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SOME RESULTS ON (STRONG) ASYMPTOTIC TOEPLITZNESS AND HANKELNESS

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Abstract. Based on the results in A. Feintuch (1989), this work sheds light upon some interesting properties of strongly asymptotically Toeplitz and Hankel operators, and relations between these two classes of operators. Indeed, among other things, two main results here are (a) vanishing Toeplitz and Hankel operators forms an ideal, and (b) finding the distance of a strongly asymptotically Toeplitz operator from the set of vanishing Toeplitz operators.

Keywords: Hardy space of the unit circle; Toeplitz operator; Hankel operator; strong operator topology

MSC 2010: 47B35, 47L80

1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane and $L^2 = L^2(\mathbb{T})$ be the Hilbert space of (equivalence classes of) square-integrable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure $\mathrm{d}\theta/2\pi$, so that the measure of the entire circle is 1. The L^2-inner product is given by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\theta.$$

Therefore, the L^2-norm of a function $f \in L^2$ is given by

$$\|f\| := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2\,\mathrm{d}\theta\right)^{1/2}.$$

For each integer n let $e_n(\mathrm{e}^{\mathrm{i}\theta}) := \mathrm{e}^{\mathrm{i}n\theta}$ be regarded as a function on $\mathbb{T}$. It is well-known that $\{e_n\}_{n\in\mathbb{Z}}$ forms an (monomial) orthonormal basis for L^2. Now, we consider the (closed) subspace H^2 of L^2 spanned by $\{e_n\}_{n=0}^{\infty}$, which consists of all

L^2-functions whose negative Fourier coefficients vanish:

$$H^2 := \{f \in L^2 \colon \langle f, e_n \rangle_{L^2} = 0 \text{ for } n < 0\},$$

that is, $f \in H^2$ if its Fourier series is of the form $\sum_{n=0}^{\infty} a_n \mathrm{e}^{in\theta}$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

On the subspace spanned by $\{e_0, e_1, \dots, e_n\}$, i.e. $\bigvee_{i=0}^{n} e_i$, we introduce the unitary operator J_n given by

$$J_n e_i := e_{n-i} \quad \text{for } 0 \leqslant i \leqslant n.$$

The operator J_n can then be extended to H^2 by defining it to be zero on the orthogonal complement of $\bigvee_{i=0}^{n} e_i$, and we will denote the resulting operator on H^2 by J_n as well.

For $\phi \in L^\infty(\mathbb{T})$, the (classical) *Toeplitz operator* T_ϕ and the (classical) *Hankel operator* H_ϕ with *symbol* ϕ are defined as

$$\begin{aligned} T_\phi &\colon H^2 \to H^2, \quad f \mapsto P(\phi f), \\ H_\phi &\colon H^2 \to L^2 \ominus H^2, \quad f \mapsto (I - P)(\phi f), \end{aligned}$$

where P and I stand for the orthogonal projection from L^2 onto H^2 and the identity operator, respectively. It is also known that Toeplitz and Hankel operators can be characterized by the *unilateral forward shift* operator U, defined on the monomial basis as $Ue_n = e_{n+1}$, and its Hilbert adjoint operator U^* (called *unilateral backward shift*), in an operator equation as

$$\begin{aligned} T \text{ is a Toeplitz operator on } H^2 &\Leftrightarrow \overbrace{U^*TU = T}^{\text{Toeplitz equation}}, \quad \text{see [2]}, \\ H \text{ is a Hankel operator on } H^2 &\Leftrightarrow \underbrace{U^*H = HU}_{\text{Hankel equation}}, \quad \text{see [4]}. \end{aligned}$$

Now, to relate the operator J_n to U and U^*, we introduce another operator, P_n on H^2, which is the orthogonal projection onto $\bigvee_{i=0}^{n} e_i$ and is defined by

$$(P_n f)(\zeta) := \sum_{i=0}^{n} a_i \zeta^i,$$

where $\sum_{i=0}^{\infty} a_i e_i$ is the Fourier expansion of f.

