

Andrea Dall'Aglio; Sergio Segura de León

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REGULARITY OF RENORMALIZED SOLUTIONS TO NONLINEAR  
ELLIPTIC EQUATIONS AWAY FROM THE SUPPORT  
OF MEASURE DATA

ANDREA DALL'AGLIO, Rome, SERGIO SEGURA DE LEÓN, València

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*Abstract.* We prove boundedness and continuity for solutions to the Dirichlet problem for the equation

$$-\operatorname{div}(a(x, \nabla u)) = h(x, u) + \mu, \quad \text{in } \Omega \subset \mathbb{R}^N,$$

where the left-hand side is a Leray-Lions operator from  $W_0^{1,p}(\Omega)$  into  $W^{-1,p'}(\Omega)$  with  $1 < p < N$ ,  $h(x, s)$  is a Carathéodory function which grows like  $|s|^{p-1}$  and  $\mu$  is a finite Radon measure. We prove that renormalized solutions, though not globally bounded, are Hölder-continuous far from the support of  $\mu$ .

*Keywords:* bounded solution;  $p$ -Laplacian; renormalized solution; measure data

*MSC 2010:* 35B45, 35B65, 35J15, 35J25, 35J60, 35J92

## 1. INTRODUCTION

In this note, we prove that solutions to the Dirichlet problem for nonlinear elliptic equations having measure datum are locally Hölder-continuous. For the sake of concreteness, consider the following simple linear problem:

$$(1.1) \quad \begin{cases} -\lambda \Delta u = f(x)(1+u) + \mu & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here  $\Omega \subset \mathbb{R}^N$  is an open and bounded set,  $f \in L^m(\Omega)$  (with  $m > N/2$ ) is small enough (in a sense to be determined) and  $\mu$  is a Radon measure having finite total

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variation and a support which does not include the whole domain. Our motivation to study this kind of problems comes from searching nonregular solutions to equations with a gradient term having “natural” growth which, by means of the Cole-Hopf change of unknown, are reduced to (1.1) (see Abdellaoui, Dall’Aglio and Peral [2] and Abdel Hamid and Bidaut-Veron [1]). In fact, we apply the results of this paper to obtain nonregular solutions to an equation involving the 1-Laplacian and a total variation term in Abdellaoui, Dall’Aglio and Segura de León, see [3].

Two features of problem (1.1) deserve a comment. The first point to note is that we have to restrict the size of  $f$  in order to get existence of a solution, even if  $\mu = 0$  and  $f(x) \equiv f_0$  is a constant. Indeed, taking  $u$  as a test function and applying Poincaré’s inequality, we may perform the following calculation:

$$\lambda \int_{\Omega} |\nabla u|^2 = f_0 \int_{\Omega} |u|^2 + f_0 \int_{\Omega} u \leq \frac{f_0}{\lambda_1} \int_{\Omega} |\nabla u|^2 + f_0 \int_{\Omega} u,$$

where  $\lambda_1$  is the first eigenvalue of the Laplacian. So, an estimate in the energy space is only possible if  $f_0 < \lambda\lambda_1$ . Observe that this bound on  $f_0$  depends on the coercivity of the principal part.

In order to deal with a general Radon measure  $\mu$ , we consider the notion of renormalized solution introduced by Dal Maso, Murat, Orsina and Prignet in [8]. We point out the related concept of entropy solution introduced by Benilan et al. in [4] for proving the existence and uniqueness of the result for  $L^1$ -data and extended by Boccardo, Gallouët and Orsina in [5] to measures which do not charge the sets of zero capacity. The existence of a renormalized solution to problem (1.1) under a smallness assumption on  $f$  has been proved in Grenon [10] and Abdel Hamid and Bidaut-Veron [1].

A classical result by Stampacchia (see [12]) shows that if  $\mu$  is actually a function belonging to  $L^m(\Omega)$  for some  $m > N/2$ , then the solution is bounded and continuous but if not, it is unbounded in general. Actually, the simple case where  $h(x, s) \equiv 0$  in (2.1) below shows that we cannot hope to prove global boundedness of solutions, but nevertheless we prove boundedness and continuity of the solution in a zone far away from the support of  $\mu$ . Heuristically, the idea is that the local boundedness of the solution only depends on the local summability of the datum  $\mu$ . In the special case where  $p = 2$  and  $f(x) = 0$ , a similar result was proved by Boccardo and Leonori in [6].

