

Shichun Yang; Florian Luca; Alain Togbé
On a divisibility problem

Mathematica Bohemica, Vol. 144 (2019), No. 2, 125–135

Persistent URL: <http://dml.cz/dmlcz/147754>

Terms of use:

© Institute of Mathematics AS CR, 2019

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON A DIVISIBILITY PROBLEM

SHICHUN YANG, Sichuan, FLORIAN LUCA, Johannesburg,
ALAIN TOGBÉ, Westville

Received June 8, 2017. Published online June 5, 2018.
Communicated by Dagmar Medková

Abstract. Let p_1, p_2, \dots be the sequence of all primes in ascending order. Using explicit estimates from the prime number theory, we show that if $k \geq 5$, then

$$(p_{k+1} - 1)! \mid \left(\frac{1}{2}(p_{k+1} - 1)\right)! p_k!,$$

which improves a previous result of the second author.

Keywords: prime; divisibility; exponent; Sándor-Luca's theorem

MSC 2010: 11A25, 11B83

1. INTRODUCTION

Let n be a positive integer. The sequence of products of n consecutive positive integers is an important arithmetic sequence in number theory. There are many interesting related problems (see for example [6]–[9]). In 1975, Erdős and Selfridge in [9] proved that the product of two or more consecutive integers is never a power. One can also refer to the results on the index decomposition of prime numbers of $n!$ obtained by Erdős and Graham (see [7], [8]). For the past ten years, many scholars such as Berend in [2], Chen and Zhu in [3], Le in [11], Luca and Stănică in [13], Moree and Roskam in [14], Sándor in [17]–[18] and others studied the arithmetic of $n!$ obtaining many important results.

Let p_k be the k th prime. In 1999, Sándor in [17]–[18] set forward the following conjecture:

Conjecture 1.1 ([17]). *For every $k \geq 5$, the integer $(p_k - 2)! p_k!$ is divisible by $(p_{k+1} - 1)!$.*

In 2000, Luca in [12] confirmed Sándor's conjecture and proved the following theorem.

Theorem 1.2 (Sándor-Luca theorem). *When $k \geq 5$, we have*

$$(1) \quad (p_{k+1} - 1)! \mid (p_k - 2)! p_k!.$$

In 2002, Atanassov in [1] gave a further strengthening of Theorem 1.2 by proving the following result.

Theorem 1.3. *If $k \geq 5$, then*

$$(2) \quad (p_{k+1} - 1)! \mid p_{k-1}! p_k!.$$

In this paper, using explicit estimates from the prime number theory we prove the following result.

Theorem 1.4. *If $k \geq 5$, we have*

$$(3) \quad (p_{k+1} - 1)! \mid \left(\frac{1}{2}(p_{k+1} - 1)\right)! p_k!.$$

Remark 1.5. When $k \geq 5$, we have $\frac{1}{2}(p_{k+1} - 1) < p_{k-1}$. Therefore, Theorem 1.4 improves Theorem 1.2 and Theorem 1.3. If $\frac{1}{2}(p_{k+1} - 1) = p$, where p is a prime, then $p \parallel \left(\frac{1}{2}(p_{k+1} - 1)\right)!$, where we use the notation $d^k \parallel n$ to mean that $d^k \mid n$ but $d^{k+1} \nmid n$. Then, we get

$$(p_{k+1} - 1)! \parallel \left(\frac{1}{2}(p_{k+1} - 1)\right)! p_k!.$$

Therefore, Theorem 1.4 shows that the best (smallest) answer to the question of what is m such that $(p_{k+1} - 1)! \mid m! p_k!$ is $m = \frac{1}{2}(p_{k+1} - 1)$. For example, taking $k = 22$, we have $82! \mid 41! 79!$. Obviously, 41 cannot be replaced by any smaller positive integer in the above divisibility relation.

We organize this paper as follows. In Section 2, we prove Theorem 1.4. In Section 3, we prove a corollary to Theorem 1.4. Furthermore, setting

$$Q_k = \frac{(p_{k+1} - 1)!}{p_k!},$$

we propose a problem concerning numbers k for which the prime factors of Q_k occupy an initial interval of primes. We suggest a conjectural answer and provide some heuristic and numerical evidence in order to support it.

