Czechoslovak Mathematical Journal, 69 (144) (2019), 337–351

BOUNDEDNESS OF LITTLEWOOD-PALEY OPERATORS RELATIVE TO NON-ISOTROPIC DILATIONS

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Received June 28, 2017. Published online August 6, 2018.

Abstract. We consider Littlewood-Paley functions associated with a non-isotropic dilation group on $\mathbb{R}^n$. We prove that certain Littlewood-Paley functions defined by kernels with no regularity concerning smoothness are bounded on weighted L^p spaces, $1 < p < \infty$, with weights of the Muckenhoupt class. This, in particular, generalizes a result of N. Rivière (1971).

Keywords: Littlewood-Paley function; non-isotropic dilation

MSC 2010: 42B25, 46E30

1. Introduction

Let P be an $n \times n$ real matrix such that

$$\langle Px, x\rangle \geqslant \langle x, x\rangle \quad \forall\, x \in \mathbb{R}^n,$$

where $\langle x, y\rangle = x_1y_1 + \ldots + x_ny_n$ is the inner product in $\mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Define a dilation group $\{\delta_t\}_{t>0}$ on $\mathbb{R}^n$ by $\delta_t = t^P = \exp((\log t)P)$. Let

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \tag{1.1}$$

be the Littlewood-Paley function on $\mathbb{R}^n$, where $\psi_t(x) = t^{-\gamma}\psi(\delta_t^{-1}x)$ with γ being trace P and ψ a function in $L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \psi(x)\,\mathrm{d}x = 0. \tag{1.2}$$

The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 25400130, Japan Society for the Promotion of Science.

When $P = E$ (the identity matrix) and g_ψ is defined with $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ in (1.1), it is known that if we further assume that $|\psi(x)| \leqslant C(1+|x|)^{-n-\varepsilon}$ for some $\varepsilon > 0$, then

$$\|g_\psi(f)\|_p \leqslant C_p\|f\|_p, \quad 1 < p < \infty, \tag{1.3}$$

where $\|f\|_p = \|f\|_{L^p}$ (see [16] and [5], [8], [10], [11], [17] for related results; also [6] for a result on the homogeneous groups including the Heisenberg group). We refer to [1] for an earlier result, which requires in addition certain regularity on ψ to get (1.3). The reverse inequality of (1.3) also holds if a certain non-degeneracy condition on ψ is further assumed (see [13], Theorem 3.8, and also [19]).

In this note we shall prove, in particular, a result analogous to (1.3) in the case of the general dilation group $\{\delta_t\}$ (see Theorem 1.1 below). To establish the generalization, we need an analogue of Lemma 3 of [16] (Lemma 3.4). It is to be noted that to prove Lemma 3.4 similarly to Lemma 3 of [16] we have a difficulty in the general case, which does not occur even in the case where P is diagonal. As can be seen in the proof of Lemma 3.4, this will be overcome by a method different from that of [16].

It is known that $|\delta_t x|$ is strictly increasing as a function of t on $\mathbb{R}_+ = (0,\infty)$ for $x \neq 0$, where $|x| = \langle x, x\rangle^{1/2}$. Define a norm function $r(x)$, $x \neq 0$, to be the unique positive real number t such that $|\delta_{t^{-1}}x| = 1$, while let $r(0) = 0$. Then $r(\delta_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. Further, the following properties of $r(x)$ and δ_t are known (see [3], [4]):

(a) $r(x+y) \leqslant r(x) + r(y)$, $r(-x) = r(x)$;
(b) $r(x) \leqslant 1$ if and only if $|x| \leqslant 1$;
(c) if $|x| \leqslant 1$, $|x| \leqslant r(x)$;
(d) $|x| \geqslant r(x)$ for $|x| \geqslant 1$;
(e) if $t \geqslant 1$, $|\delta_t x| \geqslant t|x|$ for all $x \in \mathbb{R}^n$;
(f) $|\delta_t x| \leqslant t|x|$ for all $x \in \mathbb{R}^n$ if $0 < t \leqslant 1$.

Similarly, we can also consider a norm function $r^*(x)$ associated with the dilation group $\{\delta_t^*\}_{t>0}$, where δ_t^* denotes the adjoint of δ_t; we have properties of $r^*(x)$ and δ_t^* analogous to those of $r(x)$ and δ_t mentioned above. A polar coordinates expression for the Lebesgue measure

$$\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x = \int_0^\infty \int_{S^{n-1}} \varphi(\delta_t\theta)t^{\gamma-1}\mu(\theta)\,\mathrm{d}\sigma(\theta)\,\mathrm{d}t$$

will be used in the following, where μ is a strictly positive C^∞ function on the unit sphere $S^{n-1} = \{x\colon |x| = 1\}$ and $d\sigma$ is the Lebesgue surface measure on S^{n-1}. See also [14], [21] for relevant results.

