = \{1_X\} \in \mathcal{C}(P)$, and $v^{\succ}_M(\bot) = Y \in \mathcal{C}^{\mathrm{d}}(P)$ and $v^{\succ}_M(\top) = \{1_Y\} \in \mathcal{C}^{\mathrm{d}}(P)$. As usual, when there is no danger of confusion, we omit the subscript M from $v^{\Vdash}_M$ and $v^{\succ}_M$.

Let $M = \langle P, v\rangle$ be a model and let Γ be a finite set of formulas and φ be a formula. By $M, x \Vdash \Gamma$ we mean $M, x \Vdash \psi$ for all $\psi \in \Gamma$. We say that the sequent $\Gamma \triangleright \varphi$ is *true in* or *holds in the model M* if the condition

$$M, x \Vdash \Gamma \Rightarrow M, x \Vdash \varphi$$

holds for all $x \in X$. A sequent $\Gamma \rhd \varphi$ is said to be *valid* in a bounded S-frame P if for each valuation $v\colon \mathrm{Var} \to \mathcal{C}(P)$, the sequent is true in the model $M = \langle P, v\rangle$ and we denote this by $P \vDash \Gamma \rhd \varphi$. We also say that a Gentzen-style rule (2.1) is *valid* in a bounded S-frame P when $P \vDash \Gamma_i \rhd \varphi_i$ for all $i = 1, \dots, n$ implies $P \vDash \Gamma \rhd \varphi$.

Proposition 3.4 (Soundness w.r.t bounded S-frames)**.** *Let $\Gamma \subseteq \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ be a finite subset and $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$. If the sequent $\Gamma \rhd \varphi$ is derivable in the Gentzen system $\mathcal{G}_{\mathbb{BL}}$, then it is valid over the class of all bounded S-frames.*

Proof. As usual, it is enough to show that the rules defining the Gentzen system $\mathcal{G}_{\mathbb{BL}}$ are valid in all bounded S-frames. Let P be a bounded S-frame. It is straightforward to prove directly that the structural rules are valid in P. Now we show that the rule $(\vee \rhd)$ is valid in P. So assume that $P \vDash \varphi \rhd \chi$ and $P \vDash \psi \rhd \chi$. We have to prove that $P \vDash \varphi \vee \psi \rhd \chi$. To this, let $v\colon \mathrm{Var} \to \mathcal{C}(P)$ be a valuation and $M = \langle P, v\rangle$. Let $x \in X$ and suppose that $M, x \Vdash \varphi \vee \psi$. Thus $x \in v^{\Vdash}(\varphi \vee \psi)$. Since $P \vDash \varphi \rhd \chi$ and $P \vDash \psi \rhd \chi$, it follows that $v^{\Vdash}(\varphi) \subseteq v^{\Vdash}(\chi)$ and $v^{\Vdash}(\psi) \subseteq v^{\Vdash}(\chi)$. Then we obtain $(\Psi_R \circ \Phi_R)(v^{\Vdash}(\varphi) \cup v^{\Vdash}(\psi)) \subseteq v^{\Vdash}(\chi)$. Hence, using that $v^{\Vdash}$ is a homomorphism we have

$$v^{\Vdash}(\varphi \vee \psi) = v^{\Vdash}(\varphi) \vee v^{\Vdash}(\psi) = (\Psi_R \circ \Phi_R)(v^{\Vdash}(\varphi) \cup v^{\Vdash}(\psi)) \subseteq v^{\Vdash}(\chi).$$

Thus $x \in v^{\Vdash}(\chi)$, i.e. $M, x \Vdash \chi$. Then $P \vDash \varphi \vee \psi \rhd \chi$ and therefore the rule $(\vee \rhd)$ is valid in P. By definition of $\Vdash$, it is straightforward to show directly that the rules $(\rhd \wedge)$ and $(\wedge \rhd)$ are valid in P and, since $v^{\Vdash}$ is a homomorphism, it follows that the rule $(\rhd \vee)$ is valid in P. It is clear that $P \vDash \emptyset \rhd \top$ and thus the rule $(\top)$ is valid in P. To see that the rule $(\bot)$ is valid in P, let $v\colon \mathrm{Var} \to \mathcal{C}(P)$ be a valuation and $M = \langle P, v\rangle$ and let $x \in X$. Suppose that $M, x \Vdash \bot$. So $x = 1_X$. Then $1_X \in v^{\Vdash}(\varphi)$, because $v^{\Vdash}(\varphi) \in \mathcal{C}(P)$. Thus $M, x \Vdash \varphi$. Hence $P \vDash \bot \rhd \varphi$. This completes the proof. ☐