2. PROOF OF THEOREM 1.4

When $k \geq 5$, formula (3) is equivalent to

$$(4) \quad (p_k + 1)(p_k + 2) \dots (p_{k+1} - 1) \mid \left(\frac{1}{2}(p_{k+1} - 1)\right)!$$

If p_k, p_{k+1} are two adjacent primes, then $p_k + 1, p_k + 2, \dots, p_{k+1} - 1$ are all composite numbers and each of their prime factors p satisfies $p \leq \frac{1}{2}(p_{k+1} - 1)$. Thus, for every prime $2 \leq p \leq \frac{1}{2}(p_{k+1} - 1)$, we just need to prove that

$$(5) \quad v_p((p_k + 1)(p_k + 2) \dots (p_{k+1} - 1)) \leq v_p\left(\left(\frac{1}{2}(p_{k+1} - 1)\right)!\right),$$

where $v_p(n)$ denotes the p -adic valuation of a positive integer n .

We assume that n, l are positive integers such that $l < \frac{1}{2}n$. Let p be a prime satisfying $2 \leq p \leq n$. Next, we prove that

$$(6) \quad v_p((2n - l + 2)(2n - l + 3) \dots (2n - 1)(2n)) \leq v_p(n!), \quad n > 30.$$

Writing

$$(7) \quad n = a_0 + a_1p + \dots + a_{s-1}p^{s-1} + a_s p^s, \quad 0 \leq a_0, a_1, \dots, a_s < p, \quad a_s \geq 1,$$

we get

$$(8) \quad \begin{aligned} 2n &= 2a_0 + 2a_1p + \dots + 2a_{s-1}p^{s-1} + 2a_s p^s \\ &= b_0 + b_1p + \dots + b_{r-1}p^{r-1} + b_r p^r, \end{aligned}$$

where $0 \leq b_0, b_1, \dots, b_r < p$ and $b_r \geq 1$. From (8), we have $s \leq r \leq s + 1$ and then inequality (6) is equivalent to

$$(9) \quad \sum_{j=1}^{s+1} \left\lfloor \frac{2n}{p^j} \right\rfloor - \sum_{j=1}^{s+1} \left\lfloor \frac{2n-l}{p^j} \right\rfloor \leq \sum_{j=1}^{s+1} \left\lfloor \frac{n}{p^j} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x . Since $l < \frac{1}{2}n$, and $\lfloor x \rfloor + \lfloor y \rfloor \geq \lfloor x + y \rfloor - 1$, it follows that

$$(10) \quad \begin{aligned} \left\lfloor \frac{n}{p^j} \right\rfloor + \left\lfloor \frac{2n-l}{p^j} \right\rfloor - \left\lfloor \frac{2n}{p^j} \right\rfloor &\geq \left\lfloor \frac{5n}{p^j} \right\rfloor - 1 - \left\lfloor \frac{2n}{p^j} \right\rfloor \\ &= \left\lfloor \frac{5}{2}(a_s p^{s-j} + \dots + a_{s-j+1}p + a_{s-j}) \right\rfloor \\ &\quad - 2(a_s p^{s-j} + \dots + a_{s-j+1}p + a_{s-j}) - 1 \\ &=: A_j. \end{aligned}$$

Suppose that $s \geq 4$. When $j \leq s - 2$, we have

$$A_j \geq \lfloor \frac{5}{2}(a_s p^2 + a_{s-1} p) \rfloor - 2(a_s p^2 + a_{s-1} p) - 1 \geq 2.$$

When $j = s - 1$, we obtain

$$A_j \geq \lfloor \frac{5}{2} a_s p \rfloor - 2a_s p - 1 \geq 0.$$

When $j = s$, one can see that

$$A_j \geq \lfloor \frac{5}{2} a_s \rfloor - 2a_s - 1 \geq -1.$$

When $j = s + 1$, we deduce that

$$A_j \geq \left\lfloor \frac{\frac{5}{2} a_s}{p} \right\rfloor - \frac{2a_s}{p} - 1 \geq -1.$$

Therefore, from (10) we get

$$\sum_{j=1}^{s+1} \left\lfloor \frac{n}{p^j} \right\rfloor + \sum_{j=1}^{s+1} \left\lfloor \frac{2n-l}{p^j} \right\rfloor - \sum_{j=1}^{s+1} \left\lfloor \frac{2n}{p^j} \right\rfloor \geq 2 + 0 - 1 - 1 = 0.$$

Thus, we conclude that inequality (9) holds.

