A NOTE ON THE OPEN PACKING NUMBER IN GRAPHS

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Abstract. A subset S of vertices in a graph G is an open packing set if no pair of vertices of S has a common neighbor in G. An open packing set which is not a proper subset of any open packing set is called a maximal open packing set. The maximum cardinality of an open packing set is called the open packing number and is denoted by $\varrho^{\rm o}(G)$. A subset S in a graph G with no isolated vertex is called a total dominating set if any vertex of G is adjacent to some vertex of S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. We characterize graphs of order n and minimium degree at least two with $\varrho^{\rm o}(G)=\gamma_t(G)=\frac12 n$.

Keywords: packing; open packing; total domination

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1. Introduction

In this paper, we follow the notations of [3], [7]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order n and edge set E. The *open neighborhood* of a vertex $v\in V$ is $N(v)=\{u\in V\colon uv\in E\}$ and the *closed neighborhood* of v is $N[v]=N(v)\cup\{v\}$. The *degree* of v is $\deg(v)=|N(v)|$. The *maximum* and *minimum* degrees in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex of degree one in a tree is called a *leaf* and its unique neighbor is called a *support vertex*. A *pendant* edge in a graph is an edge incident with a leaf. The *corona* graph $\operatorname{cor}(H)$ of a graph H is a graph obtained from H by adding a leaf to every vertex of H. A *matching* in a graph G is a set of edges no pair of which has a common vertex. For a subset S of vertices of G, the subgraph induced by S is denoted by $G[S]$. A subset S of vertices of a graph G is a *dominating set* of G if every vertex $x\in V-S$ is adjacent to a vertex of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S of a graph G is

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called a *total dominating set* if $G[S]$ has no isolated vertices. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A graph G is *total domination partitionable* if its vertex set can be partitioned into two total domination sets. For a comprehensive study of domination and total domination see [7], [10].

A *packing* of a graph G is a set of vertices whose closed neighborhoods are pairwise disjoint. The *packing number* of G, denoted by $\varrho(G)$, is the maximum cardinality among all packings of G. For reference on the packing number of a graph, see for example [2], [4], [11], [15]. A set S of vertices of a graph G is an *open packing* of G if the open neighborhoods of the vertices of S are pairwise disjoint in G. The *open packing number* of G, denoted by $\varrho^{\rm o}(G)$, is the maximum cardinality among all open packings of G. The open packing number of a graph has been studied in [13], [8], [9], [14], for example.

A subset S of vertices of G is an *efficient open dominating set* if $|N(v) \cap S| = 1$ for every vertex $v \in V(G)$. An *efficient open domination graph* is a graph with an efficient open dominating set. The study of efficient open domination graphs has begun by Cockayene et al. [6] and further studied in, for example, [12]. Note that the efficient open domination graphs are graphs G with $\varrho^{\rm o}(G) = \gamma_t(G)$.

Recently, Hamid and Saravanakumar in [13] continued the study of open packing in graphs, and presented several important results on the open packing number of a graph. They posed the characterization of graphs of order n with $\delta(G) \geqslant 2$ for which $\varrho^{\rm o}(G) + \gamma_t(G) = n$ as an open problem. We give a characterization of graphs of order n with minimum degree at least two for which $\varrho^{\rm o}(G) = \gamma_t(G) = \frac{1}{2}n$. We make use of the following.

Theorem 1 ([13]). *If G is a connected graph of order $n \geqslant 2$, then $\varrho^{\rm o}(G) \leqslant n/\delta(G)$.*

Theorem 2 ([5]). *If G is a graph without isolated vertices of order $n \geqslant 3$, then $\gamma_t(G) \leqslant \frac{2}{3}n$.*

Theorem 3 ([1]). *If G is a graph of order n with $\delta(G) \geqslant 3$, then $\gamma_t(G) \leqslant \frac{1}{2}n$.*

2. Main result

We begin with the following.

Lemma 4. *Let G be a connected graph of order n with $\delta(G) \geqslant 2$, and $\varrho^{\rm o}(G) + \gamma_t(G) = n$, and let S be a $\varrho^{\rm o}(G)$-set. Then:*

(1) *$\delta(G) = 2$;*

(2) $|S| \leqslant |V(G) - S|$;

(3) *Any non-support vertex of* $G[V(G)-S]$ *is adjacent to precisely one vertex of* S.