We consider a pointwise majorant of ψ defined as

$$\mathscr{H}_\psi(x) = h(r(x)) = \sup_{r(y)\geqslant r(x)} |\psi(y)| \tag{1.4}$$

and recall two seminorms from [16]:

$$\mathscr{B}_\varepsilon(\psi) = \int_{|x|>1} |\psi(x)||x|^\varepsilon \,\mathrm{d}x \quad \text{for } \varepsilon > 0, \tag{1.5}$$

$$\mathscr{C}_u(\psi) = \left(\int_{|x|<1} |\psi(x)|^u \,\mathrm{d}x\right)^{1/u} \quad \text{for } u > 1. \tag{1.6}$$

Let

$$B(x,t) = \{y \in \mathbb{R}^n\colon\ r(x-y) < t\}$$

be the ball with respect to r in $\mathbb{R}^n$ with center x and radius t; such a ball is also called an r-ball. We say that a weight function w belongs to the weight class A_p, $1 < p < \infty$, of Muckenhoupt if the quantity

$$[w]_{A_p} = \sup_B \left(|B|^{-1}\int_B w(x)\,\mathrm{d}x\right)\left(|B|^{-1}\int_B w(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1}$$

is finite, where the supremum is taken over all r-balls B in $\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of B. Let $M(f)$ be the Hardy-Littlewood maximal function defined as

$$M(f)(x) = \sup_{x\in B} |B|^{-1}\int_B |f(y)|\,\mathrm{d}y,$$

where the supremum is taken over all r-balls B in $\mathbb{R}^n$ containing x. Then we define the class A_1 to be the family of weight functions w such that $M(w) \leqslant Cw$ almost everywhere. The infimum of all such C is defined to be $[w]_{A_1}$. We refer to [2], [12] for relevant results on A_p.

We denote by L^p_w the weighted L^p space of all functions f satisfying

$$\|f\|_{L^p_w} = \|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

In this note, we shall prove the following.

Theorem 1.1. *Let $\psi \in L^1(\mathbb{R}^n)$. Let $\mathscr{H}_\psi$, $\mathscr{B}_\varepsilon(\psi)$ and $\mathscr{C}_u(\psi)$ be as in* (1.4), (1.5) *and* (1.6), *respectively. We assume that ψ satisfies* (1.2) *and that*

(1) *there exists $\varepsilon > 0$ such that $\mathscr{B}_\varepsilon(\psi) < \infty$;*
(2) *there exists $u > 1$ such that $\mathscr{C}_u(\psi) < \infty$;*
(3) *$\mathscr{H}_\psi \in L^1(\mathbb{R}^n)$.*

Then g_ψ defined by (1.1) *is bounded on L^p_w for all $p \in (1,\infty)$ and $w \in A_p$.*

Special cases of Theorem 1.1 are treated in [14], Theorem (1.4), page 269. Theorem 1.1 follows from the next result.

Theorem 1.2. *Let $\psi \in L^1(\mathbb{R}^n)$. Suppose that ψ satisfies* (1.2) *and* (1), (2) *of Theorem* 1.1. *We also assume that there exist a non-negative, non-increasing function h on $\mathbb{R}_+$ with $h(r(x)) \in L^1(\mathbb{R}^n)$ and a non-negative function Ω in $L^q(S^{n-1})$ for some q, $2 \leqslant q \leqslant \infty$, such that*

$$|\psi(x)| \leqslant h(r(x))\Omega(x'), \quad x' = \delta_{r(x)^{-1}}x.$$

Then

(1) *g_ψ is bounded on L^p_w if $p > q'$ and $w \in A_{p/q'}$, where q' denotes the exponent conjugate to q;*
(2) *g_ψ is bounded on L^2_w and $L^2_{w^{-1}}$ if $w \in A_{2/q'}$.*

If we assume that ψ is compactly supported, then we have the following results.

Theorem 1.3. *Let $\psi \in L^1(\mathbb{R}^n)$ with* (1.2). *Suppose that ψ is compactly supported. Then we have the following.*
(1) *If $\psi \in L^q(\mathbb{R}^n)$ for some $q \geqslant 2$, then*
 (α) *g_ψ is bounded on L^p_w for $p > q'$, $w \in A_{p/q'}$;*
 (β) *g_ψ is bounded on L^2_w and $L^2_{w^{-1}}$ for $w \in A_{2/q'}$.*
(2) *If $\psi \in L^q(\mathbb{R}^n)$ for some $q \in (1, 2]$, then g_ψ is bounded on L^2_w and $L^2_{w^{-1}}$ if $w^{q'/2} \in A_1$.*

Corollary 1.4. *Let ψ be a function of compact support in $L^1(\mathbb{R}^n)$ with* (1.2). *If we further suppose that $\psi \in L^q(\mathbb{R}^n)$ for some $q \in (1, 2]$, then g_ψ is bounded on $L^p(\mathbb{R}^n)$ if $0 < 1/p < 1/2 + 1/q'$.*

When $P = E$, part (2) of Theorem 1.3 is due to [8]; so is Corollary 1.4 for $p \in (1, 2)$, where its optimality is also shown.