If $s = 3$, then $A_j \geq -1$ and further

$$\begin{aligned} (11) \quad & A_1 + A_2 + A_3 + A_4 \\ & \geq \lfloor \frac{5}{2}(a_3 p^2 + a_2 p + a_1) \rfloor - 2(a_3 p^2 + a_2 p + a_1) - 1 \\ & \quad + \lfloor \frac{5}{2}(a_3 p + a_2) \rfloor - 2(a_3 p + a_2) - 1 - 1 - 1 \\ & \geq \lfloor \frac{5}{2}(4a_3 + 2a_2) \rfloor - 2(4a_3 + 2a_2) + \lfloor \frac{5}{2} \cdot 2a_3 \rfloor - 2 \cdot 2a_3 - 4 \geq 0. \end{aligned}$$

So, (9) is verified for $s = 3$ also.

If $s = 2$, when $p \geq 7$, we have

$$\begin{aligned} A_1 + A_2 + A_3 & \geq \lfloor \frac{5}{2}(a_2 p + a_1) \rfloor - 2(a_2 p + a_1) - 1 - 1 - 1 \\ & \geq \lfloor \frac{35}{2} a_2 \rfloor - 14a_2 - 3 \geq 0, \end{aligned}$$

so (9) holds. When $p \leq 5$, from (7) we get $n < 125$. For $p = 2, 3, 5$, one can directly verify the validity of (9).

If $s = 1$, by (7), we get $n = a_1 p + a_0$. When $a_1 \geq 4$, since

$$A_1 + A_2 \geq \lfloor \frac{5}{2} a_1 \rfloor - 2a_1 - 1 - 1 \geq 10 - 8 - 2 = 0,$$

it follows that formula (9) holds. When $a_1 \leq 3$, if $p \geq 7$, then (8) implies $2n = 2a_1p + 2a_0$ and $r = s = 1$. As $a_0 < p$, we then get

$$\begin{aligned} \left\lfloor \frac{2n}{p} \right\rfloor - \left\lfloor \frac{2n-l}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor &\leq \left\lfloor \frac{l}{p} \right\rfloor + 1 - \left\lfloor \frac{n}{p} \right\rfloor \leq \left\lfloor \frac{\frac{1}{2}n}{p} \right\rfloor + 1 - \left\lfloor \frac{n}{p} \right\rfloor \\ &= \left\lfloor \frac{1}{2}a_1 + \frac{\frac{1}{2}a_0}{p} \right\rfloor + 1 - a_1 \leq a_1 - 1 + 1 - a_1 = 0. \end{aligned}$$

Thus, (9) also holds. When $a_1 \leq 3$ and $p \leq 5$, it follows from (7) that $n < 20$, which contradicts the condition $n > 30$. Thus, if $n > 30$, then (9) holds.

Next, we show that

$$(12) \quad p_{k+1} - p_k < \frac{1}{4}(p_{k+1} - 1) \quad \forall k \geq 7.$$

Notice that inequality (12) is equivalent to

$$(13) \quad p_{k+1} < \frac{4}{3}p_k - \frac{1}{3} \quad \forall k \geq 7.$$

We first check (13) numerically for $k \in \{7, 8, 9\}$, and then assume $k \geq 10$, so that $p_k > 25$. From [15], the interval $(p_k, \frac{6}{5}p_k)$ contains at least one prime number, the smallest of them being p_{k+1} . Consequently,

$$p_k + 1 < \frac{6}{5}p_k \leq \frac{4}{3}p_k - \frac{1}{3},$$

as required.