Proof. Let G be a connected graph of order n with $\delta(G) \geqslant 2$ and $\varrho^{\mathrm{o}}(G) + \gamma_t(G) = n$. We consider each claim seprately:

(1) By Theorems 1 and 2, $\frac{2}{3}n \geqslant \gamma_t(G) = n - \varrho^{\mathrm{o}}(G) \geqslant n - n/\delta(G)$, and we obtain that $2 \leqslant \delta(G) \leqslant 3$. If $\delta(G) = 3$, then by Theorems 1 and 3, $\frac{1}{2}n \geqslant \gamma_t(G) = n - \varrho^{\mathrm{o}}(G) \geqslant n - n/\delta(G)$, and we obtain that $\delta(G) = 2$, a contradiction. Thus $\delta(G) = 2$.

(2) Let S be a $\varrho^{\mathrm{o}}(G)$-set. Then clearly $V(G)-S$ is a $\gamma_t(G)$-set, since $\delta(G) = 2$ and any component of $G[S]$ is K_2 or K_1. Since no pair of vertices of S have a common neighbor in $V(G) - S$, we have $|S| \leqslant |V(G) - S|$.

(3) If there is a non-support vertex x of $G[V(G) - S]$ that is not adjacent to a vertex of S, then $(V(G) - S) - \{x\}$ is a total dominating set for G, a contradiction with $\varrho^{\mathrm{o}}(G) + \gamma_t(G) = n$. Since S is an open packing set, x is adjacent to precisely one vertex of S. □

It is known that $\varrho^{\mathrm{o}}(G) \leqslant \gamma_t(G)$ for any graph G with no isolated vertex (see [12], Lemma 5). Let $\mathcal{H}_1$ be the class of all graphs G such that G is obtained from a corona $\mathrm{cor}(H)$, where H is a graph of even order and with no isolated vertex, by adding a perfect matching between the leaves of $\mathrm{cor}(H)$. Figure 1 shows a graph in the family $\mathcal{H}_1$. It is easy to see that any graph in the family $\mathcal{H}_1$ is total domination partitionable.

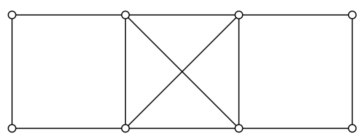

Figure 1. A graph in $\mathcal{H}_1$.

Theorem 5. *If* G *is a connected graph of order* n *with* $\delta(G) \geqslant 2$, *then* $\varrho^{\mathrm{o}}(G) = \gamma_t(G) = \frac{1}{2}n$ *if and only if* $G \in \mathcal{H}_1$.

Proof. Let G be a connected graph of order n with $\delta(G) \geqslant 2$ and $\varrho^{\mathrm{o}}(G) = \gamma_t(G) = \frac{1}{2}n$. By Lemma 4, $\delta(G) = 2$. Let S be a $\varrho^{\mathrm{o}}(G)$-set. Clearly $V(G)-S$ is a total dominating set for G, and so $\gamma_t(G) \leqslant n - \varrho^{\mathrm{o}}(G)$. Now, $n = \varrho^{\mathrm{o}}(G) + \gamma_t(G) \leqslant \varrho^{\mathrm{o}}(G) + |V(G) - S| = n$, and thus $|V(G) - S| = \gamma_t(G)$. It is evident that any component of $G[S]$ is K_1 or K_2. Since $|S| = |V(G) - S|$, any vertex of $V(G) - S$ is adjacent to precisely one vertex of S. Thus, any component of $G[S]$ is K_2. Furthermore, $\deg(x) = 2$ for any vertex $x \in S$. Let G' be obtained from G by removing all edges of $G[S]$. Then clearly $G' = \mathrm{cor}(G[V(G) - S])$. Since $V(G) - S$ is a total dominating set of G, $G[V(G) - S]$ has no isolated vertex. Consequently, $G \in \mathcal{H}_1$.

Conversely, assume that $G \in \mathcal{H}_1$. Thus G is obtained from a corona $\mathrm{cor}(H)$, where H is a graph of even order and with no isolated vertex, by adding a perfect matching M between the leaves of $\mathrm{cor}(H)$. Clearly $V(H)$ is a total dominating set for G, and thus $\gamma_t(G) \leqslant |V(H)|$. Let S be a total dominating set in G. For any edge $xy \in G[S]$, $|S \cap (N[x] \cup N[y])| \geqslant 2$. Thus $|S| \geqslant |V(H)|$. Consequently, $|V(H)| = \gamma_t(G)$. On the other hand, the vertices of the perfect matching M form an open packing for G, and so $\varrho^{\mathrm{o}}(G) \geqslant |V(H)|$. Since $\varrho^{\mathrm{o}}(G) \leqslant \gamma_t(G)$, we obtain that $\varrho^{\mathrm{o}}(G) = |V(H)|$, as desired. □

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