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ON THE ENDOMORPHISM RING AND COHEN-MACAULAYNESS
OF LOCAL COHOMOLOGY DEFINED BY A PAIR OF IDEALS

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Abstract. Let \mathfrak{a} , I , J be ideals of a Noetherian local ring (R, \mathfrak{m}, k) . Let M and N be finitely generated R -modules. We give a generalized version of the Duality Theorem for Cohen-Macaulay rings using local cohomology defined by a pair of ideals. We study the behavior of the endomorphism rings of $H_{I,J}^t(M)$ and $D(H_{I,J}^t(M))$, where t is the smallest integer such that the local cohomology with respect to a pair of ideals is nonzero and $D(-) := \text{Hom}_R(-, E_R(k))$ is the Matlis dual functor. We show that if R is a d -dimensional complete Cohen-Macaulay ring and $H_{I,J}^i(R) = 0$ for all $i \neq t$, the natural homomorphism $R \rightarrow \text{Hom}_R(H_{I,J}^t(K_R), H_{I,J}^t(K_R))$ is an isomorphism, where K_R denotes the canonical module of R . Also, we discuss the depth and Cohen-Macaulayness of the Matlis dual of the top local cohomology modules with respect to a pair of ideals.

Keywords: local cohomology; Matlis duality; endomorphism ring

MSC 2010: 13D45, 13C14

1. INTRODUCTION

Throughout this paper, (R, \mathfrak{m}, k) denotes a commutative Noetherian local ring with its residue field $k := R/\mathfrak{m}$. Let \mathfrak{a}, I, J be ideals of R , and let M and N be two finitely generated R -modules. For $i \in \mathbb{N}$, consider $H_{\mathfrak{a}}^i(M)$, the i th local cohomology module of M with respect to \mathfrak{a} (see [3], [10], [14]). This concept is an important tool in algebraic geometry and commutative algebra and has been studied by several authors.

Moreover, we will denote the Matlis dual functor $D(-) := \text{Hom}_R(-, E)$, where $E := E_R(k)$ is the injective hull of k . Let \widehat{R} be the \mathfrak{m} -adic completion of R . Since

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the module $D(M)$ admits a structure of an \widehat{R} -module, it is natural to ask the relations between $\text{Hom}_R(M, M)$ and $\text{Hom}_{\widehat{R}}(D(M), D(M))$, making the study of the endomorphism ring of the module M a topic of investigation.

Note that there are natural injective homomorphisms $M \rightarrow D(D(M))$ and

$$\text{Hom}_R(M, M) \rightarrow \text{Hom}_{\widehat{R}}(D(M), D(M))$$

such that the following diagram is commutative:

$$\begin{array}{ccc} R & \longrightarrow & \text{Hom}_R(M, M) \\ \downarrow & & \downarrow \\ \widehat{R} & \longrightarrow & \text{Hom}_{\widehat{R}}(D(M), D(M)). \end{array}$$

The investigation of endomorphism rings $\text{Hom}_R(H_I^i(M), H_I^i(M))$ was initially discussed in [13] for the case $\dim R = i$ and $I = \mathfrak{m}$. For certain ideals I and several $i \in \mathbb{N}$, see also [12], [15], [17], [23], [24] and [25].

Takahashi, Yoshino and Yoshizawa (see also [27]) introduced the notion of local cohomology module with respect to a pair of ideals. More precisely, consider $W(I, J) = \{\mathfrak{p} \in \text{Spec}(R) : I^n \subseteq \mathfrak{p} + J \text{ for some integer } n\}$ and $\widetilde{W}(I, J)$, the set of ideals \mathfrak{a} of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n . For an R -module M , the (I, J) -torsion of M is defined by

$$\Gamma_{I,J}(M) = \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

The functor $\Gamma_{I,J}(-)$ is left exact, additive and covariant, from the category of all R -modules, and is called the (I, J) -torsion functor. For an integer i , the i th right derived functor of $\Gamma_{I,J}(-)$ is denoted by $H_{I,J}^i(-)$ and will be called the i th local cohomology functor with respect to (I, J) . For an R -module M , $H_{I,J}^i(M)$ will be called the i th local cohomology module of M with respect to (I, J) while $\Gamma_{I,J}(M)$ will be referred to as the (I, J) -torsion part of M . Note that when $J = 0$ or J is a nilpotent ideal, $H_{I,J}^i(-)$ coincides with the ordinary local cohomology functor $H_I^i(-)$ with support in the closed subset $V(I)$.

Several authors have investigated the local cohomology modules with respect to a pair of ideals (see [1], [2], [5], [6], [8], [9], [18], [19], [28], [29]). The results on the behavior of endomorphism rings for local cohomology with respect to a pair of ideals were initially studied in [20] for rings, and no results are known for modules. In this sense, the main purpose of this paper is to provide a better understanding of local cohomology modules with respect to a pair of ideals by the endomorphism ring.

The organization of the paper is as follows. In Section 2, we show a generalized version of the Duality Theorem for Cohen-Macaulay rings. This is the main result of this paper (Theorem 2.3), and generalizes [27], Theorem 5.1, [17], Lemma 2.4 and [8], Theorem 6.4.

In Section 3, we will give an alternative characterization for the least integer i such that the local cohomology with respect to a pair of ideals is nonzero, denoted by $\text{depth}(I, J, M)$ (see [1]). Also, when $\text{depth}(I, J, M) = t$, we show that there is an isomorphism between the endomorphism rings $\text{Hom}_R(H_{I,J}^t(M), H_{I,J}^t(M))$ and $\text{Hom}_R(D(H_{I,J}^t(M)), D(H_{I,J}^t(M)))$, and we provide several sufficient conditions for the homomorphism

$$R \rightarrow \text{Hom}_R(D(H_{I,J}^t(M)), D(H_{I,J}^t(M)))$$

to be an isomorphism (see Theorem 3.4 and Corollary 3.5).