Thus, $p_{k+1} - p_k < \frac{1}{4}(p_{k+1} - 1)$. Therefore, we can take $n = \frac{1}{2}(p_{k+1} - 1)$ and $l = p_{k+1} - p_k$ and the inequality $l < \frac{1}{2}n$ is satisfied. If $k \geq 18$, then $n > 30$, so we can apply (5) and obtain

$$(14) \quad v_p\left(\left(\frac{1}{2}(p_{k+1} - 1)\right)!\right) \geq v_p((p_{k+1} - 1)(p_{k+1} - 2) \dots (p_{k+1} - l + 2)(p_{k+1} - l + 1)) \\ = v_p((p_{k+1} - 1)(p_{k+1} - 2) \dots (p_k + 2)(p_k + 1)).$$

Thus, (4) is verified for $k \geq 18$. For $5 \leq k \leq 17$, we verified (5) using Maple. This completes the proof of Theorem 1.4. \square

3. A COROLLARY AND A PROBLEM

Let us recall the following result.

Lemma 3.1. *Let p_n be the n th prime. If $n \geq 198$, we have*

$$(15) \quad n(\log n + \log \log n - 1) < p_n < n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} \right).$$

Proof. See (2) and (3) of [4]. □

Using Theorem 1.4 and the above prime number estimate (15), we can deduce the following corollary.

Corollary 3.2. *If $k \geq 5$, we have*

$$(16) \quad (p_{k+1} - 1)! \mid p_{\lfloor \frac{2}{3}k \rfloor}! p_k!$$

Proof. From (15), when $k \geq 198$, we have

$$(17) \quad \begin{aligned} p_{\lfloor \frac{2}{3}k \rfloor} &> \left(\frac{2}{3}k - 1\right) \left(\log\left(\frac{2}{3}k - 1\right) + \log \log\left(\frac{2}{3}k - 1\right) - 1\right) \\ &> \frac{1}{2}(k + 1) \left(\log(k + 1) + \log \log(k + 1) - \frac{1}{2}\right) \\ &> \frac{1}{2}(p_{k+1} - 1). \end{aligned}$$

Then, from (3) we get (16).

For $7 \leq k < 198$, using Maple, we can directly verify (16). This completes the proof of Corollary 3.2. □

Remark 3.3. The referee pointed out that for the proof of Corollary 3.2, the weaker upper bound $p_n \leq n(\log n + \log \log n)$ valid for $n \geq 6$ (see [16], Theorem 3) instead of the upper bound on p_n from (15) suffices.

Since for $k \geq 7$, $\lfloor \frac{2}{3}k \rfloor < k - 1$, we conclude that (16) improves (1) and (2).

Consider the standard decomposition

$$Q_k = \frac{(p_{k+1} - 1)!}{p_k!} = (p_k + 1)(p_k + 2) \dots (p_{k+1} - 1).$$

Calculations show that there are only 35 values of k such that Q_k can be written as the product of the powers of consecutive prime numbers when $k < 1000$. For

example, we have

$$\begin{aligned} Q_8 &= 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11, \\ Q_{70} &= 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, \\ Q_{85} &= 2^4 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17, \\ Q_{646} &= 2^6 \cdot 3 \cdot 5^2, \\ Q_{936} &= 2 \cdot 3 \cdot 5^2 \cdot 7^2. \end{aligned}$$

Therefore, it is natural to ask the following question:

Problem 3.4. Are there infinitely many positive integers k such that

$$(18) \quad Q_k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s},$$

where $s \geq 1$ and $\alpha_1, \alpha_2, \dots, \alpha_s \geq 1$?

We offer the following conjecture.

Conjecture 3.5. *There are infinitely many positive integers k verifying the condition of Problem 3.4 but only finitely many for which $p_{k+1} - p_k > 2$.*

Below we offer some heuristics towards Conjecture 3.5. In what follows, we use c_1, c_2, \dots for positive constants. Assume that k satisfies the condition of Problem 3.4. Crámer's conjecture asserts that $p_{k+1} - p_k = O((\log k)^2)$. We assume that it holds in order to justify our heuristic. Then

$$Q_k \leq p_{k+1}^{p_{k+1} - p_k} = \exp(O((\log k)^3)).$$

By the Prime Number Theorem

$$Q_k \geq \prod_{p \leq p_s} p \geq \exp((1 + o(1))p_s)$$

as $s \rightarrow \infty$. Thus, $p_s \leq c_1(\log k)^3$. Letting $P(m)$ be the maximal prime factor of m , it follows that

$$P(m) \leq c_1(\log k)^3$$

holds for all $m \in [p_k + 1, \dots, p_{k+1} - 1]$. Thus, all such m are $c_1(\log k)^3$ -smooth. Let $\Psi(x, y)$ be the function that counts the number of $n \leq x$ with $P(n) \leq y$. By [10], (1.14), we know that for fixed $\alpha > 1$, the following estimate holds:

$$\Psi(x, (\log x)^\alpha) = x^{1-1/\alpha+o(1)} \quad \text{as } x \rightarrow \infty.$$

