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Applications of Mathematics, Vol. 64 (2019), No. 3, 281–300

Persistent URL: `http://dml.cz/dmlcz/147718`

THEORETICAL FOUNDATION OF THE WEIGHTED LAPLACE INPAINTING PROBLEM

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Received July 31, 2018. Published online April 24, 2019.

Abstract. Laplace interpolation is a popular approach in image inpainting using partial differential equations. The classic approach considers the Laplace equation with mixed boundary conditions. Recently a more general formulation has been proposed, where the differential operator consists of a point-wise convex combination of the Laplacian and the known image data. We provide the first detailed analysis on existence and uniqueness of solutions for the arising mixed boundary value problem. Our approach considers the corresponding weak formulation and aims at using the Theorem of Lax-Milgram to assert the existence of a solution. To this end we have to resort to weighted Sobolev spaces. Our analysis shows that solutions do not exist unconditionally. The weights need some regularity and must fulfil certain growth conditions. The results from this work complement findings which were previously only available for a discrete setup.

Keywords: image inpainting; image reconstruction; Laplace equation; Laplace interpolation; mixed boundary condition; partial differential equation; weighted Sobolev space

MSC 2010: 35J15, 35J70, 46E35, 94A08

1. Introduction

Image inpainting deals with recovering lost image regions or structures by means of interpolation. It is an ill-posed process; as soon as a part of the image is lost, it cannot be recovered correctly with absolute certainty, unless the original image is completely known. The inpainting problem goes back to the works of Masnou and Morel as well as Bertalmío et al. [4], [37], although similar problems have been considered in other fields already before. There exist many inpainting techniques, often based on interpolation algorithms, but partial differential equation (PDE)-based approaches are among the most successful ones, see e.g. [21]. Among these,

DOI: 10.21136/AM.2019.0206-18

strategies based on the Laplacian stand out [5], [12], [34], [44]. In that context, the elliptic mixed boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \Omega_K, \\ u = f & \text{in } \partial\Omega_K, \\ \partial_n u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

is very popular. Here, f represents known image data in a region $\Omega_K \subsetneq \Omega$ (or on the boundary $\partial\Omega_K$) of the whole image domain Ω. Further, $\partial_n u$ denotes the derivative in outer normal direction. An exemplary sketch of the layout of the problem is given in Figure 1. Equations like (1.1), which involve different kinds of boundary conditions, are commonly referred to as mixed boundary value problems and in rare cases also as Zaremba's problem [54]. Image inpainting based on (1.1) appears under various names in the literature: Laplace interpolation [45], harmonic interpolation [48], or homogeneous diffusion inpainting [34]. The latter name is often used in combination with the steady state solution of the parabolic counterpart of (1.1).

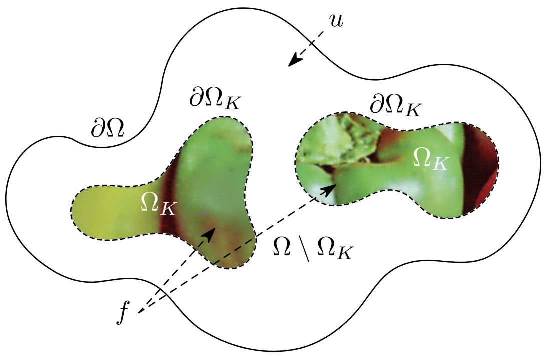
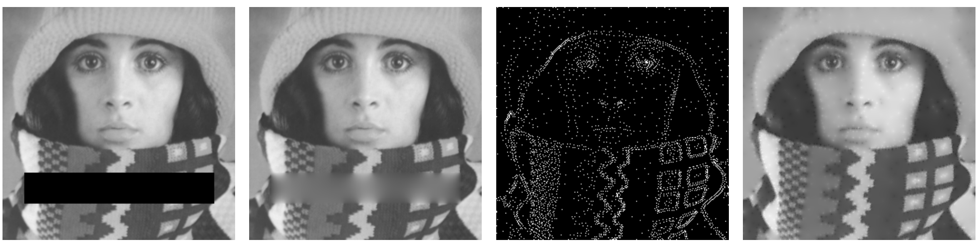