The next step is to prove the converse of the previous proposition. That is, we want to prove that if a sequent $\Gamma \rhd \varphi$ is valid in the class of all bounded S-frames, then the sequent is derivable in $\mathcal{G}_{\mathbb{BL}}$. As usual, we will show that if a sequent $\Gamma \rhd \varphi$ is not derivable in $\mathcal{G}_{\mathbb{BL}}$, then there is a bounded S-frame in which it is not valid.

4. BL-frames and completeness theorems for $\mathcal{S}_{\mathbb{BL}}$

In this section we consider a smaller class of bounded S-frames to prove the completeness theorem for the logic $\mathcal{S}_{\mathbb{BL}}$ with respect to the relational semantics considered in Section 3.

To the aim of this section, it will be important to consider the topological duality for bounded lattices developed by Moshier and Jipsen in [19] and the theory of mirrored BL-spaces introduced in [20] will also be fundamental. We sketch the topological duality between bounded lattices and the corresponding topological spaces, and then we move to the theory of mirrored BL-spaces. We refer the reader to [19] and [20] for a more detailed discussion on this subjects.

4.1. BL-spaces and mirrored BL-spaces. Let $\langle X, \tau\rangle$ be a T_0 topological space. The *specialization order* of the space X is the binary relation $\sqsubseteq$ on X defined as follows: for every $x, y \in X$,

$$x \sqsubseteq y \Leftrightarrow \forall U \in \tau\colon\ x \in U \Rightarrow y \in U.$$

Since X is a T_0-space, it follows that $\sqsubseteq$ is a partial order. A *filter* of X is a nonempty down-directed up-set with respect to $\sqsubseteq$. We denote the collection of all *open filters* of X by $\mathsf{OF}(X)$. And $\mathsf{KOF}(X)$ denotes the collection of all *compact open filters* of X. We consider the closure system on X generated by $\mathsf{OF}(X)$. We denote this closure system by $\mathsf{FSat}(X)$. Thus, the closure operator associated to $\mathsf{FSat}(X)$ is given by $\mathrm{fsat}(A) = \bigcap\{F \in \mathsf{OF}(X)\colon\ A \subseteq F\}$ for every $A \subseteq X$ and hence $\mathsf{FSat}(X)$ is a complete lattice with respect to the set-theoretic inclusion $\subseteq$. The elements of $\mathsf{FSat}(X)$ are called F-*saturated*.

Let $\langle X, \tau\rangle$ be a topological space. A closed subset A of X is said to be *irreducible* if for all closed subsets B and C of X, $A \subseteq B$ or $A \subseteq C$ whenever $A \subseteq B \cup C$. A topological space $\langle X, \tau\rangle$ is called *sober* if for every irreducible closed subset A of X there exists a unique element $x \in X$ such that $A = \mathrm{cl}(x)$ (where $\mathrm{cl}(x)$ is the topological closure of the element x). For more information about sober spaces see [14].

Definition 4.1 ([19], page 115)**.** A topological space X is called an HMS-*space* if the following conditions hold:

(1) X is a sober space,
(2) $\mathsf{KOF}(X)$ forms a base for X that is closed under finite intersections.

Proposition 4.2 ([19], Lemma 3.1)**.** *If X is an* HMS-*space, then X is a complete lattice with respect to the specialization order.*

We denote the meet and join of an HMS-space X corresponding to the specialization order by $\sqcap$ and $\sqcup$, respectively, and we indicate the bottom and top elements of X by 0_X and 1_X, respectively.