We denote by $\widehat{\psi}$ the Fourier transform defined as

$$\widehat{\psi}(\xi) = \mathscr{F}(\psi)(\xi) = \int_{\mathbb{R}^n} \psi(x) \mathrm{e}^{-2\pi\mathrm{i}\langle x,\xi\rangle} \, \mathrm{d}x.$$

Let $\mathbb{Z}$ denote the set of integers and $\mathscr{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$.

To prove the theorems above we apply the following result.

Proposition 1.5. *Let $v \in A_2$. Let $\psi \in L^1(\mathbb{R}^n)$ with* (1.2). *If we further assume that*

$$\int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 \, \mathrm{d}t \leqslant C \min(|\xi|^{\varepsilon}, |\xi|^{-\varepsilon}) \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} \tag{1.7}$$

with some $\varepsilon \in (0,1)$ and

$$\sup_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_1^2 |\psi_{t2^k} * f(x)|^2 \, \mathrm{d}t \, v(x) \, \mathrm{d}x \leqslant C_v \|f\|_{L_v^2}^2 \quad \forall f \in \mathscr{S}(\mathbb{R}^n), \tag{1.8}$$

then g_ψ is bounded on L_v^2.

Remark 1.6. It is known and also can be seen from our proof of Corollary 1.4 for $p \in (1,2]$ below that g_ψ is bounded on L^p for every p with $1 < p \leqslant 2$ if g_ψ is bounded on $L^2_{w^{-1}}$ for all $w \in A_1$. This can be applied to g_ψ of Theorem 1.2 and part (1) of Theorem 1.3.

Remark 1.7. Our results in this note may be stated also in terms of the non-isotropic dilations of [21] as in [18], which readers would infer easily.

In Section 2 we shall prove Proposition 1.5 by applying a well-known discrete parameter Littlewood-Paley decomposition with respect to the dilation group $\{\delta_t^*\}$. The method is analogous to the one in the isotropic case of [16]. We include the proof for the sake of completeness.

We shall prove Theorem 1.2 in Section 3. To prove part (2) we shall apply Proposition 1.5. Let

$$J_\varepsilon(\psi) = \sup_{|\xi|=1} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\psi(x)\psi(y)| |\langle \xi, P(x-y) \rangle|^{-\varepsilon} \, \mathrm{d}x \, \mathrm{d}y. \tag{1.9}$$

To use Proposition 1.5, we need to show $J_\varepsilon(\psi) < \infty$ for some $\varepsilon > 0$ under the assumptions of Theorem 1.2. A proof of this may be similar to the one for the case $P = E$, which is given in [16], if P is diagonal. However, in the general case, we need different methods that can be found in the proof of Lemma 3.4 below, as mentioned above. If $J_\varepsilon(\psi) < \infty$, we can deduce the decay estimate $\int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 \, \mathrm{d}t \leqslant C|\xi|^{-\varepsilon}$, $\varepsilon > 0$, from certain trigonometric integral estimates in [18] (Lemma 3.2). We shall also apply an observation of [8] concerning duality in proving part (2). Part (1) of Theorem 1.2 follows from part (2) by the extrapolation theorem of Rubio de Francia [15].

In Section 4 we shall prove Theorem 1.3 and Corollary 1.4. Part (1) of Theorem 1.3 will be shown along the lines of the proof of Theorem 1.2 by using the compactness of the support of ψ. We shall apply methods of Duoandikoetxea [8] in proving the weighted boundedness of part (2). To prove Corollary 1.4 for $p \in (1,2)$, we shall apply Theorem 1.3 (2) and methods of [8]. The proof for $p \in [2,\infty)$ will be given by adapting methods of [11], which is based on the use of vector valued inequalities.