In Section 4, we define the truncation complex using the concept of local cohomology with respect to a pair of ideals. This concept will be useful for showing that if R is a d -dimensional complete Cohen-Macaulay ring and $H_{I,J}^i(R) = 0$ for all $i \neq t$, the natural homomorphism

$$R \rightarrow \text{Hom}_R(H_{I,J}^t(K_R), H_{I,J}^t(K_R))$$

is an isomorphism, where K_R is the canonical module of R (Theorem 4.3).

In the last section, we study the depth and Cohen-Macaulayness of the Matlis dual of $H_{I,J}^d(M)$, where M is a finitely generated R -module of dimension d .

2. THE GENERALIZED DUALITY THEOREM

In this section, we fix our notation and list several results for the convenience of the reader. It is well known that for an ideal \mathfrak{a} of R (not necessarily a local ring) with $\mathfrak{a}M \neq M$, the $\text{depth}(\mathfrak{a}, M)$ is the least integer i such that $H_{\mathfrak{a}}^i(M) \neq 0$ (see [3], Theorem 6.2.7). This motivates us to consider the following definition.

Definition 2.1 ([1], Proposition 2.3). Let I, J be ideals of R and let M be an R -module. The depth of (I, J) on M is defined by

$$\text{depth}(I, J, M) = \inf\{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$$

if this infimum exists, and ∞ otherwise.

Note that if $J \neq R$, by [27], Theorem 4.3 and the previous definition we have that $H_{I,J}^i(M) \neq 0$ for all i such that

$$\text{depth}(I, J, M) \leq i \leq \dim M/JM.$$

Furthermore, by [27], Theorem 4.1 the concept of depth defined by a pair of ideals can be reformulated using the definition of the usual depth, i.e.,

$$\text{depth}(I, J, M) = \inf\{\text{depth}(M_{\mathfrak{p}}) : \mathfrak{p} \in W(I, J)\}.$$

Now, we recall a technical result showed in [26], Theorem 3.4.14.

Lemma 2.2. *Let M, N be R -modules (not necessarily finitely generated). For all $i \in \mathbb{Z}$, there are isomorphisms*

- (a) $D(\text{Tor}_i^R(N, M)) \cong \text{Ext}_R^i(N, D(M))$,
- (b) $\text{Tor}_i^R(N, D(M)) \cong D(\text{Ext}_R^i(N, M))$, provided N is finitely generated.

Also, recall that if R is a d -dimensional Cohen-Macaulay local ring with a canonical module K_R , it is well known that there are isomorphisms

$$H_{\mathfrak{m}}^i(M) = D(\text{Ext}_R^{d-i}(M, K_R))$$

for $0 \leq i \leq d$, and $H_{\mathfrak{m}}^d(R) \cong D(K_R)$. This result is called the *Duality Theorem for Cohen-Macaulay rings* (see [8], Theorem 6.4, [17], Lemma 2.4 and [27], Theorem 5.1).

Due to these comments, the next result extends the previous theorem.

Theorem 2.3 (Generalized Duality Theorem). *Assume that $H_{I,J}^i(R) = 0$ for all $i \neq t = \text{depth}(I, J, R)$. Then for any R -module M and $i \in \mathbb{Z}$, there are isomorphisms:*

- (a) $\text{Tor}_{t-i}^R(M, H_{I,J}^t(R)) \cong H_{I,J}^i(M)$,
- (b) $D(H_{I,J}^i(M)) \cong \text{Ext}_R^{t-i}(M, D(H_{I,J}^t(R)))$.

Proof. Note that if statement (a) is true, by Lemma 2.2 it follows that

$$D(H_{I,J}^i(M)) \cong D(\text{Tor}_{t-i}^R(M, H_{I,J}^t(R))) \cong \text{Ext}_R^{t-i}(M, D(H_{I,J}^t(R))),$$

and we obtain the statement (b).

So, it is sufficient to prove statement (a). Consider the families of functors $\{H_{I,J}^i(-) : i \geq 0\}$ and $\{\text{Tor}_{t-i}^R(-, H_{I,J}^t(R)) : i \geq 0\}$. We want to show that there is an isomorphism between these two families. Since $H_{I,J}^i(R) = 0$ for all $i \neq t = \text{depth}(I, J, R)$, we obtain that $H_{I,J}^t(N) \cong H_{I,J}^t(R) \otimes_R N$ (see [27], Lemma 4.8). So, if $i = t$ we have that

$$\text{Tor}_0^R(N, H_{I,J}^t(R)) \cong H_{I,J}^t(N).$$

Since both of the families are cohomological sequences of functors, by [3], Theorem 1.3.5 it is enough to show that for any projective R -module M ,

$$H_{I,J}^i(M) = 0 = \operatorname{Tor}_{t-i}^R(M, H_{I,J}^t(R))$$

for all $i \neq t$. If $M = R$, the statement is clear. Since any projective R -module over a local ring is isomorphic to a direct sum of copies of R and the functors $\operatorname{Tor}_{t-i}^R(-, H_{I,J}^t(R))$ and $H_{I,J}^i(-)$ commute with direct sums, the desired conclusion follows. \square

For the next result, recall the definition of (I, J) -Cohen-Macaulay modules introduced in [1], Definition 3.5.

Definition 2.4. An R -module M is called (I, J) -Cohen-Macaulay if either $M = 0$ or $\operatorname{depth}(I, J, M) = \dim M/JM$.

Note that if $I = \mathfrak{m}$ and $J = 0$, the definition of (I, J) -Cohen-Macaulay module coincides with the usual definition of the Cohen-Macaulay module.

The next corollary is also obtained in [8], Theorem 6.5 by a different technique.

