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STRONGLY 2-NIL-CLEAN RINGS WITH INVOLUTIONS

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Abstract. A $*$-ring R is strongly 2-nil-$*$-clean if every element in R is the sum of two projections and a nilpotent that commute. Fundamental properties of such $*$-rings are obtained. We prove that a $*$-ring R is strongly 2-nil-$*$-clean if and only if for all $a \in R$, $a^2 \in R$ is strongly nil-$*$-clean, if and only if for any $a \in R$ there exists a $*$-tripotent $e \in R$ such that $a - e \in R$ is nilpotent and $ea = ae$, if and only if R is a strongly $*$-clean SN ring, if and only if R is abelian, $J(R)$ is nil and $R/J(R)$ is $*$-tripotent. Furthermore, we explore the structure of such rings and prove that a $*$-ring R is strongly 2-nil-$*$-clean if and only if R is abelian and $R \cong R_1, R_2$ or $R_1 \times R_2$, where $R_1/J(R_1)$ is a $*$-Boolean ring and $J(R_1)$ is nil, $R_2/J(R_2)$ is a $*$-Yaqub ring and $J(R_2)$ is nil. The uniqueness of projections of such rings are thereby investigated.

Keywords: nilpotent; projection; $*$-tripotent ring; symmetry; strongly $*$-clean ring

MSC 2010: 16U99, 16E50, 16W10

1. Introduction

Throughout, all rings are associative with an identity. An element in a ring is strongly nil-clean if it is the sum of an idempotent and a nilpotent that commute. A ring R is strongly nil-clean if every element in R is strongly nil-clean. The subject of strongly nil-clean rings is interested for many mathematicians, e.g., [12] and [15]. A ring R is strongly 2-nil-clean if every element in R is the sum of two idempotents and a nilpotent that commute. Such rings were extensively studied by the authors (see [4]). An involution of a ring R is just an anti-automorphism whose square is the identity map 1_R. Thus an involution of a ring R is an operation $*\colon R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring R with an involution $*$ is called a $*$-ring. The class of $*$-rings is very large. For

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instance, all C^*-algebras, all Rickart $*$-rings, all Baer $*$-rings, etc. (see [8], [9], [11], [13]). Moreover, every commutative ring can be seen as a $*$-ring with the identity involution $*$ (see [1]).

The motivation of this paper is to characterize strongly 2-nil-clean rings with involutions, and completely determine the structure of such $*$-rings. An element $e \in R$ is a projection if $e = e^* = e^2$. A $*$-ring is called strongly 2-nil-$*$-clean if every element in R is the sum of two projections and a nilpotent that commute.

An element $e \in R$ is a $*$-tripotent if it is a self-adjoint tripotent, i.e., $e = e^* = e^3$. A $*$-ring is a $*$-tripotent if every element in R is a $*$-tripotent. An element $a \in R$ is strongly nil-$*$-clean if there exists a projection $e \in R$ such that $a - e \in R$ is a nilpotent and $ae = ea$. In Section 2, we investigate elementary properties of strongly 2-nil-$*$-clean rings. We prove that a $*$-ring R is strongly 2-nil-$*$-clean if and only if for all $a \in R$, $a^2 \in R$ is strongly nil-$*$-clean, if and only if for any $a \in R$ there exists a $*$-tripotent $e \in R$ such that $a - e \in R$ is a nilpotent and $ea = ae$. A $*$-ring R is strongly $*$-clean if every element in R is the sum of a projection and a unit that commute (see [14]). In a $*$-ring, an element u is called a symmetry if it is self-adjoint ($u = u^*$) and unitary ($u^2 = 1$) (see [1]). A $*$-ring R is an SN ring if every unit in R is the sum of a symmetry and a nilpotent that commute. In Section 3, we prove that a $*$-ring R is strongly 2-nil-$*$-clean if and only if R is a strongly $*$-clean SN ring. In Section 4, we are concerned with homomorphic images of such rings. It is proved that a $*$-ring R is strongly 2-nil-$*$-clean if and only if R is abelian, $J(R)$ is nil and $R/J(R)$ is $*$-tripotent. In Section 5, the structure of such rings is explored. We prove that a $*$-ring R is strongly 2-nil-$*$-clean if and only if R is abelian and $R \cong R_1$, R_2 or $R_1 \times R_2$, where $R_1/J(R_1)$ is a $*$-Boolean ring and $J(R_1)$ is nil, $R_2/J(R_2)$ is a $*$-Yaqub ring and $J(R_2)$ is nil. Finally, in the last section, we establish the connections between strong 2-nil-$*$-cleanness and the uniqueness of projections. We prove that a $*$-ring R is strongly 2-nil-$*$-clean if and only if $R/J(R)$ is $*$-tripotent, $J(R)$ is nil and $e - f \in J(R)$ implies $e = f$ for all projections $e, f \in R$.

We use $N(R)$ to denote the set of all nilpotent elements in R and $J(R)$ the Jacobson radical of R. $\mathbb{N}$ stands for the set of all natural numbers.

2. Elementary properties

The purpose of this section is to investigate certain elementary properties of strongly 2-nil-$*$-clean rings. We start by a simple fact which will be used frequently.