2. Proof of Proposition 1.5

Let $f \in \mathscr{S}(\mathbb{R}^n)$. We take $\Phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp}(\Phi) \subset \{1/2 \leqslant r^*(\xi) \leqslant 2\}$ and

$$\sum_{j=-\infty}^{\infty} \Phi(\delta^*_{2^j}\xi) = 1 \quad \text{for } \xi \neq 0.$$

Let

$$\widehat{D_j(f)}(\xi) = \Phi(\delta^*_{2^j}\xi)\hat{f}(\xi), \quad j \in \mathbb{Z}.$$

We use the decomposition

$$f * \psi_t(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} D_{j+k}(f * \psi_t)(x)\chi_{[2^k,2^{k+1})}(t) = \sum_{j=-\infty}^{\infty} A_j(x,t),$$

where χ_E denotes the characteristic function of a set E and

$$A_j(x,t) = \sum_{k=-\infty}^{\infty} D_{j+k}(f * \psi_t)(x)\chi_{[2^k,2^{k+1})}(t).$$

We have

$$g_\psi(f)(x) \leqslant \sum_{j=-\infty}^{\infty} S_j(f)(x)$$

with

$$S_j(f)(x) = \left(\int_0^\infty |A_j(x,t)|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}.$$

Note that

$$\|S_j(f)\|_2^2 = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |D_{j+k}(f * \psi_t)(x)|^2 \frac{\mathrm{d}t}{t}\,\mathrm{d}x.$$

Thus, if we set $U_j = \{2^{-1-j} \leqslant r^*(\xi) \leqslant 2^{1-j}\}$, Plancherel's theorem and (1.7) imply

$$\begin{aligned}
\|S_j(f)\|_2^2 &\leqslant \sum_{k=-\infty}^{\infty} C\int_{U_{j+k}} \left(\int_{2^k}^{2^{k+1}} |\widehat{\psi}(\delta^*_t\xi)|^2 \frac{\mathrm{d}t}{t}\right)|\hat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\leqslant \sum_{k=-\infty}^{\infty} C\int_{U_{j+k}} \min(|\delta^*_{2^k}\xi|^{\varepsilon}, |\delta^*_{2^k}\xi|^{-\varepsilon})|\hat{f}(\xi)|^2\,\mathrm{d}\xi.
\end{aligned}$$

Applying the δ_t^* analogues of (e), (f) of Section 1, we have

$$\min(|\delta_{2^k}^*\xi|^{\varepsilon}, |\delta_{2^k}^*\xi|^{-\varepsilon}) \leqslant C2^{-\varepsilon|j|} \quad \text{for } \xi \in U_{j+k}.$$

Thus

$$\|S_j(f)\|_2^2 \leqslant C2^{-\varepsilon|j|} \sum_{k=-\infty}^{\infty} \int_{U_{j+k}} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \leqslant C2^{-\varepsilon|j|}\|f\|_2^2, \tag{2.1}$$

where the last inequality follows from the Plancherel theorem and the fact that $\sum \chi_{U_j}(\xi) \leqslant C$ with a constant C independent of $\xi \in \mathbb{R}^n$.

While, by (1.8) we see that

$$\begin{aligned}\|S_j(f)\|_{2,v}^2 &= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |D_{j+k}(f) * \psi_t(x)|^2 \frac{\mathrm{d}t}{t} v(x)\,\mathrm{d}x \\ &\leqslant \sum_{k=-\infty}^{\infty} C \int_{\mathbb{R}^n} |D_{j+k}(f)(x)|^2 v(x)\,\mathrm{d}x.\end{aligned}$$

Since $v \in A_2$, by the L_v^2 boundedness of the discrete Littlewood-Paley operator, we have

$$\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} |D_{j+k}(f)(x)|^2 v(x)\,\mathrm{d}x \leqslant C\|f\|_{2,v}^2.$$

Thus

$$\|S_j(f)\|_{2,v}^2 \leqslant C\|f\|_{2,v}^2. \tag{2.2}$$

Interpolation with change of measures between the estimates (2.1) and (2.2) implies that

$$\|S_j(f)\|_{2,v^\theta} \leqslant C2^{-\varepsilon(1-\theta)|j|/2}\|f\|_{2,v^\theta} \tag{2.3}$$

with $\theta \in (0,1)$. Choosing θ close to 1 such that $v^{1/\theta} \in A_2$, by (2.3) we have

$$\|S_j(f)\|_{2,v} \leqslant C2^{-\varepsilon(1-\theta)|j|/2}\|f\|_{2,v}.$$

From this and the triangle inequality it follows that

$$\|g_\psi(f)\|_{2,v} \leqslant \sum_{j=-\infty}^{\infty} \|S_j(f)\|_{2,v} \leqslant C\|f\|_{2,v}.$$

This completes the proof of Proposition 1.5.

3. Proof of Theorem 1.2

We first state some results needed to show Theorem 1.2 from Proposition 1.5.