Corollary 2.5. Suppose that R is an (I, J) -Cohen-Macaulay ring, $I + J$ is an \mathfrak{m} -primary ideal and $\dim R/J = t$. Then for any R -module M and $i \in \mathbb{Z}$, we have the following isomorphisms:

- (a) $\operatorname{Tor}_{t-i}^R(M, H_{I,J}^t(R)) \cong H_{I,J}^i(M)$,
- (b) $D(H_{I,J}^i(M)) \cong \operatorname{Ext}_R^{t-i}(M, D(H_{I,J}^t(R)))$.

Proof. The result follows by [1], Corollary 3.13 and Theorem 2.3. \square

The next result shows the relation between the J -adic completion of $H_{\mathfrak{m},J}^t(R)$ and the dual of certain modules. This result generalizes [27], Theorem 5.4.

Corollary 2.6. Suppose that $I + J$ is an \mathfrak{m} -primary ideal and $H_{I,J}^i(R) = 0$ for all $i \neq t = \operatorname{depth}(I, J, R)$. Then there is a natural isomorphism

$$\varprojlim \frac{H_{I,J}^t(R)}{J^s H_{I,J}^t(R)} \cong D(\Gamma_J(D(H_{\mathfrak{m}}^t(R)))).$$

In particular, if R is a complete Cohen-Macaulay local ring, we obtain that

$$\varprojlim \frac{H_{I,J}^t(R)}{J^s H_{I,J}^t(R)} \cong D(\Gamma_J(K_R)),$$

where K_R is the canonical module of R .

Proof. First, since $I+J$ is an \mathfrak{m} -primary ideal we have that $H_{I,J}^t(R) \cong H_{\mathfrak{m},J}^t(R)$, see [27], Proposition 1.4, equations (6) and (7). We consider the isomorphisms

$$\begin{aligned} \frac{H_{\mathfrak{m},J}^t(R)}{J^s H_{\mathfrak{m},J}^t(R)} &\cong H_{\mathfrak{m},J}^t(R) \otimes_R R/J^s \\ &\cong H_{\mathfrak{m},J}^t(R/J^s) \quad (\text{by [27], Lemma 4.8}) \\ &\cong H_{\mathfrak{m}}^t(R/J^s) \quad (\text{by [27], Corollary 2.5}) \\ &\cong D(\text{Hom}_R(R/J^s, D(H_{\mathfrak{m}}^t(R)))) \quad (\text{by Theorem 2.3}). \end{aligned}$$

Applying the inverse limit we obtain that

$$\varprojlim \frac{H_{\mathfrak{m},J}^t(R)}{J^s H_{\mathfrak{m},J}^t(R)} \cong \varprojlim D(\text{Hom}_R(R/J^s, D(H_{\mathfrak{m}}^t(R)))).$$

Since

$$\varprojlim D(\text{Hom}_R(R/J^s, D(H_{\mathfrak{m}}^t(R)))) \cong D(\Gamma_J(D(H_{\mathfrak{m}}^t(R)))),$$

we complete the proof by combining the isomorphisms. \square

As a particular case of the previous result, if R is a d -dimensional complete Cohen-Macaulay local ring with canonical module K_R , and J is an ideal of R such that $\dim R/J = d - r$, then we have that

$$\varprojlim \frac{H_{I,J}^{d-r}(R)}{J^s H_{I,J}^{d-r}(R)} \cong D(H_J^r(K_R)).$$

The previous isomorphism was obtained by Takahashi et al. in [27], Theorem 5.4. Furthermore, in the same paper, the authors have shown that

$$\varprojlim \frac{H_{I,J}^{d-i}(R)}{J^s H_{I,J}^{d-i}(R)} \not\cong D(H_J^i(K_R))$$

for all integers i (see [27], Remark 5.5).

3. ENDOMORPHISM RINGS FOR A PAIR OF IDEALS

In this section, we start the investigation of the endomorphism ring of $H_{I,J}^t(M)$, where $t = \text{depth}(I, J, M)$. For this purpose, some preliminary results are useful.

Lemma 3.1. *Suppose that $\text{Supp}_R M \subseteq W(I, J)$. Then there are natural isomorphisms*

- (a) $\text{Hom}_R(M, \Gamma_{I,J}(N)) \cong \text{Hom}_R(M, N)$,
- (b) $M \otimes \text{Hom}_R(\Gamma_{I,J}(N), E) \cong M \otimes \text{Hom}_R(N, E)$.

Proof. Since $\text{Supp}_R M \subseteq W(I, J)$, we have that M is an (I, J) -torsion (see [27], Proposition 1.7). Moreover, $\Gamma_{I, J}(N) = \varinjlim_{\mathfrak{a} \in \widetilde{W}(I, J)} \Gamma_{\mathfrak{a}}(N)$ (see [27], Theorem 3.2). For statement (a), suppose initially that M is a finitely generated R -module. In this case, the proof of (a) follows from the isomorphisms

$$\text{Hom}_R(M, \Gamma_{I, J}(N)) \cong \varinjlim \text{Hom}_R(M, \Gamma_{\mathfrak{a}}(N)) \cong \text{Hom}_R(M, N),$$

where the last isomorphism follows by [24], Lemma 2.2 (a).

Now, suppose that M is not finitely generated R -module. Then $M \cong \varinjlim M_{\alpha}$, where $\{M_{\alpha}\}_{\alpha}$ is a directed family of a finitely generated R -modules such that $M_{\alpha} \subseteq M$. So $\text{Supp}_R M_{\alpha} \subseteq W(I, J)$. We complete the proof of this item by the isomorphisms

$$\text{Hom}_R(M, \Gamma_{I, J}(N)) \cong \varprojlim \text{Hom}_R(M_{\alpha}, \Gamma_{\mathfrak{a}}(N)) \cong \text{Hom}_R(M, N),$$

where the last isomorphism follows by the previous case.