Let X be an HMS-space. Since $\mathsf{KOF}(X)$ is closed under arbitrary finite intersections, it follows that $X \in \mathsf{KOF}(X)$. So X is the top element of $\mathsf{KOF}(X)$ ordered by the set-theoretic inclusion. Then $\langle \mathsf{KOF}(X), \cap, X\rangle$ is a meet-semilattice with top element X.

Definition 4.3 ([19], papge 116). A topological space X is said to be a BL-*space* if it is an HMS-space and $\mathsf{KOF}(X)$ is a sublattice of $\mathsf{FSat}(X)$.

Hence, given a BL-space X, we obtain that $\langle \mathsf{KOF}(X), \cap, \vee, X\rangle$ is a lattice with top element X, where $\vee$ is the join operation corresponding to the lattice $\mathsf{FSat}(X)$, that is, for all $A, B \in \mathsf{KOF}(X)$ we have $A \vee B = \mathrm{fsat}(A \cup B)$.

Proposition 4.4 ([19], Theorem 3.2). *Let X be an* HMS-*space. Then X is a* BL-*space if and only if* $\mathrm{fsat}(U)$ *is an open subset of X for every open subset U of X.*

Let X be a BL-space. Since 1_X is the top element of X (w.r.t. specialization order), it follows that $1_X \in F$ for all $F \in \mathsf{OF}(X)$. Then $\mathrm{fsat}(\emptyset) = \bigcap\{F\colon F \in \mathsf{OF}(X)\} = \{1_X\}$ and so, by the previous proposition, $\{1_X\}$ is an open subset of X. Hence, $\{1_X\} \in \mathsf{KOF}(X)$ and thus is the bottom element of $\mathsf{KOF}(X)$. Therefore $\langle \mathsf{KOF}(X), \cap, \vee, \{1_X\}, X\rangle$ is a bounded lattice.

Now we briefly sketch the corresponding functors between bounded lattices and BL-spaces. Given a BL-space X, the *dual bounded lattice of* X is $\mathsf{KOF}(X)$. On the other hand, given a bounded lattice L, let $X(L) := \langle \mathsf{Fi}(L), \tau_L\rangle$ be the BL-space, where τ_L is the topology generated by the base $\{U_a\colon a \in L\}$ with $U_a := \{F \in \mathsf{Fi}(L)\colon a \in F\}$. We summarize this in Figure 1.

Bounded lattices		BL-spaces
L	$\rightarrow$	$X(L) = \langle \mathsf{Fi}(L), \tau_L\rangle$
$\langle \mathsf{KOF}(X), \cap, \vee, \{1_X\}, X\rangle$	$\leftarrow$	$\langle X, \tau\rangle$

Figure 1. Dual equivalence between bounded lattices and BL-spaces.

Remark 4.5. Let L be a bounded lattice. Consider its dual BL-space $X(L) = \langle \mathsf{Fi}(L), \tau_L\rangle$. It can be proved that $\mathsf{KOF}(X(L)) = \{U_a\colon a \in L\}$. Moreover, it should be noted that for every $a, b \in L$ we have $U_a \cap U_b = U_{a\wedge b}$ and $U_a \vee U_b = U_{a\vee b}$ (see [19], Lemma 3.5). Then we obtain $\mathsf{KOF}(X(L)) \cong L$. Conversely, if X is a BL-space, then X is homeomorphic to $X(\mathsf{KOF}(X))$ (see [19], Theorem 3.7).

Remark 4.6. It is worth noting that the duality for bounded lattices due to Moshier and Jipsen (see [19]) is not a generalization of the Stone and Priestley dualities for Boolean algebras and distributive lattices, respectively. That is, if L is a Boolean algebra (distributive lattice), then the dual BL-space of L is not necessarily the dual Stone (Priestley) space of L.

Now we present the definition of polarities that we will use to prove a completeness theorem for the logic of bounded lattices.

Definition 4.7 ([20], pages 228). A polarity $\langle X, Y, R\rangle$ is said to be a *mirrored* BL-*space* if X and Y are HMS-spaces and the following conditions hold:

(1) R is open in the product topology;
(2) if $x\, R\, y_1$ and $x\, R\, y_2$, then $x\, R\, (y_1 \sqcap y_2)$;
(3) if $x_1\, R\, y$ and $x_2\, R\, y$, then $(x_1 \sqcap x_2)\, R\, y$;
(4) for every $F \in \mathsf{OF}(X)$ there exists $y \in Y$ such that $F = R^{-1}[y]$;
(5) for every $G \in \mathsf{OF}(Y)$ there exists $x \in X$ such that $G = R[x]$.