Figure 1. Generic PDE-based inpainting, as given e.g. in (1.1), with known image data f in Ω_K. The task consists in recovering a reasonable approximation u in $\Omega \setminus \Omega_K$ from the original image data f given in Ω_K. Source image: [53]

Applications of image inpainting are manifold and range from art restoration to image compression. The earliest uses of (1.1) go back to Noma and Misulia [40] (1959) and Crain [13] (1970) for generating topographic maps. Further applications include the works of Bloor and Wilson (1989) [5], who studied partial differential equations for generating blend surfaces. Finally, we refer to [48], [25] for a broad overview on PDE-based inpainting and the closely related problem of PDE-based image compression.

In the context of image reconstructions, (1.1) is often favoured over other more complex models due to its mathematically sound theory, even though the strong smoothing properties may yield undesirable blurry reconstructions. Models based on anisotropic diffusion [19], [47] or total variation [12] may be more powerful, but are much harder to grasp from a mathematical point of view. In the context of image compression, the data Ω_K used for the reconstruction can be freely chosen, since the complete image is known beforehand. The difficulty in compressing an image with a PDE lies in the fact that one has to optimise two contradicting constraints. On the one hand, the size of the data Ω_K should be small to allow an efficient coding, but on the other hand one wishes to have an accurate reconstruction from this sparse amount of information, too. The optimal data also depends on the choice of the differential operator and the simplicity of the Laplacian offers many design choices for optimization strategies to find the best Ω_K. Some of these approaches belong to the state-of-the-art methods in PDE-based image compression. We refer to [43] for a comparison of different PDE-based models and to [18], [35], [26] for data optimization strategies in the compression context. Figure 2 demonstrates the potential of such a careful data optimization. In Figure 2(a) an arbitrary rectangle (marked in black) has been removed from the image. Figure 2(b) shows the reconstruction of this missing region. The reconstruction is severely blurred and the texture of the scarf is almost completely lost. On the other hand, Figure 2(c) represents an optimized set of 5% of the data points (missing data marked in black) with the corresponding colour values. These 5% have been obtained with the method from [26]. Figure 2(d) depicts the corresponding reconstruction. Although the reconstruction has a few artefacts, its overall quality is very convincing.



(a) Arbitrary data (b) Reconstruction (c) Optimal data (d) Reconstruction

Figure 2. (a) Image data with an arbitrary missing rectangular region (marked in black). (b) Corresponding reconstruction with (1.1). The reconstruction suffers from blurring effects. (c) Remaining data (5% of all pixels) with optimal reconstruction property. Missing data is black. (d) Corresponding reconstruction with (1.1). The reconstruction is sharp although the Laplacian causes strong smoothing. Source original image: [49]

As already mentioned, finding the best pixel data is a very challenging task. Mainberger et al. [35] consider the combinatorial point of view of this task while Belhachmi et al. [3] approach the topic from the analytic side. Recently [26], the "hard" boundary conditions in (1.1) have been replaced by softer weighting schemes. These blend the given image data with the information obtained from the differential operator and can be written as

$$\begin{cases} c(u-f)+(1-c)(-\Delta)u = 0 & \text{in } \Omega, \\ \partial_n u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

with a weighting function (also called mask) $c\colon\ \Omega \to \mathbb{R}$. Optimising such a weighting function is notably simpler, at least in discrete setups. We note that (1.1) is a special case of (1.2) with $c(x) = 1$ for $x \in \Omega_K$ and $c(x) = 0$ else.

Equation (1.1) is well understood and there exist many results on existence, uniqueness and regularity of solutions, see [16], [8] for a generic analysis and [12], [11] for a more specific analysis in the inpainting context with Dirichlet boundary conditions only. Finite difference discretizations of (1.1) and (1.2) have also been object of several investigations in the past. One can show that the discrete counterpart of (1.1) admits a unique solution as soon as the Dirichlet boundary set is nonempty [34]. Similarly, the discrete finite difference formulation of (1.2) admits a unique solution if c is positive in at least one position [22].