For statement (b), we will proceed analogously to item (a). If M is finitely generated, by [3], Lemma 10.2.16, we have the isomorphism

$$M \otimes_R \text{Hom}_R(\Gamma_{I, J}(N), E) \cong \text{Hom}_R(\text{Hom}_R(M, \Gamma_{I, J}(N)), E).$$

Thus, by item (a) we conclude the proof. Now, if M is not a finitely generated R -module, then $M \cong \varinjlim M_{\alpha}$ and so

$$\text{Hom}_R(M, \Gamma_{I, J}(N)) \cong \varprojlim \text{Hom}_R(M_{\alpha}, \Gamma_{I, J}(N)).$$

By item (a) we obtain the statement. □

The next result generalizes [17], Proposition 2.1, [24], Theorem 2.3 and [20], Theorem 2.4 (a).

Theorem 3.2. *Suppose that $\text{depth}(I, J, N) = t$ and $\text{Supp}_R M \subseteq W(I, J)$. Then*

- (a) $\text{Ext}_R^t(M, N) \cong \text{Hom}_R(M, H_{I, J}^t(N))$ and $\text{Ext}_R^i(M, N) = 0$ for all $i < t$,
- (b) $\text{Tor}_t^R(M, D(N)) \cong M \otimes_R D(H_{I, J}^t(N))$ and $\text{Tor}_i^R(M, D(N)) = 0$ for all $i < t$.

Proof. Let E_R° be a minimal injective resolution of the R -module N . We can describe each E_R^i as a direct sum of indecomposable injective modules, i.e.,

$$E_R^i = \bigoplus_{\mathfrak{p} \in \text{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, N)},$$

where $E_R(R/\mathfrak{p})$ denotes the injective hull of R/\mathfrak{p} , and $\mu_i(\mathfrak{p}, N)$ is the i th Bass number of N with respect to \mathfrak{p} .

Since $\text{depth}(I, J, N) = \inf\{\text{depth}(N_{\mathfrak{p}}) : \mathfrak{p} \in W(I, J)\}$ and $\text{depth}(N_{\mathfrak{p}}) = \inf\{i : \mu_i(\mathfrak{p}, N) \neq 0\}$, it follows that $\mu_i(\mathfrak{p}, N) = 0$ for all $i < t$ and $\mathfrak{p} \in W(I, J)$. So, we can conclude that $\text{Hom}_R(M, E_R^i) = 0$ for all $i < t$. Now, since $\text{Supp}_R M \subseteq W(I, J)$, by Lemma 3.1 we obtain the isomorphism of complexes

$$\text{Hom}_R(M, E_R^\circ) \cong \text{Hom}_R(M, \Gamma_{I,J}(E_R^\circ)).$$

By the exact sequence $0 \rightarrow H_{I,J}^t(N) \rightarrow \Gamma_{I,J}(E_R^\circ)^t \rightarrow \Gamma_{I,J}(E_R^\circ)^{t+1}$ and the previous isomorphism of complexes, we can consider the following commutative diagram with exact arrows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M, H_{I,J}^t(N)) & \longrightarrow & \text{Hom}_R(M, \Gamma_{I,J}(E_R^\circ)^t) & \longrightarrow & \text{Hom}_R(M, \Gamma_{I,J}(E_R^\circ)^{t+1}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}_R^t(M, N) & \longrightarrow & \text{Hom}_R(M, E_R^\circ)^t & \longrightarrow & \text{Hom}_R(M, E_R^\circ)^{t+1}. \end{array}$$

Therefore, the first vertical arrow is an isomorphism, because the last two vertical arrows are isomorphisms (Lemma 3.1). This completes the proof of statement (a).

For statement (b), applying the Matlis dual functor $D(-)$ to the above exact sequence we obtain

$$D(\Gamma_{I,J}(E_R^{t+1})) \rightarrow D(\Gamma_{I,J}(E_R^t)) \rightarrow D(H_{I,J}^t(N)) \rightarrow 0.$$

Note that, since $D(\Gamma_{I,J}(E_R^\circ))$ is a complex of flat R -modules, the previous sequence is the beginning of a flat resolution of $D(H_{I,J}^t(N))$. Using the fact that $M \cong \varinjlim M_\alpha$, where M_α are finitely generated R -submodules, by Lemma 3.1 we obtain the isomorphisms of complexes

$$M \otimes_R D(E_R^\circ) \cong M \otimes_R D(\Gamma_{I,J}(E_R^\circ)) \cong \varinjlim \text{Hom}_R(\text{Hom}_R(M_\alpha, \Gamma_{I,J}(E_R^\circ)), E).$$

By construction, since $\text{Hom}_R(M, E_R^i) = 0$ for all $i < t$, it follows that $H_i(M \otimes_R D(E_R^\circ)) = 0$ for all $i < t$. Hence $\text{Tor}_i^R(M, D(N)) = 0$ for all $i < t$, since $D(E_R^\circ)$ is a flat resolution of $D(N)$. Therefore, by the commutative diagram with exact rows

$$\begin{array}{ccccccc} M \otimes_R D(E_R^{t+1}) & \longrightarrow & M \otimes_R D(E_R^t) & \longrightarrow & \text{Tor}_t^R(M, D(N)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ M \otimes_R D(\Gamma_{I,J}(E_R^{t+1})) & \longrightarrow & M \otimes_R D(\Gamma_{I,J}(E_R^t)) & \longrightarrow & M \otimes_R D(H_{I,J}^t(N)) & \longrightarrow & 0 \end{array}$$

the proof of statement (b) is complete, because the two vertical arrows on the left are isomorphisms and so, $\text{Tor}_t^R(M, D(N)) \cong M \otimes_R D(H_{I,J}^t(N))$. \square

Note that in the proof of the previous result, we are using a “truncation argument” that will be extended in Section 4. As a consequence, we show a characterization of depth of a module defined by a pair of ideals.