Lemma 3.1. *Let $\psi \in L^1(\mathbb{R}^n)$. Suppose that ψ satisfies* (1.2) *and condition $\mathscr{B}_\varepsilon(\psi) < \infty$ for some $\varepsilon \in (0,1]$. Then*

$$\int_1^2 |\widehat{\psi}(\delta_t^*\xi)|^2 \,\mathrm{d}t \leqslant C|\xi|^{2\varepsilon} \quad \textit{for } \xi \in \mathbb{R}^n.$$

P r o o f. We have $|\widehat{\psi}(\xi)| \leqslant C|\xi|^\varepsilon$ for all $\xi \in \mathbb{R}^n$ (see [16] for a proof). By this and the δ_t^* analogue of (f) of Section 1, we have the estimates as claimed. □

Lemma 3.2. *Let L be the degree of the minimal polynomial of P. Then for $\eta, \zeta \in \mathbb{R}^n$ we have*

$$\left| \int_1^2 \exp(\mathrm{i}\langle \delta_t^*\eta, \zeta\rangle) \frac{\mathrm{d}t}{t} \right| \leqslant C|\langle \eta, P\zeta\rangle|^{-1/L}$$

for some positive constant C independent of η and ζ.

This is from [18]. Let $J_\varepsilon(\psi)$ be as in (1.9). Applying Lemma 3.2, we have the next result.

Lemma 3.3. *Let $\psi \in L^1(\mathbb{R}^n)$. Suppose that $J_\varepsilon(\psi) < \infty$ with $\varepsilon \in (0, 1/L]$. Then*

$$\int_1^2 |\widehat{\psi}(\delta_t^*\xi)|^2 \,\mathrm{d}t \leqslant C|\xi|^{-\varepsilon} \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}.$$

P r o o f. We write

$$\int_1^2 |\widehat{\psi}(\delta_t^*\xi)|^2 \,\mathrm{d}t = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \psi(x)\overline{\psi(y)} \int_1^2 \exp(-2\pi\mathrm{i}\langle \delta_t^*\xi, x-y\rangle) \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}y.$$

Then Lemma 3.2 implies that

$$\int_1^2 |\widehat{\psi}(\delta_t^*\xi)|^2 \,\mathrm{d}t \leqslant C \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\psi(x)\psi(y)| \min(1, |\langle \xi, P(x-y)\rangle|^{-1/L}) \,\mathrm{d}x \,\mathrm{d}y.$$

We easily see that the right-hand side is bounded by $CJ_\varepsilon(\psi)|\xi|^{-\varepsilon}$ for $\varepsilon \in (0, 1/L]$. This completes the proof. □

A sufficient condition for $J_\varepsilon(\psi) < \infty$ is given in the next result.

Lemma 3.4. *Let h be a non-negative function on $\mathbb{R}_+$ and set $H(x) = h(r(x))$. Suppose that $H \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let Ω be a non-negative function in $L^v(S^{n-1})$ for some $v > 1$. For a non-negative function F on $\mathbb{R}^n$ we assume that $\mathscr{C}_u(F) < \infty$ for some $u > 1$ and that*

$$F(x) \leqslant h(r(x))\Omega(x') \quad \text{for } |x| > 1.$$

Then we have $J_\varepsilon(F) < \infty$ for $\varepsilon < \min(1/u', 1/(2v'))$.

P r o o f. We define

$$L_\varepsilon(f, g; \xi) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x)g(y)|\langle \xi, P(x-y)\rangle|^{-\varepsilon} \,\mathrm{d}x \,\mathrm{d}y,$$

where f, g are non-negative functions and $\xi \in S^{n-1}$. We write $F = E + G$, where $E(x) = F(x)$ if $|x| < 1$ and $E(x) = 0$ otherwise. We note that

$$L_\varepsilon(F, F; \xi) = L_\varepsilon(E, E; \xi) + 2L_\varepsilon(E, G; \xi) + L_\varepsilon(G, G; \xi).$$

To prove the lemma it suffices to show that

$$\sup_{\xi \in S^{n-1}} L_\varepsilon(E, E; \xi) < \infty, \tag{3.1}$$

$$\sup_{\xi \in S^{n-1}} L_\varepsilon(E, G; \xi) < \infty, \tag{3.2}$$

$$\sup_{\xi \in S^{n-1}} L_\varepsilon(G, G; \xi) < \infty \tag{3.3}$$

when $\varepsilon < \min(1/u', 1/(2v'))$.

By Hölder inequality and rotation, we see that

$$L_\varepsilon(E, E; \xi) \leqslant C\mathscr{C}_u(F)^2 \biggl(\iint_{|x|<1, |y|<1} |x_1 - y_1|^{-\varepsilon u'} \,\mathrm{d}x \,\mathrm{d}y\biggr)^{1/u'} < \infty,$$

which proves (3.1).