Lemma 2.1. *Let R be a strongly 2-nil-$*$-clean ring. Then every idempotent in R is a projection, and so R is abelian.*

P r o o f. Let $e \in R$ be an idempotent. Then we have two projections $g, h \in R$ such that $1 - e = g + h + w$, where $w \in N(R)$ and e, g, h, w commute. Hence, $e = (1 - g) - h - w$. Set $k = (1 - g) - h$. Then $k \in R$ is a projection and $e - k \in N(R)$ and $ek = ke$. Hence, $e = e^2 = k^2 + w'$ for some $w' \in N(R)$. We infer that $e - k^2 \in N(R)$ and $ek^2 = k^2e$. Since $(e - k^2)^3 = e - k^2$, we see that $e = k^2 \in R$ is a projection. By virtue of [5], Lemma 3.1, R is abelian. □

Theorem 2.2. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(1) *R is strongly 2-nil-$*$-clean.*
(2) *R is strongly $*$-clean and R is strongly 2-nil-clean.*
(3) *For all $a \in R$, $a^2 \in R$ is strongly nil-$*$-clean.*

P r o o f. (1) $\Longrightarrow$ (2) Clearly, R is strongly 2-nil-clean. In view of [4], Proposition 3.5, R is strongly clean. By virtue of Lemma 2.1, R is strongly $*$-clean.

(2) $\Longrightarrow$ (3) Due to [14], Theorem 2.2, every idempotent in R is a projection. Let $a \in R$. By using [4], Theorem 2.3, $a^2 \in R$ is strongly nil-clean, and then it is strongly nil-$*$-clean.

(3) $\Longrightarrow$ (1) In view of [4], Theorem 2.3, R is strongly 2-nil-clean. Let $e \in R$ be an idempotent. Then $e^2 \in R$ is strongly nil-$*$-clean. Hence, we have a projection $f \in R$ such that $e - f \in N(R)$ and $ef = fe$. As $(e - f)(1 - (e - f)^2) = 0$, we get $e = f$, i.e., every idempotent is a projection. This completes the proof. □

Corollary 2.3. *Every $*$-subring of a strongly 2-nil-$*$-clean ring is strongly 2-nil-$*$-clean.*

P r o o f. Let S be a $*$-subring of a $*$-ring R. As R is a strongly 2-nil-$*$-clean ring, it is strongly 2-nil-clean. In view of [4], Corollary 2.4, S is strongly 2-nil-clean. Let $e \in S$ be an idempotent. Then $e \in R$ is a projection by Lemma 2.1. Hence, $e \in S$ is a projection. Therefore S is a strongly 2-nil-$*$-clean ring. □

Corollary 2.4. *Let R be a strongly 2-nil-$*$-clean ring. Then eRe is strongly 2-nil-$*$-clean for all projections $e \in R$.*

P r o o f. Let e be a projection of R, then eRe is a $*$-subring of R. Thus we obtain the result by Corollary 2.3. □

Let $\{R_i \colon \in I\}$ be a family of $*$-rings and $|I| < \infty$. We easily see that the direct product $R = \prod\limits_{i \in I} R_i$ of $*$-rings R_i is strongly 2-nil-$*$-clean if and only if each R_i is strongly 2-nil-$*$-clean.

Theorem 2.5. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(1) *R is strongly 2-nil-$*$-clean.*

(2) *For any $a \in R$, there exists a $*$-tripotent $e \in R$ such that $a-e \in R$ is a nilpotent and $ea = ae$.*

Proof. (1) $\Longrightarrow$ (2) Let $a \in R$. Then we have two projections f, g and a $w \in N(R)$ that commute such that $1-a = f+g+w$. Set $e = (1-f)-g$. By virtue of Lemma 2.1, R is abelian. Hence, $e \in R$ is a $*$-tripotent, $ae = ea$ and $a-e = -w \in N(R)$, as desired.

(2) $\Longrightarrow$ (1) By virtue of [4], Theorem 2.8, R is strongly 2-nil-clean. Let $a \in R$. Then there exists a $*$-tripotent $e \in R$ such that $a-e \in N(R)$ and $ea = ae$. Hence, $a^2 - e^2 \in N(R)$ and $a^2e^2 = e^2a^2$. Clearly, $e^2 \in R$ is a projection. Thus, $a^2 \in R$ is strongly nill-$*$-clean. According to Theorem 2.2, we complete the proof. □

Lemma 2.6. *Let R be a $*$-ring, let $I \subseteq J(R)$, and let $e \in R$ be an idempotent. If $e - e^* \in I$, then there exists a projection $f \in R$ such that $eR = fR$ and $e - f \in I$.*

Proof. Let $z = 1 + (e^* - e)^*(e^* - e)$. Then $z \in U(R)$ and $z^* = z$. Let $t = z^{-1}$. Then $t^* = t$. We check that $ez = e(1-e-e^*+ee^*+e^*e) = (1-e-e^*+ee^*+e^*e)e = ze$, whence $et = te$ and $e^*t = te^*$. Let $f = ee^*t$. Then $f^* = f = f^2$. Hence, $f \in R$ is a projection. Obviously, $eR = fR$. Furthermore, we verify that $e-f = e(ez-ee^*)t = e(ee^*e - ee^*)t = ee^*(e-e^*)t \in I$, as asserted. □