Corollary 3.3. *Let N be an R -module. Then*

$$\text{depth}(I, J, N) = \inf\{i \in \mathbb{Z} : \text{Tor}_i^R(R/\mathfrak{a}, D(N)) \neq 0 \text{ and } \mathfrak{a} \in \widetilde{W}(I, J)\}.$$

Proof. By [1], Proposition 2.3,

$$\text{depth}(I, J, N) = \inf\{\text{depth}(\mathfrak{a}, N) : \mathfrak{a} \in \widetilde{W}(I, J)\}.$$

Then, applying the version of Theorem 3.2 or [17], Corollary 2.3, we can conclude that

$$\text{depth}(\mathfrak{a}, N) = \inf\{i \in \mathbb{Z} : \text{Tor}_i^R(R/\mathfrak{a}, D(N)) \neq 0\}.$$

This completes the proof of the statement. □

The next result shows the close relation between the endomorphism rings of $H_{I,J}^t(M)$ and $D(H_{I,J}^t(M))$. This result and the next corollary generalize [24], Theorem 1.1 and [17], Lemma 3.2, respectively.

Theorem 3.4. *Let R be a complete local ring and M an R -module. There is a natural isomorphism*

$$\text{Hom}_R(H_{I,J}^t(M), H_{I,J}^t(M)) \cong \text{Hom}_R(D(H_{I,J}^t(M)), D(H_{I,J}^t(M))),$$

where $\text{depth}(I, J, M) = t$.

Proof. First, by Lemma 2.2 consider the isomorphism

$$\text{Hom}_R(D(H_{I,J}^t(M)), D(H_{I,J}^t(M))) \cong D(H_{I,J}^t(M) \otimes_R D(H_{I,J}^t(M))).$$

Since $H_{I,J}^t(M)$ is an (I, J) -torsion R -module [27], Corollary 1.13, Equation (5) and Proposition 1.7, by Theorem 3.2 (b) we obtain that

$$\begin{aligned} D(H_{I,J}^t(M) \otimes_R D(H_{I,J}^t(M))) &\cong D(\text{Tor}_t^R(H_{I,J}^t(M), D(M))) \\ &\cong \text{Ext}_R^t(H_{I,J}^t(M), D(D(M))). \end{aligned}$$

By the Matlis Duality Theorem (see [3], Theorem 10.2.12), it follows that $D(D(M)) \cong M$. Applying again Theorem 3.2 case (a) we have that

$$\text{Ext}_R^t(H_{I,J}^t(M), D(D(M))) \cong \text{Hom}_R(H_{I,J}^t(M), H_{I,J}^t(M))$$

and so the desired conclusion follows. □

Corollary 3.5. *Consider the same hypothesis as in Theorem 3.4. Then the following conditions are equivalent.*

(a) *The natural homomorphism*

$$R \rightarrow \text{Hom}_R(H_{I,J}^t(M), H_{I,J}^t(M))$$

is an isomorphism.

(b) *The natural homomorphism*

$$R \rightarrow \text{Hom}_R(D(H_{I,J}^t(M)), D(H_{I,J}^t(M)))$$

is an isomorphism.

(c) *The natural homomorphism*

$$H_{I,J}^t(M) \otimes_R D(H_{I,J}^t(M)) \rightarrow E$$

is an isomorphism.

Proof. The result follows by the previous theorem, Matlis duality, Lemma 2.2 (a) and the fact that $D(E) \cong R$. □

It is interesting to ask when the natural homomorphisms considered in the previous corollary are isomorphisms. This question motivates the results in the next section.

4. THE TRUNCATION COMPLEX FOR A PAIR OF IDEALS

The concept of the truncation complex was introduced in [11], Section 2 when R is a d -dimensional Gorenstein ring, and using a different approach in [17], [22] and [20]. We will generalize this concept using the local cohomology defined by a pair of ideals. This definition is the key ingredient for the most important result of this section (Theorem 4.3).

Let R be a d -dimensional local ring and let $M \neq 0$ be an R -module. Consider $\text{depth}(I, J, M) = t$ and $\dim M/JM = n$. Now, we recall a construction made in Theorem 3.2.

Let $E_R^\circ(M)$ be a minimal injective resolution of an R -module M . It is well known that we can describe the modules $E_R^i(M)$ as a direct sum of indecomposable injective modules, more precisely,

$$E_R^i(M) = \bigoplus_{\mathfrak{p} \in \text{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)},$$

where $E_R(R/\mathfrak{p})$ denotes the injective hull of R/\mathfrak{p} and $\mu_i(\mathfrak{p}, N)$ is the i th Bass number of M with respect to \mathfrak{p} . By [27], Proposition 1.11, it follows that

$$\Gamma_{I,J}(E_R^i(M)) = \bigoplus_{\mathfrak{p} \in W(I,J)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}.$$

Since $\text{depth}(I, J, M) = \inf\{\text{depth}(M_{\mathfrak{p}}) : \mathfrak{p} \in W(I, J)\}$ and $\text{depth}(M_{\mathfrak{p}}) = \inf\{i : \mu_i(\mathfrak{p}, M) \neq 0\}$, we obtain that $\mu_i(\mathfrak{p}, M) = 0$ for all $i < t$ and $\mathfrak{p} \in W(I, J)$. Then for all $i < t$ we have that $\Gamma_{I,J}(E_R^i(M)) = 0$.

Therefore $H_{I,J}^t(M)$ is isomorphic to the kernel of the map

$$\Gamma_{I,J}(E_R^t(M)) \rightarrow \Gamma_{I,J}(E_R^{t+1}(M))$$

and so, there is an embedding of complexes of R -modules

$$H_{I,J}^t(M)[-t] \rightarrow \Gamma_{I,J}(E_R^{\circ}(M)).$$

Definition 4.1. We call the cokernel of the above embedding, denoted by $C_M^{\circ}(I, J)$, the truncation complex with respect to the pair of ideals (I, J) . Thus, we can consider the short exact sequence of complexes of R -modules

$$0 \rightarrow H_{I,J}^t(M)[-t] \rightarrow \Gamma_{I,J}(E_R^{\circ}(M)) \rightarrow C_M^{\circ}(I, J) \rightarrow 0.$$

Note that $H^i(C_M^{\circ}(I, J)) = 0$ for all $i < t$. Furthermore, if $i > n$ (by [27], Theorem 4.3 with $J \neq R$) and $i > \dim R/J$ we also have that $H^i(C_M^{\circ}(I, J)) = 0$.