Some straightforward consequences can be easily concluded:

(a) $P_n \xrightarrow{\text{SOT}} I$ (i.e. P_n converges to I in the Strong Operator Topology (SOT) as $n \to \infty$).

(b) $J_n = P_n J_n = J_n P_n$ and $J_n J_n = P_n$ for $n = 0, 1, 2, \ldots$
(c) $U^n U^{*n} = I - P_{n-1}$ for all $n \geqslant 1$.

The notion of asymptotic Toeplitzness was first introduced by Barría and Halmos in [1], as a natural (asymptotic) generalization of Toeplitzness on $H^2 = H^2(\mathbb{T})$. According to them, an operator[1] $T \in \mathscr{B}(H^2)$ is (*strongly*) *asymptotically Toeplitz* if the *Toeplitz sequence of* T, given by

$$(\mathcal{T}_n(T))_{n\in\mathbb{N}} := (U^{*n} T U^n)_{n\in\mathbb{N}},$$

converges in the strong operator topology (SOT).

In 1989, Feintuch [3] extended their definition considering other usual topologies on $\mathscr{B}(H^2)$. Thus, we have three flavors of asymptotic Toeplitzness: uniform, strong and weak, and each of the first two implies the next. More precisely, an operator $T \in \mathscr{B}(H^2)$ is called *uniformly asymptotically Toeplitz*, *strongly asymptotically Toeplitz*, and *weakly asymptotically Toeplitz* if its Toeplitz sequence is convergent in the uniform operator topology (UOT), the strong operator topology (SOT), and the weak operator topology (WOT), respectively. For each of them the limit-operator of the sequence $(\mathcal{T}_n(T))_{n\in\mathbb{N}}$, which we generally denote by $\mathcal{T}(T)$, is a Toeplitz operator, whose symbol is called the *asymptotic symbol* of T and is usually denoted by $\sigma(T)$.

It is worth mentioning that the class of uniformly asymptotically Toeplitz operators forms a (uniformly closed) subspace of $\mathscr{B}(H^2)$, and it contains both Toeplitz and compact operators. Hence, any compact perturbation of a Toeplitz operator belongs to the set of uniformly asymptotically Toeplitz operators. But surprisingly, Feintuch proved that these are the only ones [3], Theorem 4.1:

Theorem 1.1 (Feintuch's characterization of uniform asymptotic Toeplitzness)**.** *A bounded operator on H^2 is uniformly asymptotically Toeplitz if and only if it is a compact perturbation of a Toeplitz operator.*

Hence, if the difference of a bounded operator on H^2 from any Toeplitz operator is not a compact operator, then it does not respect uniform asymptotical Toeplitzness.

Following Barría and Halmos in [1], Feintuch introduced in [3] the notion of asymptotic Hankelness. According to him, an operator $T \in \mathscr{B}(H^2)$ is *asymptotically Hankel* if the *Hankel sequence of* T, given by

$$(\mathcal{H}_n(T))_{n\in\mathbb{N}} := (J_n T U^{n+1})_{n\in\mathbb{N}}.$$

It is not immediately obvious that the limit-operator of sequence $(\mathcal{H}_n(T))_{n\in\mathbb{N}}$ is a Hankel operator. The interested reader is referred to Feintuch's work, where he

[1] Here, $\mathscr{B}(H)$ stands for the C*-algebra of all bounded operators on a Hilbert space H.

showed, for example, that the limit-operator of Hankel sequence of an operator in WOT is a Hankel operator [3], Lemma 3.1.