Our setting is more general than problem (1.1) and includes nonlinear operators of  $p$ -Laplacian type, so it is similar to the one studied by Grenon in [10]. The only change is our restriction on the growth of function  $h(x, s)$ ; we always assume the critical exponent  $p - 1$  because of our interest in equations which appear as a consequence of the Cole-Hopf transformation. In her paper, Grenon proves the

existence of a renormalized solution under a hypothesis of smallness of  $f$ . Our main result is that every renormalized solution is Hölder-continuous outside of the support of the measure  $\mu$ . We point out that we only analyze the case  $1 < p < N$  since for  $p > N$  every renormalized solution is actually a weak solution and so it is globally bounded and Hölder-continuous.

This paper is organized as follows. The next section is devoted to introducing our notation and precise hypotheses. Section 3 deals with the definition of renormalized solutions, while Section 4 contains the results on regularity.

## 2. PRELIMINARIES AND ASSUMPTIONS

We begin by introducing our notation. From now on,  $\Omega$  is an open bounded set in  $\mathbb{R}^N$  with  $N \geq 2$  and  $|E|$  denotes the Lebesgue measure of  $E \subset \Omega$ . Symbol  $L^q(\Omega)$  stands for the usual Lebesgue space and  $q'$  denotes the conjugate of  $q$ :  $q' = q/(q - 1)$ .

We will denote by  $W_0^{1,q}(\Omega)$  the usual Sobolev space of measurable functions having weak derivatives in  $L^q(\Omega)$  and zero trace on  $\partial\Omega$ . Finally, if  $1 \leq q < N$ , we will denote by  $q^* = Nq/(N - q)$  its Sobolev conjugate exponent.

Let us state our hypotheses more precisely. We will consider the following problem:

$$(2.1) \quad \begin{cases} -\operatorname{div}(a(x, \nabla u)) = h(x, u) + \mu & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Function

$$a(x, \xi): \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

satisfies the Carathéodory conditions and there exist constants  $\lambda > 0$  and  $\nu > 0$  such that

$$(2.2) \quad a(x, \xi) \cdot \xi \geq \lambda |\xi|^p,$$

$$(2.3) \quad |a(x, \xi)| \leq \nu |\xi|^{p-1},$$

$$(2.4) \quad (a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0$$

for all  $\xi, \eta \in \mathbb{R}^N$  with  $\xi \neq \eta$  and for almost all  $x \in \Omega$ .

Function

$$h(x, s): \Omega \times \mathbb{R} \rightarrow \mathbb{R}$$

also satisfies the Carathéodory conditions and there exists a nonnegative function  $f \in L^m(\Omega)$ , for some  $m > N/p$ , such that

$$(2.5) \quad |h(x, s)| \leq f(x)(1 + |s|^{p-1})$$

for all  $s \in \mathbb{R}$  and for almost all  $x \in \Omega$ . As far as the datum  $\mu$  is concerned, we assume that

$$(2.6) \quad \mu \text{ is a Radon measure with bounded total variation.}$$

Throughout this paper, we will use two auxiliary real functions: given  $k > 0$ , we define

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k; \\ k \frac{s}{|s|} & \text{if } |s| > k; \end{cases} \quad G_k(s) = s - T_k(s).$$

### 3. RENORMALIZED SOLUTIONS

In this Section, we define a renormalized solution to problem (2.1); we refer to [8] for a detailed study of renormalized solutions and several equivalent definitions.

**Definition 3.1.** Given the measure  $\mu$ , we decompose it as  $\mu = \mu_0 + \mu_s^+ - \mu_s^-$ , where  $\mu_0$  is absolutely continuous with respect to the  $p$ -capacity, while  $\mu_s^+$  and  $\mu_s^-$  are two nonnegative measures which are concentrated on two disjoint subsets of zero  $p$ -capacity.