An important question that arises in this context is what these discrete requirements relate to in the continuous setting. If we consider for example the following model problem that one may extract from the formulation (1.1),

$$\begin{cases} -\Delta u = 0 & \text{in } B_1 \setminus B_\varepsilon, \\ u = 0 & \text{in } \partial B_1, \\ u = 1 & \text{in } \partial B_\varepsilon, \end{cases} \tag{1.3}$$

where $B_r \subset \mathbb{R}^2$ is a ball or radius r with the centre at the origin and where ∂B_r is its boundary, then one can show that a smooth solution exists for every $\varepsilon > 0$, but that no solution in the classic sense (i.e. twice differentiable and fulfilling all boundary conditions) exists in the limiting case $\varepsilon \to 0$. Indeed, the solution is given by

$$u(x,y) = \frac{\ln(x^2+y^2)}{2\ln(\varepsilon)}. \tag{1.4}$$

Yet, the discrete formulation will admit a unique solution independently of the choice of ε. It suffices that the corresponding matrix is block irreducible. We refer to [22], [34] for a detailed discussion on the existence of solutions. To remedy the situation for the continuous formulation in (1.1), the authors of [3] have required that the set Ω_K should have positive α-capacity. The α-capacity ($\alpha > 0$) of a subset $E \subset D$ of a smooth, bounded and open set D is given by

$$\inf_{u \in U_E} \int_D (|\nabla u|^2 + \alpha |u|^2)\,\mathrm{d}x, \tag{1.5}$$

where U_E is the set of all functions u of the Sobolev space $H_0^1(D)$ such that $u \geqslant 1$ almost everywhere in a neighbourhood of E. If Ω_K has positive α-capacity, then a solution of (1.1) exists in the Sobolev space $H^1(\Omega)$ [3]. This requirement, that Ω_K must have positive capacity, can be understood as requiring that image pixels are "fat enough" to allow a reconstruction. It reconciles the continuous and discrete worlds and leads to a consistent theory on both sides. A higher regularity than $H^1(\Omega)$ can be achieved for specific constellations of the boundary data. A rather general theory is given in [17], [36], [2]. The author of [38] shows that a Hölder continuous solution exists if the data is regular enough. Finally, [7] discusses the regularity of solutions on Lipschitz domains. Let us mention that Caselles et al. [9] have also discussed this inability of the Laplacian to recover images from isolated points and that they suggested absolutely minimizing Lipschitz extensions as an alternative.

The authors of this manuscript are not aware of any similar theory that would remedy the apparent discrepancy between (1.2) and its discrete counterpart. The discrete setup is almost always solvable. On the other hand, solutions for the continuous model are only known for some special cases such as c being bounded between two positive constants in the interval $(0, 1)$, or c being itself a constant [8], [16]. For inpainting purposes it is important that c may map to the whole unit interval and even beyond. Regions with $c \equiv 1$ keep the data fixed and if c exceeds the value 1, then contrast enhancing in the reconstruction is possible, see [23], [27].

Here, we attempt to bridge that gap between the discrete setup and the continuous model for the case when c maps to $[0, 1]$. We show that a weak solution exists if certain assumptions on the weight functions are met. Special interest will be paid to occurring requirements on c and whether they correspond to discrete counterparts. We aim at applying the Theorem of Lax-Milgram in purpose-built weighted Sobolev spaces. As such, the contributed novelties of this manuscript are twofold. First, we complement the well-posedness study of (1.1) and $c > 1$, which has recently been discussed in [24], with the missing case where c maps to $[0, 1]$, and secondly, we introduce weighted Sobolev spaces to the image processing community. These spaces bear a certain number of interesting properties that can also be useful for other image analysis tasks, see e.g. [6].

In the next section we first derive the weak formulation corresponding to (1.1) and introduce the weighted Sobolev spaces where the solution is sought. Then we will state the necessary conditions on the weight function c that must be fulfilled to assert the existence of a solution. Finally, we show that a unique solution exists.