For the rest of this paper, we introduce the following subspaces of $\mathscr{B}(H^2)$:

$$\begin{aligned}
\mathfrak{H}_S &:= \{T \in \mathscr{B}(H^2)\colon\ (\mathcal{H}_n(T))_{n\in\mathbb{N}} \text{ is convergent in SOT}\},\\
\mathfrak{H}_S^0 &:= \{T \in \mathscr{B}(H^2)\colon\ \mathcal{H}_n(T) \xrightarrow{\text{SOT}} 0 \text{ as } n \to \infty\},\\
\mathfrak{T}_S &:= \{T \in \mathscr{B}(H^2)\colon\ (\mathcal{T}_n(T))_{n\in\mathbb{N}} \text{ is convergent in SOT}\},\\
\mathfrak{T}_S^0 &:= \{T \in \mathscr{B}(H^2)\colon\ \mathcal{T}_n(T) \xrightarrow{\text{SOT}} 0 \text{ as } n \to \infty\},
\end{aligned}$$

where we shall call each element in $\mathfrak{H}_S^0$ and $\mathfrak{T}_S^0$ a *vanishing Hankel operator* and *vanishing Toeplitz operator*, respectively.

Remark 1.2. Here we list some basic and main results of elements in $\mathfrak{T}_S$ and $\mathfrak{T}_S^0$:

(1) $\mathfrak{T}_S^0$ contains all compact operators. Indeed, for a compact operator K we have

$$KU^n e_j \xrightarrow{\text{SOT}} 0 \qquad (\text{as } n \to \infty \text{ and for } j = 0, 1, 2, \ldots).$$

(2) $\mathfrak{T}_S^0$ also contains all Hankel operators, since $U^{*n} \xrightarrow{\text{SOT}} 0$, for a Hankel operator H we have

$$\mathcal{T}_n(H)e_j = (U^{*n}HU^n)e_j = (U^{*2n}H)e_j \xrightarrow{\text{SOT}} 0 \ \ (\text{as } n \to \infty \text{ and for } j = 0, 1, 2, \ldots).$$

(3) $\mathfrak{T}_S$ contains the norm-closed algebra generated by all Toeplitz operators and all Hankel operators together [1], Theorem 4.

(4) Obviously, any Toeplitz operator is in $\mathfrak{T}_S$. But those elements in the norm-closed algebra generated by all Toeplitz operators belong to $\mathfrak{T}_S^0$ which are in the commutator ideal[2] of the same algebra [1], Theorem 7.

2. Main results

The following statement shows that the set of strongly asymptotically Toeplitz operators is topologically well-behaved:

[2] The *commutator ideal* of a C*-algebra $\mathcal{A}$ is the closed ideal generated by the commutators $ab - ba$ for all $a, b \in \mathcal{A}$.

Proposition 2.1. *$\mathfrak{T}_S$ is uniformly closed in $\mathscr{B}(H^2)$.*

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a uniformly convergent sequence in $\mathfrak{T}_S$ to an operator $T \in \mathscr{B}(H^2)$. We shall show that $T \in \mathfrak{T}_S$. Since $\|\mathcal{T}_n(T)\| \leqslant \|T\|$ for each n it suffices to show that for $f \in H^2$, the sequence $(\mathcal{T}_n(T)f)_{n\in\mathbb{N}}$ is a Cauchy sequence. So fix $f \in H^2 \setminus \{0\}$, $\varepsilon > 0$, and choose k so that $\|T - T_k\| < \varepsilon/4\|f\|$. Then

$$\begin{aligned}\|[\mathcal{T}_n(T) - \mathcal{T}_m(T)]f\| &= \|\mathcal{T}_n(T - T_k)f + [\mathcal{T}_n(T_k) - \mathcal{T}_m(T_k)]f + \mathcal{T}_m(T_k - T)f\| \\ &\leqslant \|\mathcal{T}_n(T - T_k)f\| + \|[\mathcal{T}_n(T_k) - \mathcal{T}_m(T_k)]f\| + \|\mathcal{T}_m(T_k - T)f\| \\ &\leqslant \|U^{*n}(T - T_k)U^n\|\|f\| + \|[\mathcal{T}_n(T_k) - \mathcal{T}_m(T_k)]f\| \\ &\quad + \|U^{*m}(T - T_k)U^m\|\|f\| \\ &\leqslant 2\|T - T_k\|\|f\| + \|[\mathcal{T}_n(T_k) - \mathcal{T}_m(T_k)]f\|.\end{aligned}$$

Since $T_k \in \mathfrak{T}_S$, there exists a natural number N such that for $n, m > N$,

$$\|[\mathcal{T}_n(T_k) - \mathcal{T}_m(T_k)]f\| < \varepsilon/2.$$

Thus,

$$\|[\mathcal{T}_n(T) - \mathcal{T}_m(T)]f\| < \varepsilon,$$

and the proof is complete. □

The next result, along with [3], Lemma 6.1, shows that vanishing Toeplitz and Hankel operators form a left ideal.