A measurable function  $u: \Omega \rightarrow \mathbb{R}$  is a renormalized solution to problem (2.1) if the following conditions hold:

- (1) Function  $u$  is finite almost everywhere and  $T_k(u) \in W_0^{1,p}(\Omega)$  for all  $k > 0$ . (As a consequence, a generalized gradient  $\nabla u$  can be defined, see [4], Lemma 2.1.)
- (2) The gradient satisfies  $|\nabla u|^{p-1} \in L^q(\Omega)$  for every  $q < N/(N-1)$ .
- (3)  $|u|^{p-1} \in L^s(\Omega)$  for every  $s < N/(N-p)$ . In particular, by assumption (2.5), this implies that the function  $h(x, u)$  belongs to  $L^1(\Omega)$ .
- (4) For every  $S \in W^{1,\infty}(\mathbb{R})$  such that  $S'$  has compact support in  $\mathbb{R}$  (consequently  $S$  is constant for  $|s|$  large and so the limits  $S(+\infty) = \lim_{s \rightarrow +\infty} S(s)$  and  $S(-\infty) = \lim_{s \rightarrow -\infty} S(s)$  exist) we have

$$\begin{aligned} & \int_{\Omega} S'(u) \varphi a(x, \nabla u) \cdot \nabla u + \int_{\Omega} S(u) a(x, \nabla u) \cdot \nabla \varphi \\ &= \int_{\Omega} h(x, u) S(u) \varphi + \int_{\Omega} S(u) \varphi d\mu_0 + S(+\infty) \int_{\Omega} \varphi d\mu_s^+ - S(-\infty) \int_{\Omega} \varphi d\mu_s^- \end{aligned}$$

for all  $\varphi \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ , with  $r > N$ , such that  $S(u)\varphi \in W_0^{1,p}(\Omega)$ .

In [1] and [10] it was proved that under a smallness assumption on  $f$  there exists a renormalized solution to problem (2.1). In particular, the next theorem can be found in [1]:

**Theorem 3.2.** *Assume that hypotheses (2.2)–(2.6) are true and that*

$$\frac{1}{\lambda} < \lambda_1(f) := \inf \left\{ \frac{\int_{\Omega} |\nabla w|^p}{\int_{\Omega} f|w|^p} : w \in W_0^{1,p}(\Omega), \int_{\Omega} f|w|^p \neq 0 \right\}.$$

*Then there exists a renormalized solution  $u$  to problem (2.1).*

#### 4. REGULARITY AWAY FROM THE SUPPORT OF $\mu$

In the next results we aim to prove that a renormalized solution of problem (2.1) is “regular enough” (in particular, it is bounded) far from the set where the measure  $\mu$  is concentrated. We start by proving that it belongs to all  $L^q$  spaces.

We recall that a measure  $\mu$  is said to be concentrated on a set  $A$  if  $\mu(E) = \mu(E \cap A)$  for every measurable set  $E$ .

**Proposition 4.1.** *Assume that hypotheses (2.2)–(2.6) are true and that the measure  $\mu$  is concentrated on a set  $A \subset \Omega$ .*

*Then for every open set  $U \subset \Omega$  having positive distance from  $A$ , and for every  $q < \infty$ ,  $|u|^q \in W^{1,p}(U)$ . It follows that  $u \in L^q(U)$  for every  $q < \infty$ .*

In order to prove Proposition 4.1, we need an iteration lemma inspired in the Brezis-Kato approach (see [7]), which allows to improve the summability of a renormalized solution of problem (2.1):

**Lemma 4.2.** *Under the same hypotheses as in Proposition 4.1 assume that  $U$  and  $V$  are two open bounded sets in  $\mathbb{R}^N$  such that  $\overline{U} \subset V \subset \mathbb{R}^N \setminus A$ . Define*

$$M = \frac{N}{(N-p)m'} > 1$$

*and assume that there exist  $\theta > (p-1)/Mp$  and  $k \in \mathbb{N}$  such that*

$$(1 + |u|)^{\theta M^k} \in W^{1,p}(V \cap \Omega).$$

*Then*

$$(1 + |u|)^{\theta M^{k+1}} \in W^{1,p}(U \cap \Omega).$$

Proof. First of all, by standard inclusions between Lebesgue spaces, we can always assume that  $f$  satisfies the assumption

$$f \in L^m(\Omega) \quad \text{with} \quad \frac{N}{p} < m \leq \frac{N}{p-1}.$$