In this paper, we will use the terminology of BL-*frames* instead of mirrored BL-spaces, given the use that these polarities have for us.

Let X be an HMS-space. For every $x \in X$ we define the set $\psi_x := \{F \in \mathsf{OF}(X)\colon x \in F\}$. Then we consider the topology on $\mathsf{OF}(X)$ generated by the collection $\{\psi_x\colon x \in X\}$. We denote this topological space simply by $\mathsf{OF}(X)$. Notice that $\psi_x \cap \psi_y = \psi_{x \sqcap y}$ for all $x, y \in X$ and $\psi_{1_X} = \mathsf{OF}(X)$. Thus, the collection $\{\psi_x\colon x \in X\}$ is closed under finite intersection and hence it is a base for $\mathsf{OF}(X)$. Moreover, the specialization order of the topological space $\mathsf{OF}(X)$ is the set-theoretic inclusion.

Proposition 4.8 ([20], Lemma 5.1). *Let $\langle X, Y, R\rangle$ be a* BL-*frame. Then the map $y \mapsto R^{-1}[y]$ from Y onto $\mathsf{OF}(X)$ and the map $x \mapsto R[x]$ from X to $\mathsf{OF}(Y)$ are homeomorphisms. Moreover, X and Y are* BL-*spaces.*

Remark 4.9. Let $P = \langle X, Y, R\rangle$ be a BL-frame. By Proposition 4.8 we have that the maps $y \mapsto R^{-1}[y]$ and $x \mapsto R[x]$ are homeomorphisms. Thus, it is clear that P is an S-frame. Since X and Y are HMS-spaces, we have that they are complete lattices and thus they have top elements 1_X and 1_Y, respectively. From conditions (4) and (5) of Definition 4.7, we obtain that $1_X \in R^{-1}[y]$ for all $y \in Y$ and $1_Y \in R[x]$ for all $x \in X$. Therefore P is a bounded S-frame.

The following corollary is an immediate consequence of Proposition 3.4.

Corollary 4.10 (Soundness w.r.t. BL-frames). *Let $\Gamma \subseteq \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ be a finite subset and $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$. If the sequent $\Gamma \rhd \varphi$ is derivable in the Gentzen system $\mathcal{G}_{\mathbb{BL}}$, then it is valid over the class of all* BL-*frames.*

The idea in [20] of considering BL-frames (mirrored BL-spaces) is that they represent a bounded lattice L and its opposite L^{∂}. In fact, there is a categorical dual equivalence between bounded lattices and BL-frames. If the BL-frame $\langle X, Y, R\rangle$ is the corresponding dual (categorically) to a bounded lattice L, then X is the dual BL-space of L and Y is the dual BL-space of the opposite lattice L^{∂} of L.

Let L be a bounded lattice and $X = \langle \mathsf{Fi}(L), \tau_L\rangle$ its dual BL-space. Then the dual (categorical) BL-frame of L is $\langle X, \mathsf{OF}(X), R\rangle$, where R is defined as follows: for every

$x \in X$ and every $F \in \mathsf{OF}(X)$, $xRF \Leftrightarrow x \in F$. The reader should keep in mind this construction of a BL-frame from a bounded lattice, since it will play an important role in constructing the canonical BL-frame in the next section.

4.2. Complete relational semantics for $\mathcal{S}_{\mathbb{BL}}$. Let $P = \langle X, Y, R\rangle$ be a BL-frame. By (1) of Proposition 2.6 and by conditions (4) and (5) of Definition 4.7 we have for every $A \subseteq X$ and $B \subseteq Y$, that $\Phi_R(A) \in \mathsf{FSat}(X)$ and $\Psi_R(B) \in \mathsf{FSat}(Y)$. This implies that $\mathcal{C}(P) \subseteq \mathsf{FSat}(X)$ and $\mathcal{C}^{\mathrm{d}}(P) \subseteq \mathsf{FSat}(Y)$. Now we show that the Galois closed subsets of X and the dual Galois closed subsets of Y are exactly the F-saturated of X and the F-saturated of Y, respectively.