Theorem 2.7. *Let R be a $*$-ring. Then R is strongly 2-nil-$*$-clean if and only if*

(1) *R is abelian;*

(2) *$a - a^* \in N(R)$ for all $a \in R$;*

(3) *R is strongly 2-nil-clean.*

Proof. $\Longrightarrow$ In light of Lemma 2.1, R is abelian. Let $a \in R$. By virtue of Theorem 2.5, we have a $*$-tripotent $e \in R$ such that $a - e \in N(R)$ and $ae = ea$. Hence, $a - a^3$, $a - a^* \in N(R)$, as $N(R)^* \subseteq N(R)$.

$\Longleftarrow$ In view of [4], Theorem 3.6, $N(R)$ forms an ideal of R. Hence, $N(R) \subseteq J(R)$. Let $a \in R$. Then we can find two idempotents $e, f \in R$ such that $a - e - f \in N(R)$. As $e - e^*, f - f^* \in N(R)$, it follows by Lemma 2.6 that $e - g, f - h \in N(R)$ for some projections $g, h \in R$. Hence, $a - g - h \in N(R)$. As $g, h \in R$ are central, we conclude that R is strongly 2-nil-$*$-clean. □

We note that the above conditions are necessary as the following examples show.

Example 2.8. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Define $*\colon R \to R$, $(a,b)^* = (b,a)$. Then R is abelian and strongly 2-nil-clean, but it is not strongly 2-nil-$*$-clean.

P r o o f. Clearly, R is Boolean, and so it is abelian and strongly 2-nil-clean. But $(1,0)-(1,0)^* = (1,1) \notin N(R)$. Hence, R is not strongly 2-nil-$*$-clean by Theorem 2.7. $\square$

Example 2.9. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Define $\sigma\colon R \to R$ by $\sigma(x,y) = (y,x)$. Consider the ring $T_2(R,\sigma) = \left\{\begin{pmatrix} a & b \\ 0 & a\end{pmatrix} : a, b \in R\right\}$ with the following operations:

$$\begin{pmatrix} a & b \\ 0 & a\end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c\end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & a+c\end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & a\end{pmatrix} \cdot \begin{pmatrix} c & d \\ 0 & c\end{pmatrix} = \begin{pmatrix} ac & ad+b\sigma(c) \\ 0 & ac\end{pmatrix}.$$

Define $*\colon T_2(R,\sigma) \to T_2(R,\sigma)$ by $\begin{pmatrix} a & b \\ 0 & a\end{pmatrix}^* = \begin{pmatrix} a & \sigma(b) \\ 0 & a\end{pmatrix}$. Then $T_2(R,\sigma)$ is strongly 2-nil-clean and $a - a^* \in N(T_2(R,\sigma))$, but it is not strongly 2-nil-$*$-clean.

P r o o f. Let $A = \begin{pmatrix} a & b \\ 0 & a\end{pmatrix} \in T_2(R,\sigma)$. Then $A - A^3 \in N(T_2(R,\sigma))$. In light of [4], Theorem 2.3, $T_2(R,\sigma)$ is strongly 2-nil-clean. Additionally, we easily check that $A - A^* \in N(T_2(R,\sigma))$. Let $E = \begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (0,1)\end{pmatrix}$. We check that $E^2 = E \in T_2(R,\sigma)$ is not central, and so $T_2(R,\sigma)$ is not abelian. Therefore the ring $T_2(R,\sigma)$ is not strongly 2-nil-$*$-clean, by Lemma 2.1. $\square$

3. Symmetry and nilpotent decompositions

The aim of this section is to characterize strongly 2-nil-$*$-clean rings in terms of the decompositions of symmetries and nilpotents. Using the techniques already known, developed in [6], Proposition 2.6, equation (1) and [7], Corollary 2.16, we now derive:

Lemma 3.1. *Let R be an* SN *ring. Then $J(R)$ is nil.*

P r o o f. Let $x \in J(R)$, then $1+x = u+w$, where $w \in N(R)$ and u is a symmetry. So $(1-w)+x = u$ and $(1-w)^2+x^2+2(1-w)x = 1$. This implies that $x^2+2x \in N(R)$. Similarly, as $x^2 \in J(R)$, $2x^2 + x^4 \in N(R)$. Also $x(x^2+2x) = x^3+2x^2 \in N(R)$. Then $x^4 - x^3 = x^4 + 2x^2 - (x^3+2x^2) = x^3(1-x) \in N(R)$. As $1-x \in U(R)$, we have $x^3 \in N(R)$ and so $x \in N(R)$. This completes the proof. $\square$

Lemma 3.2. *Let R be a strongly $*$-clean* SN *ring. If $2 \in N(R)$, then for any $a \in R$, $a^4 - a^6 \in N(R)$.*