Let  $\varphi$  be a function in  $C_0^\infty(V)$  such that  $0 \leq \varphi \leq 1$  in  $V$ ,  $\varphi \equiv 1$  in  $\overline{U}$ . For  $\alpha > 0$  (to be chosen later) let us take  $w = \varphi^p[(1 + |T_L u|)^{\alpha(p-1)} - 1] \text{sign } u$  as a test function in the definition of renormalized solution. We obtain

$$(4.1) \quad \begin{aligned} & \alpha(p-1)\lambda \int_{V \cap \Omega} (1 + |T_L(u)|)^{\alpha(p-1)-1} |\nabla T_L(u)|^p \varphi^p \\ & \leq p\nu \int_{V \cap \Omega} \varphi^{p-1} |\nabla u|^{p-1} |\nabla \varphi| (1 + |T_L(u)|)^{\alpha(p-1)} \\ & \quad + \int_{V \cap \Omega} f(1 + |u|)^{p-1} (1 + |T_L(u)|)^{\alpha(p-1)} \varphi^p. \end{aligned}$$

Note that the measure  $\mu$  disappears due to the presence of  $\varphi$ . By the monotone convergence theorem it is easy to pass to the limit for  $L \rightarrow \infty$  in all the integrals in (4.1), thus we obtain

$$(4.2) \quad \begin{aligned} & \alpha(p-1)\lambda \int_{V \cap \Omega} (1 + |u|)^{\alpha(p-1)-1} |\nabla u|^p \varphi^p \\ & \leq p\nu \int_{V \cap \Omega} \varphi^{p-1} |\nabla u|^{p-1} |\nabla \varphi| (1 + |u|)^{\alpha(p-1)} \\ & \quad + \int_{V \cap \Omega} f(1 + |u|)^{(p-1)(\alpha+1)} \varphi^p. \end{aligned}$$

We only have to check that the last two integrals in (4.2) are finite. Let us start with the last one, which is finite if  $\int_{V \cap \Omega} f(1 + |u|)^{(p-1)(\alpha+1)}$  is finite. By the assumptions, we know that  $f \in L^m(\Omega)$  and, using Sobolev's inequality, that  $1 + |u| \in L^{p^* \theta M^k}(V \cap \Omega)$ . Therefore the integral is finite if we choose  $\alpha$  such that

$$\frac{1}{m} + \frac{(\alpha+1)(p-1)}{p^* \theta M^k} = 1,$$

that is,

$$(4.3) \quad \alpha = \frac{p}{p-1} \theta M^{k+1} - 1.$$

We point out that  $\alpha > 0$  due to our assumption

$$\theta > \frac{p-1}{Mp} > \frac{p-1}{M^{k+1}p}.$$

We now need to check that with this choice of  $\alpha$ , the second integral in (4.2) is finite, that is, we have to make sure that  $\int_{V \cap \Omega} |\nabla u|^{p-1} (1 + |u|)^{\alpha(p-1)} < \infty$ . Indeed, using Young's inequality,

$$\begin{aligned} \int_{V \cap \Omega} |\nabla u|^{p-1} (1 + |u|)^{\alpha(p-1)} &= \int_{V \cap \Omega} |\nabla u|^{p-1} (1 + |u|)^{p\theta M^{k+1} - p + 1} \\ &= \int_{V \cap \Omega} |\nabla u|^{p-1} (1 + |u|)^{(\theta M^k - 1)(p-1)} (1 + |u|)^{\theta M^k (pM - p + 1)} \\ &\leq \int_{V \cap \Omega} |\nabla u|^p (1 + |u|)^{(\theta M^k - 1)p} + \int_{V \cap \Omega} (1 + |u|)^{\theta p M^k (pM - p + 1)} \\ &= \frac{1}{(\theta M^k)^p} \int_{V \cap \Omega} |\nabla (1 + |u|)^{\theta M^k}|^p + \int_{V \cap \Omega} (1 + |u|)^{\theta p M^k (pM - p + 1)}. \end{aligned}$$

Using the assumption on  $u$ , Sobolev's inequality and the fact that

$$\theta p M^k (pM - p + 1) \leq \theta p^* M^k$$

due to the assumption  $m \leq N/(p-1)$ , we obtain that the last two integrals are finite. The Lemma is thus proved.  $\square$