Proposition 4.11. *Let $P = \langle X, Y, R\rangle$ be a* BL*-frame. Then*

$$\mathcal{C}(P) = \mathsf{FSat}(X) \quad \textit{and} \quad (\Psi_R \circ \Phi_R)(A) = \mathrm{fsat}(A)$$

for every $A \subseteq X$ and

$$\mathcal{C}^{\mathrm{d}}(P) = \mathsf{FSat}(Y) \quad \textit{and} \quad (\Phi_R \circ \Psi_R)(B) = \mathrm{fsat}(B)$$

for every $B \subseteq Y$.

Proof. We only prove the first part. The second part of the proposition is similar to the first one, and thus we leave the details to the reader. Let $S \in \mathsf{FSat}(X)$. So $S = \bigcap\{F \in \mathsf{OF}(X) \colon S \subseteq F\}$. Then, by Propositions 4.8 and 2.6, we have

$$\begin{aligned} S &= \bigcap\{R^{-1}[y] \colon y \in Y \text{ and } S \subseteq R^{-1}[y]\} \\ &= \bigcap\{R^{-1}[y] \colon y \in \Phi_R(S)\} = (\Psi_R \circ \Phi_R)(S). \end{aligned}$$

Hence $S \in \mathcal{C}(P)$ and therefore $\mathcal{C}(P) = \mathsf{FSat}(X)$. Now, let $A \subseteq X$. Then, by Proposition 4.8 again, we have

$$\begin{aligned} (\Psi_R \circ \Phi_R)(A) = \bigcap\{R^{-1}[y] \colon y \in \Phi_R(A)\} &= \bigcap\{R^{-1}[y] \colon A \subseteq R^{-1}[y]\} \\ &= \bigcap\{F \in \mathsf{OF}(X) \colon A \subseteq F\} = \mathrm{fsat}(A). \end{aligned}$$

□

Now we want to build our canonical BL-frame that we will use to prove a completeness theorem for the logic of bounded lattices $\mathcal{S}_{\mathbb{BL}}$. We need to consider some notions of abstract algebraic logic (see [6], [7], [5]).

Let $\mathcal{L}$ be an arbitrary algebraic language and $\mathcal{S} = \langle \mathrm{Fm}(\mathcal{L}), \vdash_{\mathcal{S}} \rangle$ be a sentential logic. The binary relation $\Lambda\mathcal{S}$ on $\mathrm{Fm}(\mathcal{L})$ defined by

$$(\varphi, \psi) \in \Lambda\mathcal{S} \Leftrightarrow \varphi \vdash_{\mathcal{S}} \psi \text{ and } \psi \vdash_{\mathcal{S}} \varphi$$

is called the *Frege relation* of $\mathcal{S}$. In other words, the Frege relation is just the interderivability between formulas of $\mathrm{Fm}(\mathcal{L})$. A sentential logic $\mathcal{S}$ is said to be *selfextensional* if the Frege relation $\Lambda\mathcal{S}$ is a congruence on the algebra of formulas $\mathrm{Fm}(\mathcal{L})$.

Proposition 4.12 ([6], 5.1.2)**.** *The sentential logic $\mathcal{S}_{\mathbb{BL}} = \langle \mathrm{Fm}(\mathcal{L}_{\mathrm{b}}), \vdash_{\mathbb{BL}} \rangle$ is selfextensional.*

Now, we consider the quotient algebra $\langle \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})/\Lambda\mathcal{S}_{\mathbb{BL}}, \wedge, \vee, [\bot], [\top] \rangle$. From the rules of $\mathcal{G}_{\mathbb{BL}}$, it is straightforward to show directly that the algebra $L := \langle \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})/\Lambda\mathcal{S}_{\mathbb{BL}}, \wedge, \vee, [\bot], [\top] \rangle$ is a bounded lattice. Notice that the lattice order of L is given by

$$[\varphi] \leqslant [\psi] \Leftrightarrow \varphi \vdash_{\mathbb{BL}} \psi \Leftrightarrow \text{the sequent } \varphi \rhd \psi \text{ is derivable on } \mathcal{G}_{\mathbb{BL}}.$$