Let $\zeta = P^*\xi$. Then if $Y = E$ or $Y = G$, we see that

$$L_\varepsilon(Y, G; \xi) \leqslant \int_{\mathbb{R}^n} Y(x)\biggl(\int_{\mathbb{R}^n} G(y)|\langle \zeta, x-y\rangle|^{-\varepsilon} \,\mathrm{d}y\biggr) \mathrm{d}x.$$

We prove

$$\sup_{x \in \mathbb{R}^n, \xi \in S^{n-1}} \int_{\mathbb{R}^n} G(y)|\langle \zeta, x-y\rangle|^{-\varepsilon} \,\mathrm{d}y < \infty \tag{3.4}$$

if $\varepsilon < 1/(2v')$, which will imply (3.2) and (3.3) since $Y \in L^1$.

If $n = 1$, we have

$$\int_{\mathbb{R}} G(y)|\langle\zeta, x-y\rangle|^{-\varepsilon}\,\mathrm{d}y \leqslant C\int_{|x-y|>1} G(y)|x-y|^{-\varepsilon}\,\mathrm{d}y + C\int_{|x-y|\leqslant 1} G(y)|x-y|^{-\varepsilon}\,\mathrm{d}y$$
$$\leqslant C\|H\|_1 + C\|H\|_\infty \int_{|y|\leqslant 1} |y|^{-\varepsilon}\,\mathrm{d}y,$$

which proves (3.4) when $n = 1$.

Let $n \geqslant 2$. For $x \in \mathbb{R}^n$, $s > 0$, let

$$I_\varepsilon(\zeta, x, s) = \int_{S^{n-1}} |\langle\zeta, x - \delta_s\omega\rangle|^{-\varepsilon}\Omega(\omega)\mu(\omega)\,\mathrm{d}\sigma(\omega).$$

Then by Hölder inequality

$$I_\varepsilon(\zeta, x, s) \leqslant C(N_{\varepsilon v'}(\zeta, x, s))^{1/v'}\|\Omega\|_v,$$

where

$$N_\varepsilon(\zeta, x, s) = \int_{S^{n-1}} |\langle\zeta, x - \delta_s\omega\rangle|^{-\varepsilon}\mu(\omega)\,\mathrm{d}\sigma(\omega).$$

Thus

$$\int_{\mathbb{R}^n} G(y)|\langle\zeta, x-y\rangle|^{-\varepsilon}\,\mathrm{d}y \leqslant \int_1^\infty h(s)s^{\gamma-1}I_\varepsilon(\zeta, x, s)\,\mathrm{d}s$$
$$\leqslant C\|\Omega\|_v \int_1^\infty h(s)s^{\gamma-1}(N_{\varepsilon v'}(\zeta, x, s))^{1/v'}\,\mathrm{d}s. \tag{3.5}$$

We shall see that

$$\sup_{x\in\mathbb{R}^n} N_\varepsilon(\zeta, x, s) \leqslant C|\delta_s^*\zeta|^{-\varepsilon} \tag{3.6}$$

if $0 < \varepsilon < 1/2$. Then (3.4) follows from (3.5) and (3.6). We can deduce (3.6) from Lemma 3.5 below. This completes the proof. □

The following result is used in the proof of Lemma 3.4.

Lemma 3.5. *Let $n \geqslant 2$ and $\eta \in S^{n-1}$. Suppose that $0 < \delta < 1/2$. Then*

$$\sup_{a\in\mathbb{R}} \int_{S^{n-1}} |a - \langle\eta, \omega\rangle|^{-\delta}\,\mathrm{d}\sigma(\omega) \leqslant C$$

with a constant C independent of η.

P r o o f. By rotation, we may assume that $\eta = (1, 0, \ldots, 0)$. Thus we have to estimate

$$I(a) = \int_{S^{n-1}} |a - \omega_1|^{-\delta} \,\mathrm{d}\sigma(\omega), \quad a \in \mathbb{R}.$$

We see that

$$I(a) = c_{n-2} \int_0^{\pi} |a - \cos\theta|^{-\delta} (\sin\theta)^{n-2} \,\mathrm{d}\theta \leqslant c_{n-2} \int_0^{\pi} |a - \cos\theta|^{-\delta} \,\mathrm{d}\theta,$$

where $c_{n-2} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$. Thus the conclusion follows from an elementary fact that

$$\sup_{a\in\mathbb{R}} \int_0^{\pi} |a - \cos\theta|^{-\delta} \,\mathrm{d}\theta < \infty \quad \text{if } 0 < \delta < 1/2.$$

□

Let

$$M_\Omega(f)(x) = \sup_{r>0} r^{-\gamma} \int_{r(y)<r} |f(x-y)| \Omega(\delta_{r(y)^{-1}} y) \,\mathrm{d}y,$$

where Ω is a non-negative function on S^{n-1}.