**Proof of Proposition 4.1.** In order to apply Lemma 4.2, we need a startpoint, that is, we need to verify that the assumption of Lemma 4.2 is valid for  $k = 0$ . In other words, we need to show that there exists a number  $\theta > (p-1)/Mp$  such that  $(1 + |u|)^\theta$  is in  $W^{1,p}$  far from the support of  $\mu$ . To this aim, assume that  $V$  is an open set such that  $\overline{U} \subset V \subset \mathbb{R}^N \setminus A$ . Let us again consider a cut-off function  $\varphi$  which vanishes outside  $V$  and is 1 on  $U$ .

Multiplying the equation by  $\varphi^p [(1 + |T_L(u)|)^{\alpha(p-1)} - 1] \text{sign } u$  and letting  $L$  go to infinity, we obtain

$$\begin{aligned} (4.4) \quad \alpha(p-1)\lambda \int_{V \cap \Omega} (1 + |u|)^{\alpha(p-1)-1} |\nabla u|^p \varphi^p \\ \leq p\nu \int_{V \cap \Omega} \varphi^{p-1} |\nabla u|^{p-1} |\nabla \varphi| (1 + |u|)^{\alpha(p-1)} \\ + \int_{V \cap \Omega} f(1 + |u|)^{(\alpha+1)(p-1)} \varphi^p. \end{aligned}$$

We choose  $\alpha$  such that

$$(4.5) \quad 0 < \alpha < \frac{N}{m'(N-p)} - 1.$$

With this choice, the last integral in (4.4) is finite since  $u^{p-1} \in L^s(\Omega)$  for every  $s < N/(N-p)$ . As far as the second integral is concerned, by the definition of



renormalized solution we know that  $|\nabla u|^{p-1} \in L^r(\Omega)$  for every  $r < N/(N-1)$ . Therefore

$$\int_{V \cap \Omega} \varphi^{p-1} |\nabla u|^{p-1} |\nabla \varphi| (1 + |u|)^{\alpha(p-1)} < \infty$$

as long as

$$\frac{N-1}{N} + \frac{\alpha(N-p)}{N} < 1,$$

which corresponds to

$$\alpha < \frac{1}{N-p}.$$

It is easy to see that under the condition  $m \leq N/(p-1)$ , which can be assumed without loss of generality, one has

$$\frac{N}{m'(N-p)} - 1 \leq \frac{1}{N-p},$$

therefore all the integrals in (4.4) are finite for every  $\alpha$  as in (4.5). From there it easily follows that  $(1 + |u|)^\theta \in W^{1,p}(U)$  for every  $\theta$  such that

$$\theta < \frac{N(p-1)}{m'p(N-p)} = \frac{M(p-1)}{p}.$$

In order to apply Lemma 4.2 it is enough to choose  $\theta$  satisfying

$$\frac{p-1}{Mp} < \theta < \frac{M(p-1)}{p},$$

which is possible since  $M > 1$ . So, we have completed the proof of Proposition 4.1.  $\square$

In order to prove the following two results, we need to use some Caccioppoli estimate techniques. To obtain the estimates up to the boundary of  $\Omega$ , it is convenient to extend the renormalized solution  $u$  to be zero outside of  $\Omega$ . We therefore define

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega; \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

**Theorem 4.3.** *Assume that hypotheses (2.2)–(2.6) are true and that  $\mu$  is concentrated on a set  $A \subset \Omega$ . Let  $u$  be a renormalized solution of problem (2.1).*

*Then for every open set  $U \subset \Omega$  having positive distance from  $A$ ,*

$$u \in L^\infty(U).$$

*P r o o f.* Our aim is to see that for every  $x_0$  far from the support of  $\mu$  there exists a ball centered at  $x_0$  in which  $\tilde{u}$  is bounded. To this end, we get a Caccioppoli type inequality that allows us to deduce a  $L_{\text{loc}}^\infty$ -estimate far from the support of  $\mu$ .