Since L is a bounded lattice, we can consider its dual BL-frame $P_{\mathbb{BL}} = \langle X, Y, R \rangle$, where $X = \mathsf{Fi}(L)$, $Y = \mathsf{OF}(X)$ and the relation $R \subseteq X \times Y$ is defined as: $xRF \Leftrightarrow x \in F$. The BL-frame $P_{\mathbb{BL}}$ is called the *canonical* BL-*frame* of $\mathcal{S}_{\mathbb{BL}}$. Now we define the valuation $v\colon \mathrm{Var} \to \mathsf{KOF}(X)$ as follows:

$$v(p) = \{F \in \mathsf{Fi}(L)\colon\ [p] \in F\}.$$

Then we consider the *canonical* BL-*model* $M_{\mathbb{BL}} := \langle P_{\mathbb{BL}}, v \rangle$ for $\mathcal{S}_{\mathbb{BL}}$. Recall, by Definition 3.2, that $v^{\Vdash}_{M_{\mathbb{BL}}}\colon\ \mathrm{Fm}(\mathcal{L}_{\mathrm{b}}) \to \mathsf{FSat}(X)$ is the unique extension homomorphism of v and (by Proposition 3.3) such that

$$v^{\Vdash}_{M_{\mathbb{BL}}}(\varphi) = \{x \in X\colon\ M_{\mathbb{BL}}, x \Vdash \varphi\}$$

for every formula φ. In fact, since the range of the valuation v is $\mathsf{KOF}(X)$ and $\mathsf{KOF}(X)$ is a sublattice of $\mathsf{FSat}(X)$, it follows that $v^{\Vdash}_{M_{\mathbb{BL}}}(\varphi) \in \mathsf{KOF}(X)$ for every formula φ and thus $v^{\Vdash}_{M_{\mathbb{BL}}}\colon\ \mathrm{Fm}(\mathcal{L}_{\mathrm{b}}) \to \mathsf{KOF}(X)$.

From now on, we omit the subscript $\mathbb{BL}$ on $M_{\mathbb{BL}}$ and the subscript $M_{\mathbb{BL}}$ on $v^{\Vdash}_{M_{\mathbb{BL}}}$, i.e. we write $M := M_{\mathbb{BL}}$ and $v^{\Vdash} := v^{\Vdash}_{M_{\mathbb{BL}}}$.

Proposition 4.13. *For every $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ we have $v^{\Vdash}(\varphi) = \{F \in \mathsf{Fi}(L)\colon [\varphi] \in F\}$.*

Proof. We proceed by induction on the complexity of formulas. For $p \in \mathrm{Var}$ it is trivial by definition of v. Now let $\varphi, \psi \in \mathrm{Fm}(\mathcal{L}_\mathrm{b})$ be such that $v^{\Vdash}(\varphi) = \{F \in \mathsf{Fi}(L)\colon [\varphi] \in F\}$ and $v^{\Vdash}(\psi) = \{F \in \mathsf{Fi}(L)\colon [\psi] \in F\}$.

Since $v^{\Vdash}$ is a homomorphism and using the inductive hypothesis, we have

$$\begin{aligned} v^{\Vdash}(\varphi \wedge \psi) &= v^{\Vdash}(\varphi) \cap v^{\Vdash}(\psi) = \{F \in \mathsf{Fi}(L)\colon [\varphi] \in F\} \cap \{F \in \mathsf{Fi}(L)\colon [\psi] \in F\} \\ &= \{F \in \mathsf{Fi}(L)\colon [\varphi], [\psi] \in F\} = \{F \in \mathsf{Fi}(L)\colon [\varphi \wedge \psi] \in F\}. \end{aligned}$$