Lemma 4.2. *Let R be a d -dimensional complete local ring, and let M be an R -module such that $\text{depth}(I, J, M) = t$ and $H_{I,J}^i(M) = 0$ for all $i \neq t$. Then for all integers $i \neq c$:*

- (a) *There are isomorphisms*
 - (i) $\text{Ext}_R^{i-t}(H_{I,J}^t(M), H_{I,J}^t(M)) \cong \text{Ext}_R^i(H_{I,J}^t(M), M)$.
 - (ii) $\text{Tor}_{i-t}^R(H_{I,J}^t(M), D(H_{I,J}^t(M))) \cong \text{Tor}_i^R(H_{I,J}^t(M), D(M))$.
- (b) *The following conditions are equivalent*
 - (i) $\text{Ext}_R^{i-t}(H_{I,J}^t(M), H_{I,J}^t(M)) = 0$.
 - (ii) $\text{Ext}_R^{i-t}(D(H_{I,J}^t(M)), D(H_{I,J}^t(M))) = 0$.
 - (iii) $\text{Tor}_{i-t}^R(H_{I,J}^t(M), D(H_{I,J}^t(M))) = 0$.

Proof. Let $E_R^{\circ}(M)$ be a minimal injective resolution of the R -module M . Note that the complex $\Gamma_{I,J}(E_R^{\circ}(M))$ is a minimal injective resolution of $H_{I,J}^t(M)[-t]$,

because $H_{I,J}^i(M) = 0$ for all $i \neq t$. By [27], Corollary 1.13 and Proposition 1.7, we have that $\text{Supp}_R(H_{I,J}^t(M)) \subseteq W(I, J)$. So, by Lemma 3.1,

$$\text{Ext}_R^{i-t}(H_{I,J}^t(M), H_{I,J}^t(M)) \cong \text{Ext}_R^i(H_{I,J}^t(M), M)$$

for all integer i . Thus, we obtain the first statement of (a). For the second claim, first note that the complexes $D(E_R^\circ(M))$ and $D(\Gamma_{I,J}(E_R^\circ(M)))$ are flat resolutions of $D(M)$ and $D(H_{I,J}^t(M)[t])$, respectively.

Since $\text{Supp}_R(H_{I,J}^t(M)) \subseteq W(I, J)$, by Lemma 3.1 we obtain the isomorphism

$$H_{I,J}^t(M) \otimes_R D(E_R^\circ(M)) \cong H_{I,J}^t(M) \otimes_R D(\Gamma_{I,J}(E_R^\circ(M))),$$

which induces for all integers i the following isomorphisms on homology:

$$\text{Tor}_{i-t}^R(H_{I,J}^t(M), D(H_{I,J}^t(M))) \cong \text{Tor}_i^R(H_{I,J}^t(M), D(M)).$$

This completes the proof of case (a).

Now, by Lemma 2.2 there are isomorphisms

$$(4.1) \quad \text{Ext}_R^{i-t}(D(H_{I,J}^t(M)), D(H_{I,J}^t(M))) \cong D(\text{Tor}_{i-t}^R(D(H_{I,J}^t(M)), H_{I,J}^t(M)))$$

and

$$(4.2) \quad D(\text{Tor}_i^R(H_{I,J}^t(M), D(M))) \cong \text{Ext}_R^i(H_{I,J}^t(M), M)$$

for all integer i . Therefore, by case (a) and the isomorphisms (4.1) and (4.2), we obtain the vanishing desired in case (b). \square

The next theorem improves [17], Theorem 3.4 and is the main result of this section.

Theorem 4.3. *Let R be a d -dimensional complete Cohen Macaulay local ring. If $\text{depth}(I, J, R) = t$ and $H_{I,J}^i(R) = 0$ for all $i \neq t$, then:*

(a) *The natural homomorphism*

$$R \rightarrow \text{Hom}_R(H_{I,J}^t(K_R), H_{I,J}^t(K_R))$$

is an isomorphism and for all $i \neq t$

$$\text{Ext}_R^{i-t}(H_{I,J}^t(K_R), H_{I,J}^t(K_R)) = 0.$$

(b) *The natural homomorphism*

$$R \rightarrow \text{Hom}_R(D(H_{I,J}^t(K_R)), D(H_{I,J}^t(K_R)))$$

is an isomorphism and for all $i \neq t$

$$\text{Ext}_R^{i-t}(D(H_{I,J}^t(K_R)), D(H_{I,J}^t(K_R))) = 0.$$

(c) *The natural homomorphism*

$$H_{I,J}^t(K_R) \otimes_R D(H_{I,J}^t(K_R)) \rightarrow E$$

is an isomorphism and for all $i \neq c$

$$\text{Tor}_{i-t}^R(H_{I,J}^t(K_R), D(H_{I,J}^t(K_R))) = 0.$$

Proof. First, we will show the item (a). By [1], Definition 2.1 we have that

$$\text{depth}(I, J, K_R) = \inf\{\text{depth}(\mathfrak{a}, K_R) : \mathfrak{a} \in \widetilde{W}(I, J)\}.$$

By hypothesis, R has a canonical module K_R . Note that for all $\mathfrak{a} \in \widetilde{W}(I, J)$, by [4], Theorem 2.1.2

$$\text{depth}(\mathfrak{a}, K_R) = \dim_R(K_R) - \dim_R(K_R/\mathfrak{a}K_R),$$

and so $t = \text{depth}(I, J, K_R)$, because $\text{depth}(\mathfrak{a}, K_R) = \text{depth}(\mathfrak{a}, R)$.