From now on,  $B_\varrho \subset B_R$  stand for concentric open balls; we will always assume that  $B_R$  has positive distance from  $A$ . In addition, if  $u$  is the renormalized solution and  $k \geq 0$ , we will write

$$A(k, \varrho) = \{x \in B_\varrho : |\tilde{u}(x)| \geq k\}.$$

Fix  $R > 0$  such that  $|B_R|$  is small enough (it will be determined later). Let  $\varphi \in C_0^\infty(B_R)$  satisfy  $0 \leq \varphi \leq 1$ . Given  $k > 0$ , since we know that  $u \in W^{1,p}(B_R \cap \Omega)$ , we may choose  $T_L(G_k(u))\varphi^p$  as a test function in (2.1) and then let  $L$  go to  $\infty$ . We obtain

$$(4.6) \quad \lambda \int_{\Omega} \varphi^p |\nabla G_k(u)|^p \leq p\nu \int_{\Omega} \varphi^{p-1} |G_k(u)| |\nabla G_k(u)|^{p-1} |\nabla \varphi| + \int_{\Omega} f(1+|u|)^{p-1} |G_k(u)| \varphi^p.$$

We analyze the right-hand side of (4.6). By applying Young's inequality, we get

$$(4.7) \quad p\nu \int_{\Omega} \varphi^{p-1} |G_k(u)| |\nabla G_k(u)|^{p-1} |\nabla \varphi| \leq \varepsilon \int_{\Omega} \varphi^p |\nabla G_k(u)|^p + C(\varepsilon) \int_{\Omega} |G_k(u)|^p |\nabla \varphi|^p.$$

Observe that if we choose  $\varepsilon = \lambda/2$ , the first term can be absorbed by the left-hand side of (4.6).

The last term on the right-hand side of (4.6) is estimated as follows:

$$(4.8) \quad \int_{\Omega} f(1+|u|)^{p-1} |G_k(u)| \varphi^p \leq \int_{\Omega} f[2^{p-1}((1+k)^{p-1} + |G_k(u)|^{p-1})] |G_k(u)| \varphi^p \\ = 2^{p-1} \int_{\Omega} f(1+k)^{p-1} |G_k(u)| \varphi^p + 2^{p-1} \int_{\Omega} f |G_k(u)|^p \varphi^p \\ \leq 2^{p-1} \frac{p-1}{p} \int_{A(k,R)} f(1+k)^p \varphi^p + 2^{p-1} \left(1 + \frac{1}{p}\right) \int_{\Omega} f |G_k(u)|^p \varphi^p,$$

where in the last step we have applied Young's inequality. Hence, on account of (4.7) and (4.8), it follows from (4.6) that

$$(4.9) \quad \int_{\Omega} \varphi^p |\nabla G_k(u)|^p \leq C \int_{\Omega} |G_k(u)|^p |\nabla \varphi|^p + C \int_{A(k,R)} f(1+k)^p \varphi^p + C \int_{\Omega} f |G_k(u)|^p \varphi^p$$

for a constant  $C > 0$ . The next step is to estimate the last term on the right-hand side. We apply Hölder's and Sobolev's inequalities to deduce

$$\begin{aligned} \int_{\Omega} f|G_k(u)|^p \varphi^p &\leq \|f\|_m |A(k, R)|^{1/m' - p/p^*} \left[ \int_{\Omega} |G_k(u)|^{p^*} \varphi^{p^*} \right]^{p/p^*} \\ &\leq \|f\|_m |B_R|^{1/m' - p/p^*} S_{N,p} \int_{\Omega} |\nabla(G_k(u)\varphi)|^p \\ &= \|f\|_m |B_R|^{1/m' - p/p^*} S_{N,p} 2^{p-1} \left[ \int_{\Omega} \varphi^p |\nabla G_k(u)|^p + \int_{\Omega} |G_k(u)|^p |\nabla \varphi|^p \right]. \end{aligned}$$

Choosing  $R$  such that  $C\|f\|_m |B_R|^{1/m' - p/p^*} S_{N,p} 2^{p-1} \leq \frac{1}{2}$ , where  $C$  is the same constant occurring in (4.9), this first term can be absorbed by the left-hand side of (4.9). Thus, (4.9) becomes

$$(4.10) \quad \int_{\Omega} \varphi^p |\nabla G_k(u)|^p \leq C \int_{\Omega} |G_k(u)|^p |\nabla \varphi|^p + C \int_{A(k,R)} f(1+k)^p \varphi^p$$

for a different constant  $C > 0$ .