Next we use again that $v^{\Vdash}$ is a homomorphism and the inductive hypothesis. Moreover, since $X = \langle \mathsf{Fi}(L), \tau_L \rangle$ is the dual BL-space of $L = \mathrm{Fm}(\mathcal{L}_\mathrm{b})/\Lambda\mathcal{S}_{\mathbb{BL}}$, we have by Remark 4.5 that $U_{[\chi_1]} \vee U_{[\chi_2]} = U_{[\chi_1] \vee [\chi_2]}$ for all $[\chi_1], [\chi_2] \in L$. Then

$$\begin{aligned} v^{\Vdash}(\varphi \vee \psi) &= v^{\Vdash}(\varphi) \vee v^{\Vdash}(\psi) = \{F \in \mathsf{Fi}(L)\colon [\varphi] \in F\} \vee \{F \in \mathsf{Fi}(L)\colon [\psi] \in F\} \\ &= U_{[\varphi]} \vee U_{[\psi]} = U_{[\varphi] \vee [\psi]} = U_{[\varphi \vee \psi]} = \{F \in \mathsf{Fi}(L)\colon [\varphi \vee \psi] \in F\}. \end{aligned}$$

Notice that the top element of $X = \mathsf{Fi}(L)$ is L, in other words $1_X = L$. Then

$$v^{\Vdash}(\bot) = \{1_X\} = \{L\} = \{F \in \mathsf{Fi}(L)\colon [\bot] \in F\}.$$

For $\top$ we have $v^{\Vdash}(\top) = X = \mathsf{Fi}(L) = \{F \in \mathsf{Fi}(L)\colon [\top] \in F\}$. This completes the proof. $\square$

Corollary 4.14. *Let $\varphi \in \mathrm{Fm}(\mathcal{L}_\mathrm{b})$ and $F \in X = \mathsf{Fi}(L)$. Then*

$$M, F \Vdash \varphi \Leftrightarrow [\varphi] \in F.$$

Theorem 4.15. *Let $\Gamma \subseteq \mathrm{Fm}(\mathcal{L}_\mathrm{b})$ be finite and $\varphi \in \mathrm{Fm}(\mathcal{L}_\mathrm{b})$. If the sequent $\Gamma \rhd \varphi$ cannot be derived in $\mathcal{G}_{\mathbb{BL}}$, then it is not valid over the canonical* BL*-model M.*

Proof. Let $F := \mathsf{Fi}_L(\{[\psi]\colon \psi \in \Gamma\})$ be the filter of the lattice $L = \mathrm{Fm}(\mathcal{L}_\mathrm{b})/\Lambda S_{\mathbb{BL}}$ generated by $\{[\psi]\colon \psi \in \Gamma\}$. From the previous corollary we have that $M, F \Vdash \Gamma$. Suppose that $M, F \Vdash \varphi$. So, by the previous corollary, we obtain $[\varphi] \in F$. Then, by definition of the filter F, there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $[\psi_1] \wedge \ldots \wedge [\psi_n] \leqslant [\varphi]$. Thus $[\psi_1 \wedge \ldots \wedge \psi_n] \leqslant [\varphi]$. This implies that the sequent $\psi_1 \wedge \ldots \wedge \psi_n \rhd \varphi$ is derivable in $\mathcal{G}_{\mathbb{BL}}$ and thus $\Gamma \rhd \varphi$ is derivable. This is a contradiction and hence $M, F \nVdash \varphi$. Therefore $\Gamma \rhd \varphi$ is not valid over the canonical BL-model M.

$\square$

Now, by Proposition 3.4 and Theorem 4.15, we can enunciate the two completeness theorems for the logic of bounded lattices $\mathcal{S}_{\mathbb{BL}}$ with respect to the class of all BL-frames and the class of all bounded S-frames.

Theorem 4.16 (Completeness w.r.t. the BL-frames)**.** *Let $\Gamma \subseteq \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ be finite and $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$. The sequent $\Gamma \rhd \varphi$ is derivable in $\mathcal{G}_{\mathbb{BL}}$ if and only if it is valid in the class of all* BL*-frames.*

Then, as an immediate consequence, we obtain the following result.

Theorem 4.17 (Completeness w.r.t. bounded S-frames)**.** *Let $\Gamma \subseteq \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$ be finite and $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{b}})$. The sequent $\Gamma \rhd \varphi$ is derivable in $\mathcal{G}_{\mathbb{BL}}$ if and only if it is valid in the class of all bounded S-frames.*

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