Proposition 2.2. *If $T \in \mathfrak{T}_S^0 \cap \mathfrak{H}_S^0$, then $AT \in \mathfrak{T}_S^0$ for any $A \in \mathscr{B}(H^2)$.*

Proof. For $n = 1, 2, \ldots$ we have

$$\begin{aligned}\mathcal{T}_n(AT) &= U^{*n}ATU^n \\ &= U^{*n}A(I - P_{n-1})TU^n + U^{*n}AP_{n-1}TU^n \\ &= (U^{*n}AU^n)(U^{*n}TU^n) + (U^{*n}AJ_{n-1})(J_{n-1}TU^n) \\ &= \mathcal{T}_n(A)\mathcal{T}_n(T) + (\mathcal{H}_{n-1}(A^*))^*\mathcal{H}_{n-1}(T).\end{aligned}$$

Since $\|\mathcal{T}_n(A)\|$ and $\|(\mathcal{H}_{n-1}(A^*))^*\|$ are uniformly bounded by $\|A\|$, we have the required result. □

The following result gives a sufficient condition for a vanishing Toeplitz operator to be a vanishing Hankel operator.

Proposition 2.3. *Let $T \in \mathfrak{T}_S^0$ be such that*

$$TU^n \xrightarrow{\text{SOT}} 0 \quad \text{as } n \to \infty.$$

Then $T \in \mathfrak{H}_S^0$.

P r o o f. For any $f \in H^2$ and $n = 1, 2, \ldots$ we have

$$\begin{aligned}\|TU^n(f)\|^2 &= \|P_{n-1}TU^n(f)\|^2 + \|(I - P_{n-1})TU^n(f)\|^2 \\ &= \|J_{n-1}TU^n(f)\|^2 + \|U^{*n}TU^n(f)\|^2 \\ &= \|(\mathcal{H}_{n-1}(T))f\|^2 + \|(\mathcal{T}_n(T))f\|^2.\end{aligned}$$

Thus, if $\mathcal{T}_n(T) \xrightarrow{\text{SOT}} 0$ and $TU^n \xrightarrow{\text{SOT}} 0$ as $n \to \infty$, it follows that $\mathcal{H}_n(T) \xrightarrow{\text{SOT}} 0$, i.e. $T \in \mathfrak{H}_S^0$. □

It is obvious that the product of two Toeplitz operators is not generally Toeplitz, however, such operators are in the norm-closed algebra generated by all Toeplitz operators and all Hankel operators together, whose limit operator may be obtained as a differentiation formula on this algebra.

Proposition 2.4. *If $\varphi, \phi \in L^\infty(\mathbb{T})$, then $\mathcal{T}(T_\varphi T_\phi) = T_\varphi T_\phi + H_{\overline{\varphi}^*} H_\phi$.*

P r o o f. For $n = 1, 2, 3 \ldots$,

$$\begin{aligned}\mathcal{T}_n(T_\varphi T_\phi) &= U^{*n} T_\varphi T_\phi U^n \\ &= U^{*n} T_\varphi (I - P_{n-1}) T_\phi U^n + U^{*n} T_\varphi P_{n-1} T_\phi U^n \\ &= U^{*n} T_\varphi U^n U^{*n} T_\phi U^n + (U^{*n} T_\varphi J_{n-1})(J_{n-1} T_\phi U^n) \\ &= T_\varphi T_\phi + (J_{n-1} T_{\overline{\varphi}} U^n)^* \mathcal{H}_{n-1}(T_\phi) \\ &= T_\varphi T_\phi + (\mathcal{H}_{n-1}(T_{\overline{\varphi}}))^* \mathcal{H}_{n-1}(T_\phi).\end{aligned}$$