Now, take  $0 < \varrho < R$ , consider  $B_{\varrho}$  a ball centered at the same point as  $B_R$  and choose  $\varphi \in C_0^{\infty}(B_R)$  satisfying  $\varphi(x) = 1$  for all  $x \in B_{\varrho}$  and  $|\nabla \varphi| \leq 2/(R - \varrho)$ . Then we deduce from (4.10) that

$$(4.11) \quad \begin{aligned} \int_{B_{\varrho}} |\nabla G_k(\tilde{u})|^p &\leq \frac{C}{(R - \varrho)^p} \int_{B_R} |G_k(\tilde{u})|^p + C(1+k)^p \int_{A(k,R)} f \\ &\leq \frac{C}{(R - \varrho)^p} \int_{B_R} |G_k(\tilde{u})|^p + C(1+k)^p \|f\|_m |A(k, R)|^{1/m'}, \end{aligned}$$

which is the desired Caccioppoli type inequality. Since  $m > N/p$ , it yields

$$\frac{1}{m'} > 1 - \frac{p}{N},$$

so (4.11) is similar to the estimate found in [9], Theorem 7.1. Now we may follow the same arguments of [9], Chapter 7, to infer that  $\tilde{u} \in L^{\infty}(B_{R/2})$ .

Let  $U$  be an open set  $U \subset \Omega$  having positive distance from  $A$ . We have seen that for every  $x \in \bar{U}$  there exists  $r > 0$  (depending on  $x$ ) such that  $\tilde{u} \in L^{\infty}(B_r(x))$ . The compactness of  $\bar{U}$  implies the desired conclusion.  $\square$

**Remark 4.4.** If we assume  $h(x, s) = f(x)s^{p-1}$  (where  $f$  is a nonnegative function belonging to  $L^m(\Omega)$  for some  $m > N/p$ ) and  $\mu \geq 0$ , then we may apply [11], Theorem 2.4, to obtain a pointwise estimate of the solution. It is easy to check that this estimate is bounded far from the support of  $\mu$ . Therefore in this case we may deduce Theorem 4.3 above from [11], Theorem 2.4. The authors thank the referee for bringing [11] to their attention.

It is now straightforward to prove that the renormalized solution is actually continuous outside the support of  $\mu$ .

**Theorem 4.5.** *Assume that hypotheses (2.2)–(2.6) are true and that  $\mu$  is concentrated on a closed set  $A \subset \Omega$ . Let  $u$  be a renormalized solution of problem (2.1). Then  $u$  is Hölder-continuous in  $\overline{\Omega} \setminus A$ .*

**Proof.** Let  $B_\varrho \Subset B_R$  be a pair of concentric balls having positive distance from  $A$ . We have proved in Theorem 4.3 that  $u$  is bounded in  $\Omega \cap B_R$ . It follows that (4.11) can be rewritten as

$$\int_{B_\varrho} |\nabla G_k(\tilde{u})|^p \leq \frac{C}{(R-\varrho)^p} \int_{B_R} |G_k(\tilde{u})|^p + C(1 + \|\tilde{u}\|_{L^\infty(B_R)})^p \|f\|_m |A(k, R)|^{1/m'}$$

for all  $k < \|\tilde{u}\|_{L^\infty(B_R)}$ . Therefore it is possible to apply Theorem 7.6 of [9] to obtain that  $\tilde{u}$  is Hölder-continuous far from  $A$ .  $\square$

**Remark 4.6.** We point out that the smallness assumption on  $f$ , which is needed in the existence result, is not used in the proof of the summability/boundedness/continuity of the renormalized solution.

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*Authors' addresses:* Andrea Dall'Aglio, Dipartimento di Matematica “G. Castelnuovo”, Università di Roma “La Sapienza”, Piazzale A. Moro 2, I-00185 Roma, Italy, e-mail: [dallaglio@mat.uniroma1.it](mailto:dallaglio@mat.uniroma1.it); Sergio Segura de León, Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner 50, 46100 Burjassot, València, Spain, e-mail: [sergio.segura@uv.es](mailto:sergio.segura@uv.es).