Taking $x = p_{k+1}$ and $\alpha = 3 + \varepsilon$ for any $\varepsilon > 0$ small but fixed, and using the fact that $k < p_{k+1}$, we get

$$\Psi(p_{k+1}, c_1(\log k)^3) \leq \Psi(p_{k+1}, (\log p_{k+1})^{3+\varepsilon}) \leq p_{k+1}^{1-1/(3+\varepsilon)+o(1)}$$

for $k \geq c_2(\varepsilon)$. Making ε tend to 0 and k tend to infinity we get that

$$\Psi(p_{k+1}, c_1(\log k)^3) \leq p_{k+1}^{2/3+o(1)}$$

as $k \rightarrow \infty$. We interpret this as saying that the probability of a number $m \leq p_{k+1}$ to be $c_1(\log k)^3$ -smooth is

$$(19) \quad \frac{p_{k+1}^{2/3+o(1)}}{p_{k+1}} = \frac{1}{p_{k+1}^{1/3+o(1)}} \quad \text{as } k \rightarrow \infty.$$

Assume now that $p_{k+1} - p_k \geq 6$. Then $p_k + 1, p_k + 2, p_k + 3, p_k + 4$ are all in $[p_k + 1, p_{k+1} - 1]$ and are all $c_1(\log k)^3$ -smooth as $k \rightarrow \infty$. Assuming that the events “ $n + i \leq x$ is y -smooth” are independent for $i = 1, 2, 3, 4$, it follows that the “probability” that $p_k + i$ are all $c_1(\log k)^3$ -smooth for $i = 1, 2, 3, 4$ is at most

$$\left(\frac{1}{p_{k+1}^{1/3+o(1)}} \right)^4 = \frac{1}{p_{k+1}^{4/3+o(1)}} \quad \text{as } k \rightarrow \infty.$$

The above number is smaller than $p_{k+1}^{-5/4}$ for large k and since the series

$$(20) \quad \sum_{k \geq 1} \frac{1}{p_{k+1}^{5/4}}$$

is convergent, we infer that there should be only finitely many instances of k satisfying Problem 3.4 with $p_{k+1} - p_k \geq 6$. Assume next that $p_{k+1} - p_k = 4$. Then

$$Q_k = (p_k + 1)(p_k + 2)(p_k + 3) < p_{k+1}^3 = \exp(3 \log p_{k+1}).$$

Since $Q_k \geq \exp((1 + o(1))p_s)$, we get that $p_s \leq (3 + o(1)) \log k \leq (3 + o(1)) \log p_{k+1}$ as $k \rightarrow \infty$. By a result of de Bruijn (see [19], Theorem 2), if we put

$$Z = \frac{\log x}{\log y} \log \left(1 + \frac{y}{\log x} \right) + \frac{y}{\log y} \log \left(1 + \frac{\log x}{y} \right),$$

then

$$\log \Psi(x, y) = (1 + o(1))Z$$

uniformly as x and y tend to infinity. For us, taking $x = p_{k+1}$ and $y = (3 + o(1)) \times \log p_{k+1}$ as $k \rightarrow \infty$, a quick calculation shows that

$$Z = (c_3 + o(1)) \frac{\log p_{k+1}}{\log \log k}, \quad \text{where } c_3 = \log(1 + 3) + 3 \log(1 + \frac{1}{3}) = \log \frac{256}{27}$$

as $k \rightarrow \infty$. In particular, $\Psi(p_{k+1}, (3 + o(1)) \log p_{k+1}) = p_{k+1}^{o(1)}$ as $k \rightarrow \infty$. We interpret this by saying that the probability that $m \leq p_{k+1}$ has $P(m) \leq (3 + o(1)) \log p_{k+1}$ is

$$\frac{p_{k+1}^{o(1)}}{p_{k+1}} = \frac{1}{p_{k+1}^{1+o(1)}} \quad \text{as } k \rightarrow \infty.$$