Since for a Toeplitz operator T_f, $\mathcal{H}_n(T_f) = P_n H_f$, which is a finite rank, and $\mathcal{H}_n(T_f)$ converges strongly to H_f, we have

$$\begin{aligned}\mathcal{T}_n(T_\varphi T_\phi) &= T_\varphi T_\phi + (P_{n-1} H_{\overline{\varphi}})^* P_{n-1} H_\phi \\ &= T_\varphi T_\phi + H_{\overline{\varphi}^*}\ P_{n-1} H_\phi \xrightarrow{\text{SOT}} T_\varphi T_\phi + H_{\overline{\varphi}^*} H_\phi\ ,\end{aligned}$$

where $\overline{\varphi}^*(\mathrm{e}^{\mathrm{i}\theta}) = \varphi(\mathrm{e}^{-\mathrm{i}\theta})$. This gives the required formula. □

2.1. Distance formula. Now, we give a distance formula of asymptotically Toeplitz operators from the set of vanishing Toeplitz operators. But before stating it, let us bring the following simple fact to our attention.

Lemma 2.5. *If $T \in \mathfrak{T}_S$, then*

$$\|\mathcal{T}(T)\| \leqslant \liminf_{n \to \infty} \|\mathcal{T}_n(T)\| \leqslant \|T\|,$$

where $\mathcal{T}(T)$ stands for the limit Toeplitz operator of $(\mathcal{T}_n(T))_{n \in \mathbb{N}}$ in the strong operator topology.

P r o o f. Since $T \in \mathfrak{T}_S$, $\mathcal{T}_n(T) \xrightarrow{\text{SOT}} \mathcal{T}(T)$. Hence, for every $f \in H^2$ with $\|f\| = 1$ we have

$$\|(\mathcal{T}(T))f\| \leqslant \|(\mathcal{T}_n(T) - \mathcal{T}(T))f\| + \|\mathcal{T}_n(T)\|,$$

for $n = 0, 1, 2, \ldots$ Therefore

$$\|(\mathcal{T}(T))f\| \leqslant \liminf_{n\to\infty} \|\mathcal{T}_n(T)\|.$$

This in turn implies that

$$\|\mathcal{T}(T)\| \leqslant \liminf_{n\to\infty} \|\mathcal{T}_n(T)\|.$$

Since $\|\mathcal{T}_n(T)\| \leqslant \|T\|$, the proof is complete. □

Theorem 2.6. *For $T \in \mathfrak{T}_S$, $\operatorname{dist}(T, \mathfrak{T}_S^0) = \|\sigma(T)\|$, where $\mathcal{T}_n(T) \xrightarrow{\text{SOT}} T_{\sigma(T)}$.*

P r o o f. Since $T \in \mathfrak{T}_S$, the sequence $\mathcal{T}_n(T)$ converges strongly to a Toeplitz operator $T_{\sigma(T)}$ for some $\sigma(T) \in L^\infty(\mathbb{T})$, and thus $(T - T_{\sigma(T)}) \in \mathfrak{T}_S^0$. Therefore

$$\|\sigma(T)\| = \|T_{\sigma(T)}\| = \|T - (T - T_{\sigma(T)})\| \geqslant \operatorname{dist}(T, \mathfrak{T}_S^0).$$

For the opposite inequality, considering $L \in \mathfrak{T}_S^0$ along with Lemma 2.5, we have

$$\|\mathcal{T}(T)\| = \|\mathcal{T}(T - L)\| \leqslant \|T - L\|.$$

Recall that $\mathcal{T}(T)$, the limit-operator of the sequence $(\mathcal{T}_n(T))_{n\in\mathbb{N}}$, is simply $T_{\sigma(T)}$.

Thus,

$$\|\sigma(T)\| = \|\mathcal{T}(T)\| \leqslant \inf\{\|T - L\| \colon L \in \mathfrak{T}_S^0\} = \operatorname{dist}(T, \mathfrak{T}_S^0),$$

and the proof is complete. □

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