We also need the following result (see [7]) for proving Theorem 1.2.

Lemma 3.6. *Let $\Omega \in L^q(S^{n-1})$, $\Omega \geqslant 0$. Then M_Ω is bounded on L^2_w if $q \geqslant 2$ and $w \in A_{2/q'}$.*

P r o o f of Theorem 1.2. Applying the assumed pointwise majorization of ψ, we can prove that

$$\sup_{t>0} |\psi_t * f(x)| \leqslant C\, M_\Omega(f)(x)$$

as in [20], pages 63–64. Thus, Lemma 3.6 implies

$$\int_{\mathbb{R}^n} |\psi_t * f(x)|^2 v(x) \,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} |f(x)|^2 v(x) \,\mathrm{d}x \tag{3.7}$$

for $v = w$ with $w \in A_{2/q'}$ and also for $v = w^{-1}$ with $w \in A_{2/q'}$ by duality with a constant C independent of $t > 0$. From (3.7) we see that (1.8) holds for ψ of Theorem 1.2 for $v = w$ and $v = w^{-1}$ with $w \in A_{2/q'}$.

Also by Lemma 3.4 we have $J_\varepsilon(\psi) < \infty$ for $\varepsilon < \min(1/u', 1/(2q'))$. Here we note that the function h of Theorem 1.2 is bounded on $[1, \infty)$. Thus (1.7) follows from Lemmas 3.1 and 3.3.

Therefore Proposition 1.5 can be applied to get the boundedness of g_ψ on L^2_w and $L^2_{w^{-1}}$ with $w \in A_{2/q'}$, since $A_{2/q'} \subset A_2$. Thus part (2) of Theorem 1.2 follows. Also, by the boundedness of g_ψ on L^2_w, $w \in A_{2/q'}$, and the extrapolation theorem of Rubio de Francia [15] we have part (1). □

4. Proofs of Theorem 1.3 and Corollary 1.4

P r o o f of Theorem 1.3. To prove (α) and (β) of part (1) of Theorem 1.3, similarly to the proof of Theorem 1.2 in Section 3, it suffices to show that ψ satisfies conditions (1.7) and (1.8) for $v = w$, $v = w^{-1}$ with $w \in A_{2/q'}$.

To prove (1.7), we first note that $\mathscr{B}_1(\psi) < \infty$. Without loss of generality, we may assume that $\operatorname{supp}(\psi) \subset B(0,1)$ in what follows. By applying Lemma 3.4 we can see that $J_\varepsilon(\psi) < \infty$ if $\varepsilon < 1/q'$. Thus, by Lemmas 3.1 and 3.3 we have (1.7).

Next we deal with condition (1.8). Let $q > 2$. Then we have

$$\sup_{t>0} |\psi_t * f(x)| \leqslant C\, M(|f|^{q'})(x)^{1/q'}$$

by Hölder inequality and it implies (3.7) with $v = w$, $w \in A_{2/q'}$. To prove (3.7) when $q = 2$ and $v = w \in A_1$, we may assume that $t = 1$ by dilation invariance. Now applying Schwarz inequality, we have

$$|\psi * f(x)|^2 \leqslant \|\psi\|_2^2 \int_{r(y)<1} |f(x-y)|^2 \, \mathrm{d}y.$$

Therefore we see that

$$\int |\psi * f(x)|^2 w(x) \, \mathrm{d}x \leqslant \|\psi\|_2^2 \int |f(y)|^2 \left(\int_{r(x-y)<1} w(x) \, \mathrm{d}x \right) \mathrm{d}y.$$

Since $w \in A_1$, we have

$$\int_{r(x-y)<1} w(x) \, \mathrm{d}x \leqslant C[w]_{A_1} w(y) \quad \text{a.e.},$$

and hence

$$\int |\psi * f(x)|^2 w(x) \, \mathrm{d}x \leqslant C\|\psi\|_2^2 \int |f(y)|^2 w(y) \, \mathrm{d}y.$$

So (3.7) holds with $v = w$, $w \in A_{2/q'}$, $q \geqslant 2$. Inequality (3.7) for $v = w^{-1}$ follows by duality. Thus, we have (1.8) for $v = w$ and $v = w^{-1}$ with $w \in A_{2/q'}$, $q \geqslant 2$.