P r o o f. Let $a \in R$. Then $a = e + u$ for some projection e and unit u. Since R is an SN ring, $u = v + w$ for some tripotent v and nilpotent w, where $eu = ue$. As $vw = wv$, we have $ev = ve$. Hence $a = e + v + w$, which implies $a + N(R) = (e+v) + N(R)$, $a^4 + N(R) = (e+v)^4 + N(R) = (e+v)^2(e+v)^2 + N(R) =$

$(e+1+2ev)(e+1+2ev)+N(R) = (7e+1+8ev)+N(R) = 7e+1+N(R)$, as $8 = 2^3 \in N(R)$. By a similar argument we have $a^6+N(R) = 15e+1$. Then $a^4-a^6+N(R) = N(R)$. We obtain the result. □

Lemma 3.3. *Let R be a strongly $*$-clean* SN *ring. If $3 \in N(R)$, then for any $a \in R$, $a-a^3 \in N(R)$.*

Proof. Let $a \in R$, then $a = e+u = a+v+w$ for some projection e and an involution v and $w \in N(R)$, where $wv = vw$ and as $ue = eu$, $ev = ve$. It is clear that $a^3+N(R) = e+v+3e+3ev+N(R)$, as $ev = ve$. Since $3 \in N(R)$, we have $a^3-a = 3e(e+v) \in N(R)$. □

A ring R is $*$-periodic if R is a periodic ring in which every idempotent is a projection. We now have at our disposal all the information necessary to prove the following theorem.

Theorem 3.4. *Let R be a $*$-ring. Then the following properties are equivalent:*

(1) *R is strongly 2-nil-$*$-clean.*
(2) *R is a strongly $*$-clean* SN *ring.*
(3) *R is a $*$-periodic* SN *ring.*

Proof. (1) $\Longrightarrow$ (3) As R is strongly 2-nil-$*$-clean, so it is strongly $*$-clean. In light of [14], Theorem 2.2, every idempotent in R is a projection. Furthermore, R is strongly 2-nil-clean. In view of [4], Proposition 3.5, R is periodic. Accordingly, R is $*$-periodic.

Now let $u \in U(R)$, $u = e+w$ for some $*$-tripotent e and nilpotent w such that $ew = we$. Hence $uw = u(u-e) = (u-e)u = wu$, since e is a central idempotent. This implies that u and w commute and so $u-w$ is a unit. This implies that e is a unit. As e is an idempotent, we see that $e = 1$ which is an involution. Therefore R is an SN ring, as desired.

(3) $\Longrightarrow$ (2) As R is $*$-periodic, it is periodic and every idempotent in R is a projection. In view of [14], Theorem 2.2, R is strongly $*$-clean, as required.

(2) $\Longrightarrow$ (1) In light of [14], Theorem 2.2, R is abelian. Write $3 = e+u$ for a projection e and a unit u in R. Since R is an SN ring, we can find an involution $v \in R$ and a nilpotent $w \in R$ such that $u = v+w$. Hence, $3-v = e+w$, and then $(3-v)^2 = e+q$ for some $q \in N(R)$. Thus, $9-6v+v^2 = 3-v+r$ for some $r \in N(R)$. It follows that $7 = 5v+r$; hence, $49 = 25v^2+t$ for some $t \in N(R)$. Thus, $24 \in N(R)$, and so $6 \in N(R)$. Write $2^n3^n = 0$ for some $n \in \mathbb{N}$. Clearly, $2^nR \cap 3^nR = \{0\}$. As $2^nR+3^nR = R$, by the Chinese Reminder Theorem, $R \cong R/2^nR \oplus R/3^nR$. Since $3 \in N(R/3^nR)$, it follows from Lemma 3.3 that for any

$a \in R/3^nR$, $a - a^3 \in N(R/3^nR)$. By using [4], Theorem 2.3, we deduce that $R/3^nR$ is strongly 2-nil-clean, and by Theorem 2.2, it is strongly 2-nil-$*$-clean. For any $a \in R/2^nR$, by Lemma 3.2, $a^4 - a^6 = a^4(1-a^2) \in N(R/2^nR)$, since $2 \in N(R/2^nR)$. Then $(1-a^2)^4a^4(1-a)^2a \in N(R/2^nR)$, and by using the same argument we conclude that $R/2^nR$ is strongly 2-nil-$*$-clean. Therefore R is strongly 2-nil-$*$-clean. □

Corollary 3.5. *A $*$-ring R is $*$-tripotent if and only if for any $a \in R$ there exists a symmetry $u \in R$ such that $a = a^* = aua$.*

P r o o f. $\Longrightarrow$ Let $a \in R$. Then $a = a^* = a^3$. Let $u = 1 - a^2 + a$. Then $u = u^* = u^{-1}$, and so u is a symmetry. We directly verify that $a = a^* = aua$, as desired.