Let $E_R^\circ(K_R)$ be a minimal injective resolution of K_R . Applying the functor $\text{Hom}_R(-, E_R^\circ(K_R))$ to the short exact sequence associated to the truncation complex of K_R with respect to the pair of ideals (I, J) , we obtain an exact sequence

$$(4.3) \quad 0 \rightarrow \text{Hom}_R(C_{K_R}^\circ(I, J), E_R^\circ(K_R)) \rightarrow \text{Hom}_R(\Gamma_{I,J}(E_R^\circ(K_R)), E_R^\circ(K_R)) \\ \rightarrow \text{Hom}_R(H_{I,J}^t(K_R), E_R^\circ(K_R))[t] \rightarrow 0.$$

Note that for each $\mathfrak{a} \in \widetilde{W}(I, J)$ and due to the definition of the functor $\Gamma_{\mathfrak{a}}(-)$ we have the isomorphisms

$$\text{Hom}_R(\Gamma_{\mathfrak{a}}(E_R^\circ(K_R)), E_R^\circ(K_R)) \cong \varprojlim \text{Hom}_R(\text{Hom}_R(R/\mathfrak{a}^r, E_R^\circ(K_R)), E_R^\circ(K_R)) \\ \cong \varprojlim (R/\mathfrak{a}^r \otimes_R Y).$$

where $Y := \text{Hom}_R(E_R^\circ(K_R), E_R^\circ(K_R))$.

In the previous isomorphisms, note that R/\mathfrak{a}^r is a finitely generated R -module for all $r \geq 1$. Since R is Cohen-Macaulay one has that the minimal injective resolution $E_R^\circ(K_R)$ of K_R is bounded and isomorphic to a dualizing complex for R . Therefore the natural chain map

$$R \rightarrow \text{Hom}_R(E_R^\circ(K_R), E_R^\circ(K_R))$$

is a quasi-isomorphism. We claim that Y is a flat resolution of R . From the definition of Hom of complexes, we can conclude that

$$Y^j \cong \prod_{i \in \mathbb{Z}} \text{Hom}_R(E_R^i(K_R), E_R^{i+j}(K_R)),$$

and so, Y^j is a flat R -module for all $j \in \mathbb{Z}$. Since R is a d -dimensional Cohen-Macaulay local ring, $H_{\mathfrak{m}}^i(R) = 0$ for all $i \neq d$. Hence $H^j(Y) \cong \text{Ext}_R^j(K_R, K_R) = 0$ for all $j \neq 0$, and $H^0(Y) = \text{Hom}_R(K_R, K_R) \cong R$ by Theorem 2.3 (in the case $I = \mathfrak{m}$ and $J = 0$). Moreover, if $\mathfrak{p} \in \text{Spec}(R)$ we have that

$$E_R^i(K_R) \cong \bigoplus_{\text{height } \mathfrak{p}=i} E_R(R/\mathfrak{p}).$$

Since R is Cohen Macaulay, K_R has finite injective dimension. So, $Y^j = 0$ for all $j > 0$. Hence, we can conclude that Y is a flat resolution of R . Now, we can see that the i th cohomology module of the complex

$$\varprojlim (R/\mathfrak{a}^r \otimes_R \text{Hom}_R(E_R^\circ(K_R), E_R^\circ(K_R)))$$

is zero for all $i \neq 0$, and R for $i = 0$ because R is complete. Thus, we conclude that $\text{Hom}_R(\Gamma_{\mathfrak{a}}(E_R^\circ(K_R)), E_R^\circ(K_R)) \cong R$.

Now, applying $\varprojlim_{\mathfrak{a} \in \widetilde{W}(I, J)}$ in the previous isomorphism and using [27], Theorem 3.2, we obtain that

$$\text{Hom}_R(\Gamma_{I, J}(E_R^\circ(K_R)), E_R^\circ(K_R)) \cong R.$$

Furthermore, $H_{I, J}^i(K_R) = 0$ for all $i \neq t$. Thus the complex

$$\text{Hom}_R(C_{K_R}^\circ(I, J), E_R^\circ(K_R))$$

is an exact complex. Considering the long exact sequence of the cohomology of the short exact sequence (4.3), we obtain the isomorphism

$$R \xrightarrow{\cong} \text{Ext}_R^t(H_{I, J}^t(K_R), K_R)$$

and for all $i \neq t$,

$$\text{Ext}_R^i(H_{I, J}^t(K_R), K_R) = 0.$$

By Theorem 3.2 and Lemma 4.2 statement (a), we have the proof of statement (a). The proof of statements (b) and (c) follows immediately from statement (a), Lemma 4.2, Theorem 3.4 and Corollary 3.5. \square

5. LOCAL COHOMOLOGY AND COHEN-MACAULAYNESS

In this section, we prove some results involving local cohomology with respect to a pair of ideals, Matlis duality, depth and Cohen-Macaulayness.

Remember that, if (R, \mathfrak{m}) is a complete local ring, the Matlis dual functor $D(-) = \text{Hom}_R(-, E(R/\mathfrak{m}))$ establishes a close relationship between Artinian and Noetherian R -modules (see [3], Theorem 10.2.12).

For local cohomology defined by a pair of ideals, Chu and Wang [6] have shown that, if (R, \mathfrak{m}) is a local ring and M is a finitely generated R -module of dimension d , then $H_{I,J}^d(M)$ is Artinian. These facts will be important for the next results.

The next result is a generalization of [16], Lemma 3.3.