Assuming again that $n + i \leq x$ being y -smooth are independent events for $i = 1, 2$ and taking $n = p_k$, we get that the probability that both $p_k + 1$ and $p_k + 2$ are $(3 + o(1)) \log p_{k+1}$ -smooth is at most

$$\left(\frac{1}{p_{k+1}^{1+o(1)}} \right)^2 = \frac{1}{p_{k+1}^{2+o(1)}}$$

as $k \rightarrow \infty$. By the heuristic used at the convergence of series (20), it follows that there should be only finitely many k with $p_{k+1} - p_k = 4$ satisfying Problem 3.4.

When $p_{k+1} - p_k = 2$, then $Q_k = p_k + 1$. So, instead of asking Problem 3.4, we can reformulate the problem by saying:

Problem 3.6. Are there infinitely many numbers of the form

$$M = p_1^{\alpha_1} \dots p_s^{\alpha_s}$$

for positive s and $\alpha_1, \dots, \alpha_s$ such that both $M - 1$ and $M + 1$ are primes?

If the answer to Problem 3.6 is affirmative, letting k be such that $M - 1 = p_k$, then certainly $M + 1 = p_{k+1}$, therefore for this k we have that $Q_k = M$ satisfies the requirement of Problem 3.4. To see why perhaps there are infinitely many solutions to Problem 3.6, we just take any s , let $p_s = p$ and search for numbers

$$M = p_1^{\alpha_1} \dots p_s^{\alpha_s}, \quad \alpha_1, \dots, \alpha_s \in \{1, 2, \dots, a\}$$

where a is some fixed number. By the Prime Number Theorem, $M \leq \exp((a + o(1))p)$ as $s \rightarrow \infty$, so

$$(21) \quad \log M \leq 2ap$$

as $s \rightarrow \infty$. Also by the Prime Number Theorem, the “probability” that a number n is prime is $(1 + o(1))/\log n$ as $n \rightarrow \infty$. Assuming that $M - 1$ and $M + 1$ being

primes are “almost” independent events, that is, the probability of them being both prime is some constant times $((1 + o(1))/\log M)^2$ (an assumption backed up by the Twin Prime Conjecture or the Bateman-Horn heuristics), we would get that the probability that both $M - 1$ and $M + 1$ are primes is greater than

$$c_4 \left(\frac{1}{\log M} \right)^2 \geq \frac{c_5}{(ap)^2} \quad \text{as } p \rightarrow \infty,$$

with $c_5 = \frac{1}{4}c_4$ by (21). Now we vary $(\alpha_1, \dots, \alpha_s) \in [1, a]$ obtaining

$$a^s = a^{\pi(p)} = \exp\left((\log a + o(1))\frac{p}{\log p}\right)$$

choices for M as $p \rightarrow \infty$. Thus, the number of “twin primes” obtainable in this way should be at least

$$\frac{c_5 \exp((\log a + o(1))p/\log p)}{(ap)^2} = c_5 \exp\left((\log a + o(1))\frac{p}{\log p} - 2 \log(ap)\right),$$

an amount which tends to infinity with p . For fun, we took $p = 29$, and $a = 2$ obtaining 41 examples. The largest is



$$M = 181986654305686230 = 2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2.$$

When we increased $a = 2$ to $a = 4$ (keeping the same p), we got 13746 examples. The largest one is

$$\begin{aligned} M &= 9368998021767173907463983973766430000 \\ &= 2^4 \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13^4 \cdot 17^3 \cdot 19^4 \cdot 23^4 \cdot 29^4. \end{aligned}$$

Acknowledgements. We thank the referee for a careful reading of the manuscript and many specific suggestions which increased the quality of our paper. The first author was supported by the Innovative Research Team Project of Sichuan Provincial Universities (No. 18TD0047). The second author was supported in part by grant CPRR160325161141 and an A-rated scientist award both from the NRF of South Africa and by grant No. 17-02804S of the Grant Agency of the Czech Republic. The third author was supported in part by Purdue University Northwest.