We next prove part (2). We apply ideas of [8]. To prove (1.8) for $v = w$ and $v = w^{-1}$ with $w^{q'/2} \in A_1$, it suffices to show estimate (3.7) for $v = w$ and $v = w^{-1}$ under the condition $w^{q'/2} \in A_1$ with a constant C independent of $t > 0$. If we have (3.7) for $v = w$, then the result for $v = w^{-1}$ follows by duality. To prove (3.7) for $v = w$ with $w^{q'/2} \in A_1$, we may assume that $t = 1$ by dilation invariance as above. Now, by applying Hölder inequality twice and the condition $w^{q'/2} \in A_1$, we have

$$
\begin{aligned}
\int |f*\psi(x)|^2 w(x)\,\mathrm{d}x &\leqslant \int \left(\|\psi\|_q^q \int |f(x-y)|^2 |\psi(y)|^{2-q}\,\mathrm{d}y \right) w(x)\,\mathrm{d}x \\
&= \|\psi\|_q^q \int |f(y)|^2 \left(\int |\psi(x-y)|^{2-q} w(x)\,\mathrm{d}x \right) \mathrm{d}y \\
&\leqslant \|\psi\|_q^q \int |f(y)|^2 \|\psi\|_q^{2-q} \left(\int_{r(y-x)<1} w(x)^{q'/2}\,\mathrm{d}x \right)^{2/q'} \mathrm{d}y \\
&\leqslant C\|\psi\|_q^2 \int |f(y)|^2 w(y)\,\mathrm{d}y.
\end{aligned}
$$

Thus, since condition (1.7) also holds when $q \in (1,2]$, Proposition 1.5 implies that g_ψ is bounded on L^2_w and $L^2_{w^{-1}}$ if $w^{q'/2} \in A_1$. This completes the proof of Theorem 1.3. □

Proof of Corollary 1.4. The proof of [11], Corollary 3 (ii) can be adapted to prove the boundedness of g_ψ on L^p when $2 \leqslant p < \infty$ and $\psi \in L^q$ for some $q \in (1,\infty)$, since the proof is based on vector valued inequalities derived from the boundedness on L^r, $1 < r < \infty$, of the maximal function

$$
N_\psi(f)(x) = \sup_{k\in\mathbb{Z}} |p_{2^k} * f(x)|, \quad p(x) = \int_1^2 |\psi_t(x)|\,\mathrm{d}t/t
$$

and the boundedness follows by the methods of [9] together with some Fourier transform estimates in the present context:

$$
\begin{gathered}
\left| \int_1^2 \mathscr{F}(|\psi|)(\delta_t^*\xi)\,\mathrm{d}t \right| \leqslant C|\xi|^{-\varepsilon} \quad \text{for some } \varepsilon > 0, \\
\left| \int_1^2 \mathscr{F}(|\psi|)(\delta_t^*\xi)\,\mathrm{d}t - \|\psi\|_1 \right| \leqslant C|\xi|^{\varepsilon} \quad \text{for some } \varepsilon > 0,
\end{gathered}
$$

which can be shown by the methods of this note.

Let us assume that $1 < p < 2$ and apply the methods of [8]. Let $s > q'/2$ and define $M_s(f) = M(|f|^s)^{1/s}$. Then $M_s(|f|^{2-p})^{q'/2}$ is in A_1 (we may assume that $0 < M_s(|f|^{2-p}) < \infty$) and M_s is bounded on $L^{p/(2-p)}$ if $s < p/(2-p)$. So by Hölder inequality and the $L^2_{w^{-1}}$ boundedness of g_ψ of Theorem 1.3 (2) with $w = M_s(|f|^{2-p})$, if $q'/2 < s < p/(2-p)$, we have

$$
\begin{aligned}
&\int g_\psi(f)(x)^p\,\mathrm{d}x = \int g_\psi(f)(x)^p M_s(|f|^{2-p})(x)^{-p/2} M_s(|f|^{2-p})(x)^{p/2}\,\mathrm{d}x \\
&\leqslant \left(\int g_\psi(f)(x)^2 M_s(|f|^{2-p})(x)^{-1}\,\mathrm{d}x \right)^{p/2} \left(\int M_s(|f|^{2-p})(x)^{p/(2-p)}\,\mathrm{d}x \right)^{1-p/2}
\end{aligned}
$$

$$\leqslant C\bigg(\int |f(x)|^2 M_s(|f|^{2-p})(x)^{-1}\,\mathrm{d}x\bigg)^{p/2} \|f\|_p^{p(1-p/2)}$$

$$\leqslant C\bigg(\int |f(x)|^2 |f(x)|^{p-2}\,\mathrm{d}x\bigg)^{p/2} \|f\|_p^{p(1-p/2)} = C\|f\|_p^p.$$

This completes the proof of Corollary 1.4. □

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