Lemma 5.1. *Let R be a complete local ring, and let M be a nonzero finitely generated R -module of dimension d . If $d \leq 2$ and $H_{I,J}^d(M) \neq 0$, then $D(H_{I,J}^d(M))$ is a Cohen-Macaulay R -module of dimension d .*

Proof. By [5], Theorem 2.3 there exists a quotient module L of M satisfying $\text{Supp}_R L \subseteq V(J)$ and $\dim L = d$ such that $H_{I,J}^d(M) \cong H_I^d(L)$. Since $\text{Hom}_R(H_I^d(L), E(R/\mathfrak{m}))$ is a Cohen-Macaulay R -module of dimension d (see [16], Lemma 3.3) and

$$\text{Hom}_R(H_{I,J}^d(M), E(R/\mathfrak{m})) \cong \text{Hom}_R(H_I^d(L), E(R/\mathfrak{m})),$$

we have the desired conclusion. □

Now we examine the depth of Matlis dual of top local cohomology modules with respect to a pair of ideals. The next result improves [21], Proposition 2.4.

Theorem 5.2. *Let R be a complete local ring, and let M be a nonzero finitely generated R -module of dimension d . If $H_{I,J}^d(M) \neq 0$, then*

$$\text{depth}(D(H_{I,J}^d(M))) \geq \min\{2, d\}.$$

Proof. As in the previous lemma, there exists a quotient module L of M satisfying $\text{Supp}_R L \subseteq V(J)$ and $\dim L = d$ such that $H_{I,J}^d(M) \cong H_I^d(L)$ (see [5], Theorem 2.3). By [7], Theorem 2.3 and Lemma 3.2,

$$H_I^d(L) \cong H_{\mathfrak{m}}^d(L/Q_I(L)),$$

where $Q_I(L)$ is the intersection of all primary components of the zero submodule of L such that $\dim R/\mathfrak{p} = \dim L$ and $\dim R/I + \mathfrak{p} = 0$, and

$$\text{Ass}_R L/Q_i(L) = \{\mathfrak{p} \in \text{Ass}_R L : \dim R/\mathfrak{p} = d \text{ and } \dim R/I + \mathfrak{p} = 0\}.$$

We claim that $\text{depth}(H_{\mathfrak{m}}^d(L/Q_I(L))) \geq \min\{2, d\}$. We will proceed by induction on d . If $d \leq 2$ the result follows by Lemma 5.1. So, assume that $d \geq 3$.

Note that $\mathfrak{m} \notin \text{Ass}_R L/Q_i(L)$. So we may consider an element $x \in \mathfrak{m}$ which is a nonzero divisor on $L/Q_i(L)$, since $\text{depth}(L/Q_i(L)) > 0$. The short exact sequence

$$0 \rightarrow L/Q_i(L) \xrightarrow{x} L/Q_i(L) \rightarrow L/(x, Q_i(L)) \rightarrow 0$$

induces the long exact sequence

$$0 \rightarrow D(H_{\mathfrak{m}}^d(L/Q_I(L))) \xrightarrow{x} D(H_{\mathfrak{m}}^d(L/Q_I(L))) \rightarrow D(H_{\mathfrak{m}}^{d-1}(L/(x, Q_i(L)))) \rightarrow \dots$$

Now, from the induction hypothesis, $\text{depth}(H_{\mathfrak{m}}^{d-1}(L/(x, Q_i(L)))) \geq 2$, because $\dim L/(x, Q_i(L)) = d - 1$.

So there are elements $x_1, x_2 \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \text{Ass}_R(D(H_{\mathfrak{m}}^d(L/Q_I(L)))/x D(H_{\mathfrak{m}}^d(L/Q_I(L))))} \mathfrak{p}$. Therefore $\text{depth}(H_{I,J}^d(M)) = \text{depth}(H_{\mathfrak{m}}^d(L/Q_I(L))) \geq 2$, as desired. \square

Corollary 5.3. *With the same assumptions as in the previous result,*

$$\text{depth}(\text{Hom}_R(H_{I,J}^d(M), H_{I,J}^d(M))) \geq \min\{2, d\}.$$

Proof. Since $H_{I,J}^d(M)$ is Artinian and R is complete, by Lemma 2.2 case (a)

$$\begin{aligned} \text{Hom}_R(H_{I,J}^d(M), H_{I,J}^d(M)) &\cong \text{Hom}_R(H_{I,J}^d(M), D(D(H_{I,J}^d(M)))) \\ &\cong D(H_{I,J}^d(M) \otimes_R D(H_{I,J}^d(M))) \\ &\cong D(D(H_{I,J}^d(M)) \otimes_R H_{I,J}^d(M)) \\ &\cong \text{Hom}_R(D(H_{I,J}^d(M)), D(H_{I,J}^d(M))). \end{aligned}$$

Now by [4], Exercise 1.4.19 and Theorem 5.2 the proof is complete. \square

Proposition 5.4. *Let R be a complete local ring, and let M be a nonzero finitely generated R -module of dimension d . If $H_{I,J}^d(M) \neq 0$, then $\text{Hom}_R(H_{I,J}^d(M), H_{I,J}^d(M))$ is a commutative semi-local Noetherian ring.*

Proof. Proceeding as in the proof of Theorem 5.2, there is a quotient module L of M such that

$$H_{I,J}^d(M) \cong H_I^d(L) \cong H_{\mathfrak{m}}^d(L/Q_I(L)).$$

Since $D(H_{\mathfrak{m}}^d(L/Q_I(L))) \cong K_{L/Q_I(L)}$, where $K_{L/Q_I(L)}$ is the canonical module of $L/Q_I(L)$, by the isomorphism shown in the proof of Corollary 5.3 we obtain that

$$\begin{aligned} \text{Hom}_R(H_{I,J}^d(M), H_{I,J}^d(M)) &\cong \text{Hom}_R(H_{\mathfrak{m}}^d(L/Q_I(L)), H_{\mathfrak{m}}^d(L/Q_I(L))) \\ &\cong \text{Hom}_R(K_{L/Q_I(L)}, K_{L/Q_I(L)}). \end{aligned}$$

Now the result follows by [13], Remark 2.2 case (f). \square

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