References

- [1] *K. T. Atanassov*: Remark on József Sándor and Florian Luca’s theorem. *C. R. Acad. Bulg. Sci.* 55 (2002), 9–14. 
- [2] *D. Berend*: On the parity of exponents in the factorization of $n!$. *J. Number Theory* 64 (1997), 13–19. 

- [3] *Y. G. Chen, Y. C. Zhu*: On the prime power factorization of $n!$. *J. Number Theory* 82 (2000), 1–11. [zbl](#) [MR](#) [doi](#)
- [4] *P. Dusart*: Explicit inequalities for $\psi(X)$, $\theta(X)$, $\pi(X)$ and prime numbers. *C. R. Math. Acad. Sci., Soc. R. Can.* 21 (1999), 53–59. (In French.) [zbl](#) [MR](#)
- [5] *P. Dusart*: The k -th prime is greater than $k(\log k + \log \log k - 1)$ for $k \geq 2$. *Math. Comput.* 68 (1999), 411–415. [zbl](#) [MR](#) [doi](#)
- [6] *P. Erdős*: Note on products of consecutive integers. *J. Lond. Math. Soc.* 14 (1939), 194–198. [zbl](#) [MR](#) [doi](#)
- [7] *P. Erdős*: On a conjecture of Klee. *Am. Math. Monthly* 58 (1951), 98–101. [zbl](#) [MR](#) [doi](#)
- [8] *P. Erdős, R. L. Graham*: Old and New Problems and Results in Combinatorial Number Theory. Monographs of L’Enseignement Mathématique 28. L’Enseignement Mathématique, Université de Genève, Genève, 1980. [zbl](#) [MR](#)
- [9] *P. Erdős, J. L. Selfridge*: The product of consecutive integers is never a power. *Ill. J. Math.* 19 (1975), 292–301. [zbl](#) [MR](#)
- [10] *A. Hildebrand, G. Tenenbaum*: Integers without large prime factors. *J. Théor. Nombres Bordx.* 5 (1993), 411–484. [zbl](#) [MR](#) [doi](#)
- [11] *M. Le*: A conjecture concerning the Smarandache dual function. *Smarandache Notion J.* 14 (2004), 153–155. [zbl](#)
- [12] *F. Luca*: On a divisibility property involving factorials. *C. R. Acad. Bulg. Sci.* 53 (2000), 35–38. [zbl](#) [MR](#)
- [13] *F. Luca, P. Štănică*: On the prime power factorization of $n!$. *J. Number Theory* 102 (2003), 298–305. [zbl](#) [MR](#) [doi](#)
- [14] *P. Moree, H. Roskam*: On an arithmetical function related to Euler’s totient and the discriminant. *Fibonacci Q.* 33 (1995), 332–340. [zbl](#) [MR](#)
- [15] *J. Nagura*: On the interval containing at least one prime number. *Proc. Japan Acad.* 28 (1952), 177–181. [zbl](#) [MR](#) [doi](#)
- [16] *J. B. Rosser, L. Schoenfeld*: Approximate formulas for some functions of prime numbers. *Ill. J. Math.* 6 (1962), 64–94. [zbl](#) [MR](#)
- [17] *J. Sándor*: On values of arithmetical functions at factorials I. *Smarandache Notions J.* 10 (1999), 87–94. [zbl](#) [MR](#)
- [18] *J. Sándor*: On certain generalizations of the Smarandache function. *Smarandache Notions J.* 11 (2000), 202–212. [MR](#)
- [19] *G. Tenenbaum*: Introduction to Analytic and Probabilistic Number Theory. Cambridge Studies in Advanced Mathematics 46. Cambridge University Press, Cambridge, 1995. [zbl](#) [MR](#)

Authors’ addresses: *Shichun Yang*, College of Mathematics and Computer Science, Institute of Mathematics, ABa Teachers University, Sichuan 623000, P. R. China, e-mail: ysc1020@sina.com; *Florian Luca*, School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa and Department of Mathematics, Faculty of Sciences, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic, e-mail: florian.luca@wits.ac.za; *Alain Togbé*, Department of Mathematics, Statistics and Computer Science, Purdue University Northwest, 1401 S, U.S. 421, Westville IN 46391 USA, e-mail: atogbe@pnw.edu.