$\Longleftarrow$ Let $u \in U(R)$, there exists a symmetry $v \in U(R)$ such that $u = u^* = uvu$, then $vu = 1$, and so $v = u^{-1}$. As $v^2 = 1$, we see that $u^2 = 1$. Hence, u is a symmetry and then R is an SN ring. Now let $a \in J(R)$, so $a(1-ua) = 0$ for some unit $u \in U(R)$. As $1 - ua \in U(R)$, we deduce that $a = 0$ and so $J(R) = 0$. It is clear that R is strongly clean and as for any $a \in R$, $a = a^*$, hence it is strongly $*$-clean. According to Theorem 3.4, R is strongly 2-nil-$*$-clean. By virtue of [4], Theorem 3.3, $R/J(R)$ is tripotent, and so R is tripotent. For any $a \in R$, $a = a^*$, so we conclude that R is $*$-tripotent. □

Corollary 3.6. *Let R be a $*$-ring. Then R is strongly 2-nil-$*$-clean if and only if*

(1) *for any $a \in R$, there exists $e = e^* = e^n$ $(n \geqslant 2)$ such that $a - e \in N(R)$ and $ae = ea$;*

(2) *R is an* SN *ring.*

P r o o f. $\Longrightarrow$ Choose $n = 3$. Then we prove (1). And (2) easily follows from Theorem 3.4.

$\Longleftarrow$ From (1) we deduce that R is periodic. Let $f \in R$ be an idempotent. Then there exists $e = e^* = e^n$ $(n \geqslant 2)$ such that $f - e \in N(R)$ and $fe = ef$. Hence, $f - e^{n-1} \in N(R)$. Clearly, $(f - e^{n-1})^3 = f - e^{n-1}$, and then $f = e^{n-1} \in R$ is a projection. Hence, R is $*$-periodic. The result follows by Theorem 3.4. □

4. Homomorphic images

In this section, we investigate various homomorphic images of strongly 2-nil-$*$-clean rings. We say that an ideal I of a $*$-ring R is a $*$-ideal provided $I^* \subseteq I$. If I is a $*$-ideal of a $*$-ring, it is easy to check that R/I is also a $*$-ring.

Lemma 4.1. *Let I be a nil $*$-ideal of a ring R. Then R is strongly 2-nil-$*$-clean if and only if R is abelian and R/I is strongly 2-nil-$*$-clean.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Let $a \in R$. Then there exist two projections $\overline{e}, \overline{f} \in R/I$ and a nilpotent $\overline{w} \in R/I$ such that $\overline{a} = \overline{e} + \overline{f} + \overline{w}$. Since I is nil, every idempotent lifts modulo I. Thus, we may assume that $e, f \in R$ are idempotents. Clearly, $e-e^*$, $f-f^* \in I$. Since $I \subseteq J(R)$, by Lemma 2.6 we have two projections $g, h \in R$ such that $e-g, f-h \in I$. Clearly, $w \in N(R)$. Thus, $a = g + h + w'$ for some $w' \in N(R)$. Therefore R is strongly 2-nil-$*$-clean. □

Theorem 4.2. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(1) *R is strongly 2-nil-$*$-clean.*
(2) *R is abelian, $J(R)$ is nil and $R/J(R)$ is $*$-tripotent.*
(3) *R is abelian, R is $*$-periodic and $R/J(R)$ is $*$-tripotent.*

Proof. (1) $\Longrightarrow$ (3) In light of Lemma 2.1, R is abelian. By virtue of Theorem 3.4, R is $*$-periodic.

In view of [4], Theorem 3.3, $R/J(R)$ is tripotent. Let $a \in R$. Then $a - a^* \in N(R) \subseteq J(R)$ by Theorem 2.7. Therefore $R/J(R)$ is $*$-tripotent, as desired.

(3) $\Longrightarrow$ (2) This is obvious as the Jacobson radical of every periodic ring is nil.

(2) $\Longrightarrow$ (1) Clearly, $R/J(R)$ is strongly 2-nil-$*$-clean by Theorem 2.5. Therefore we obtain the result by Lemma 4.1. □

A $*$-ring R is 2-$*$-Boolean if $a^2 \in R$ is a projection for all $a \in R$. For instance, every $*$-Boolean ring R, i.e., such that every element in R is a projection, is 2-$*$-Boolean. A ring R is strongly π-$*$-regular provided that for any $a \in R$ there exist a projection $e \in R$, a unit $u \in R$ and $n \in \mathbb{N}$ such that $a^n = eu$ where a, e and u commute with each other (see [5]). We have the following corollary.

Corollary 4.3. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(1) *R is strongly 2-nil-$*$-clean.*
(2) *R is abelian, $J(R)$ is nil and $R/J(R)$ is 2-$*$-Boolean.*
(3) *R is abelian, strongly π-$*$-regular, and $R/J(R)$ is 2-$*$-Boolean.*

P r o o f. (1) $\Longrightarrow$ (3) In view of Theorem 4.2, R is abelian, strongly π-$*$-regular, and $R/J(R)$ is $*$-tripotent. We easily see that every $*$-tripotent ring is 2-$*$-Boolean, as required.

(3) $\Longrightarrow$ (2) This is obvious as the Jacobson radical of every strongly π-$*$-regular ring is nil.

(2) $\Longrightarrow$ (1) Let $\overline{a} \in R/J(R)$, then $\overline{a}^2 = \overline{e}$ for some projection $\overline{e} \in R/J(R)$. This implies that $\overline{a}^2$ is strongly nil-$*$-clean, then by Theorem 2.2, $R/J(R)$ is strongly 2-nil-$*$-clean. This completes the proof by Lemma 4.1. □

5. Structure theorems

A $*$-ring R is a $*$-Yaqub ring if it is isomorphic to the subdirect product of $\mathbb{Z}_3$'s and every element in R is self-adjoint (i.e., $a = a^*$ for all $a \in R$). Next, we are concerned with the structure of strongly 2-nil-$*$-clean rings. Our starting point is this lemma.

Lemma 5.1. *A $*$-ring R is $*$-tripotent if and only if R is a $*$-Boolean ring, a $*$-Yaqub ring, or the product of such $*$-rings.*

P r o o f. $\Longrightarrow$ Since R is $*$-tripotent, $2^3 = 2$; hence, $6 = 0$. Thus, $R \cong R/2R \times R/3R$. Let $R_1 = R/2R$ and $R_2 = R/3R$. By Birkhoff's Theorem, R_i $(i = 1, 2)$ is the subdirect product of some subdirectly irreducible rings S_{ij}. Here, a ring is subdirectly irreducible if and only if the intersection of all its non-zero ideals is non-zero. As a homomorphic image of R_i, each S_{ji} is $*$-tripotent. Thus, S_{ji} is a commutative ring in which every element is the sum of two projections, by [10], Theorem 1. Since S_{ji} is subdirectly irreducible, it has no central projections except for 0 and 1. Thus, $S_{ji} = \{0, 1, -1\}$. As $2 \in N(R_1)$ and $3 \in N(R_2)$, we see that $S_{j1} \cong \mathbb{Z}_2$ and $S_{j2} \cong \mathbb{Z}_3$. Let R_1 and R_2 be the product of $\mathbb{Z}_2$'s and $\mathbb{Z}_3$'s, respectively. Therefore R is R_1, R_2 or $R_1 \times R_2$, as desired.

$\Longrightarrow$ Since $*$-Boolean rings and $*$-Yaqub rings are all $*$-tripotent, we easily obtain the result. □

Theorem 5.2. *A ring R is strongly 2-nil-$*$-clean if and only if*

(1) *R is abelian;*
(2) *$J(R)$ is nil;*
(3) *$R/J(R)$ is isomorphic to a $*$-Booelan ring, a $*$-Yaqub ring, or the product of such $*$-rings.*

P r o o f. Combining Theorem 4.2 and Lemma 5.1, we complete the proof. □

Lemma 5.3. *A ring R is $*$-Boolean if and only if*

(1) $2 \in R$ *is nilpotent;*
(2) R *is $*$-tripotent.*

Proof. $\Longrightarrow$ This is clear.

$\Longleftarrow$ Let $a \in R$. By virtue of Lemma 5.1, R is a $*$-Boolean ring, a $*$-Yaqub ring, or the product of such $*$-rings. Since $2 \in N(R)$, R is $*$-Boolean. □

Lemma 5.4. *A ring R is a $*$-Yaqub ring if and only if*

(1) $3 \in R$ *is nilpotent;*
(2) R *is $*$-tripotent.*

Proof. $\Longrightarrow$ Since $3 \in \mathbb{Z}_3$ is nilpotent, we see that $3 \in N(R)$. As $\mathbb{Z}_3$ is $*$-tripotent, so is R, as required.

$\Longleftarrow$ Let $a \in R$. In view of Lemma 5.1, R is a $*$- Boolean ring, a $*$-Yaqub ring, or the product of such $*$-rings. As $3 \in N(R)$, R is a $*$-Yaqub ring, as asserted. □

We have accumulated all the information necessary to prove the following theorem.

Theorem 5.5. *A ring R is strongly 2-nil-$*$-clean if and only if R is abelian and $R \cong R_1, R_2$ or $R_1 \times R_2$, where*

(1) $R_1/J(R_1)$ *is a $*$-Boolean ring and* $J(R_1)$ *is nil;*
(2) $R_2/J(R_2)$ *is a $*$-Yaqub ring and* $J(R_2)$ *is nil.*

Proof. $\Longrightarrow$ In view of Lemma 2.1, R is abelian. Since R is strongly 2-nil-clean, it follows by [4], Theorem 3.6 that $6 \in N(R)$. Write $6^n = 0$ ($n \in \mathbb{N}$). Then $2^nR + 3^nR = R$; hence, $R \cong R_1, R_2$ or $R_1 \times R_2$, where $R_1 = R/2^nR$ and $R_2 = R/3^nR$. As the homomorphic images of R, R_1 and R_2 are strongly 2-nil-$*$-clean. In light of Theorem 4.2, $R_1/J(R_1)$ is $*$-tripotent. As $2 \in N(R_1/J(R_1))$, it follows by Lemma 5.3 that $R_1/J(R_1)$ is $*$-Boolean. Likewise, $R_2/J(R_2)$ is $*$-tripotent and $3 \in N(R_2/J(R_2))$. By using Lemma 5.4, $R_2/J(R_2)$ is a $*$-Yaqub ring. In light of Theorem 4.2, $J(R)$ is nil; whence, $J(R_1)$ and $J(R_2)$ are both nil, as required.

$\Longleftarrow$ By hypothesis, $R/J(R)$ is a $*$-Boolean ring $R_1/J(R_1)$, a $*$-Yaqub ring $R_2/J(R_2)$, or the direct product of such $*$-rings. According to Lemma 5.1, $R/J(R)$ is $*$-tripotent. Clearly, $J(R) \cong J(R_1) \times J(R_2)$ is nil. Therefore R is strongly 2-nil-$*$-clean, by virtue of Theorem 4.2. □

A $*$-ring R is a strongly weakly nil-$*$-clean (nil-clean) ring if every element in R is the sum or the difference of a nilpotent and a projection (idempotent) that commute. As a consequence of Theorem 5.5, we now derive this corollary.

Corollary 5.6. *A $*$-ring R is strongly weakly nil-$*$-clean if and only if*

(1) *R has no homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$;*

(2) *R is strongly 2-nil-$*$-clean.*

Proof. $\Longrightarrow$ Since R is strongly weakly nil-clean, so is every homomorphic image of R. But $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not strongly weakly nil-clean, as $(1,-1)$ cannot be written as the sum or difference of a nilpotent and an idempotent, and this proves (1). We easily prove (2).

$\Longleftarrow$ Since R is strongly 2-nil-$*$-clean, by Theorem 5.5, $R \cong R_1, R_2$ or $R_1 \times R_2$, where $R_1/J(R_1)$ is a $*$-Boolean ring and $J(R_1)$ is nil; $R_2/J(R_2)$ is a $*$-Yaqub ring and $J(R_2)$ is nil. Since R has no homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$, we see that neither does $R_2/J(R_2)$. This forces that $R_2/J(R_2) \cong \mathbb{Z}_3$. Therefore R is strongly weakly nil-$*$-clean, as in [12], Theorem 1. $\square$

6. Uniqueness for projections

In this section, we observe that the condition "R is abelian" in Theorem 4.2 could be replaced by the unique property of projections. An element a in a $*$-ring R is uniquely $*$-clean provided that there exists a unique projection e such that $a - e$ is invertible (see [3]). The next result is the goal we will be striving for throughout this section.

Theorem 6.1. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(1) *R is strongly 2-nil-$*$-clean.*

(2) *$R/J(R)$ is $*$-tripotent, $J(R)$ is nil and $e - f \in J(R)$ implies $e = f$ for all projections $e, f \in R$.*

(3) *$R/J(R)$ is $*$-tripotent, $J(R)$ is nil and $a^2 \in R$ is uniquely $*$-clean for all $a \in R$.*

Proof. (1) $\Longrightarrow$ (3) In view of Theorem 4.2, $R/J(R)$ is $*$-tripotent and $J(R)$ is nil. Let $a \in R$. Then there exist a $*$-tripotent $e \in R$ and a $w \in N(R)$ such that $a = e + w$ with $ae = ea$ by Theorem 2.5. Hence, $a^2 = e^2 + w'$ where $w' \in N(R)$. This implies that $a^2 = (1 - e^2) + ((2e^2 - 1) + w')$. Clearly, $(2e^2 - 1) + w' = (2e^2 - 1)(1 + (2e^2 - 1)w') \in U(R)$. Thus, $a^2 \in R$ is $*$-clean. Assume that $a^2 = f + v$, where $f \in R$ is a projection and $v \in U(R)$. Then $e^2 - f \in U(R)$. In view of Lemma 2.1, R is abelian; hence, it follows from $(e^2 - f)^3 = e^2 - f$ that $(e^2 - f) \times (1 - (e^2 - f)^2) = 0$. Thus, $1 - e^2 + 2e^2 f - f = 0$, and so $f = (1 - 2e^2)^{-1}(1 - e^2) = 1 - e^2$. Therefore $a^2 \in R$ is uniquely $*$-clean.

(3) $\Longrightarrow$ (2) Let $e, f \in R$ be projections with $e - f \in J(R)$. By hypothesis, e^2 is uniquely $*$-clean. Obviously, $e^2 = (1 - e) + (2e - 1) = (1 - f) + ((2f - 1) + (e - f))$.

We see that both $1-e$, $1-f$ are projections, $(2e-1)^2=1$ and $(2f-1)+(e-f)=(2f-1)(1+(2f-1)(e-f))\in U(R)$. Thus $1-e=1-f$, and so $e=f$, as needed.

(2) $\Longrightarrow$ (1) Let $e\in R$ be an idempotent. Then $e-e^*\in J(R)$.

Set $z=1+(e-e^*)^*(e-e^*)$. Write $t=z^{-1}$. It follows from $z^*=z$ that $t^*=t$. Since $e^*z=e^*ee^*=ze^*$, we get $e^*t=te^*$ and $et=te$. Set $f=e^*et=te^*e$. Then $f^*=f$, $f^2=e^*ete^*et=e^*ee^*(tet)=e^*ztet=e^*et=f$, $fe=f$ and $ef=ee^*et=ezt=e$. Now $e=f+(e-f)$ and $e-f=e-e^*et=ee^*et-e^*et=(e-e^*)e^*et\in J(R)$. Here $f=f^*=f^2$, where $f=e^*e(1+(e^*-e)(e-e^*))^{-1}$.

Set $z'=1+(e^*-e)^*(e^*-e)$. Write $t'=(z')^{-1}$. Since $(z')^*=z'$, we have $(t')^*=t'$. Also $ez'=ee^*e=z'e$. Set $f'=ee^*t'=t'ee^*$. As in the preceding proof, we see that $f'=(f')^2=(f')^*$ and $ef'=f'$, $f'e=e$. In addition,

$$e-f'=f'e-f'=t'ee^*(e-e^*)\in J(R),$$

where $f'=(1+(e-e^*)(e^*-e))^{-1}ee^*$.

Thus $e-f, e-f'\in J(R)$, f and f' are projections. Hence, $f-f'=(e-f')-(e-f)\in J(R)$. By hypothesis, $f=f'$, and so

$$e^*e(1+(e^*-e)(e-e^*))^{-1}=(1+(e-e^*)(e^*-e))^{-1}ee^*.$$

This implies that

$$(1+(e-e^*)(e^*-e))e^*e=ee^*(1+(e^*-e)(e-e^*)).$$

Clearly, $(e-e^*)(e^*-e)e^*e=-e^*e+e^*ee^*e$ and $ee^*(e^*-e)(e-e^*)=-ee^*+ee^*ee^*$. Thus, $e^*ee^*e=ee^*ee^*$. One easily checks that

$$(e-e^*)^3-(e-e^*)=-ee^*e+e^*ee^*;$$
$$((e-e^*)^3-(e-e^*))(e+e^*)=(e-e^*)^3-(e-e^*).$$

Thus $(e-e^*)((e-e^*)^2-1)((e+e^*)-1)=0$.

As $e-f\in J(R)$, we get $e^*-f\in J(R)$. Thus, $(e+e^*)-2f\in J(R)$. This implies that $(e+e^*)-1=(2f-1)+((e+e^*)-2f)\in U(R)$, as $(2f-1)^2=1$. Since $(e-e^*)^2-1, (e+e^*)-1\in U(R)$, we get $e=e^*$. Therefore every idempotent in R is a projection. In light of [14], Theorem 2.1, R is abelian. Accordingly, R is strongly 2-nil-$*$-clean, by Theorem 4.2. □

Projections e, f in R are said to be equivalent, written $e\sim f$, in case there exists $w\in R$ such that $w^*w=e$ and $ww^*=f$ (see [2]). Let R be a $*$-ring. An element $a\in R$ is called a partial isometry provided that $a=aa^*a$. An element $u\in R$ is called a unitary element provided that $uu^*=u^*u=1$.

Corollary 6.2. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(1) *R is strongly 2-nil-$*$-clean.*

(2) *$R/J(R)$ is $*$-tripotent, $J(R)$ is nil and $e \sim f$ implies $e = f$ for all projections $e, f \in R$.*

(3) *$R/J(R)$ is $*$-tripotent, $J(R)$ is nil and for any partial isometry $a \in R$ there exist a projection e and a unitary u such that $a = eu = ue$.*

Proof. (1) $\Longrightarrow$ (3) In view of Theorem 4.2, $R/J(R)$ is $*$-tripotent and $J(R)$ is nil. Let $w \in R$ be a partial isometry. Then $w = ww^*w$. Hence, $w^* = w^*ww^*$, ww^* and w^*w are projections with $ww^*R \cong w^*wR$. By Lemma 2.1, R is abelian; hence $ww^* = w^*w$. Let $u = 1 - w^*w + w$. Then $u^* = 1 - w^*w + w^*$ and $uu^* = u^*u = 1$, i.e., $u \in R$ is a unitary element. Let $e = ww^*$. Then $e \in R$ is a projection. Furthermore, $w = ww^*(1 - ww^* + w) = eu = ue$, as desired.

(3) $\Longrightarrow$ (2) Suppose $e \sim f$ for projections $e, f \in R$. Write $e = w^*w$ and $f = ww^*$. We may assume that $w \in fRe$ and $w^* \in eRf$. Then $ww^*w = we = w$, i.e., $w \in R$ is a partial isometry. By hypothesis, there exist a projection g and a unitary u such that $w = gu = ug$. Accordingly, $e = w^*w = (u^*g)(gu) = u^*gu = (u^*u)g = g$ and $f = ww^* = (gu)(u^*g) = g(uu^*)g = g$, and then $e = f$, as desired.

(2) $\Longrightarrow$ (1) Let $e, f \in R$ be projections such that $e - f \in J(R)$. Set $u = 1 - e - f$. Then $eu = -ef = uf$. Clearly, $u = u^* = u^{-1} \in U(R)$. Set $w = fu^{-1}e$. Then $f = u^{-1}eu = ww^*$ and $e = ufu^{-1} = w^*w$. We infer that $e \sim f$. By hypothesis, $e = f$. By virtue of Theorem 6.1, R is strongly 2-nil-$*$-clean. □

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