2. Inpainting with the weighted Laplacian

We assume the following requirements on our domain Ω and our data f. These assumptions will hold throughout the whole paper, unless mentioned otherwise. Even though some of these are stronger than necessary, they are not uncommon in the image processing context. Further, they help us to keep the discussion on PDE-based models low on technical details.

(1) Ω is an open, connected and bounded subset of $\mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary $\partial\Omega$.
(2) $\Omega_K \subsetneqq \Omega$ is a closed subset of Ω with positive Lebesgue measure. It represents the known data locations used to recover the missing information on $\Omega \setminus \Omega_K$. The interpolation data is given by $f(\Omega_K)$. The boundary $\partial\Omega_K$ is assumed to be $\mathcal{C}^\infty$, too.
(3) $f\colon \overline{\Omega} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function representing the given image data to be interpolated by the underlying PDE. Here, $\overline{\Omega}$ denotes the closure of Ω.
(4) The boundaries $\partial\Omega$ and $\partial\Omega_K$ do not intersect and neither of the boundaries $\partial\Omega$ or $\partial\Omega_K$ are empty.

As already mentioned in the previous section, the classic formulation for PDE-based inpainting with the Laplacian reads

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \Omega_K, \\ u = f & \text{in } \partial\Omega_K, \\ \partial_n u = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.1}$$

Using the findings from [16], [24], it is easy to show that (2.1) is well-posed and that a unique weak solution exists in a subspace of $H^1(\Omega)$. If we define a function $c\colon \Omega \to \{0,1\}$ with $c(x) \equiv 1$ when $x \in \Omega_K$ and $c(x) \equiv 0$ else, then (2.1) can also be rewritten as

$$\begin{cases} c(u-f) + (1-c)(-\Delta)u = 0 & \text{in } \Omega, \\ \partial_n u = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.2}$$

Interestingly, the latter formulation also makes sense if we generalise to $c\colon \Omega \to \mathbb{R}$ a fact which was first exploited in [26]. If c has binary values in the set $\{0,1\}$, then (2.2) is equivalent to (2.1) with the Dirichlet boundary conditions specified by f at those regions where c equals 1. Equation (2.2) can also be interpreted from a physical or chemical point of view. We are in the presence of a stationary reaction-diffusion equation. The diffusive term $(1-c)(-\Delta)u$ is responsible for spreading the information generated by the reactive term $c(u-f)$. The weight c is responsible for the speed at which information is generated and spread.

If c is bounded between two non-negative numbers strictly smaller than one, then it follows from [8], [24] that a solution exists in $C^{2,\alpha}(\overline{\Omega})$ if the data f and c are regular enough. We refer to the cited references for the concrete requirements. For inpainting purposes it is however important to allow $c(x) = 1$ or even $c(x) > 1$.

In order to derive the weak formulation of (2.2), we follow the presentation in [24], where the setup in (2.2) with $c > 1$ was discussed by outlining its relationship to the Helmholtz equation. We also introduce the following additional requirements on the function c.

(5) We have $c \equiv 1$ inside the set of known data Ω_K.
(6) The function $c\colon \Omega \to [0,1]$ is an element of $H^1(\Omega, [0,1])$ and the function $1 - c$ is an element of the $A_2(\mathbb{R}^2)$ Muckenhoupt class (see next page for the precise definition). Finally, $\nabla f / \sqrt{1-c}$ is an element of $L^2(\Omega \setminus \Omega_K)$.
(7) The function c has a trace on $\partial\Omega$ and $c|_{\partial\Omega} \equiv 0$ holds.

Let us briefly comment on these requirements. The first part of item (1) is trivially fulfilled by images. Its second part is more restrictive. Assuming the boundary of Ω to be piecewise $\mathcal{C}^\infty$ would be more realistic, but this would in general also reduce the regularity of solutions of PDEs. Item (2) and item (3) do not impose any severe restrictions for image processing tasks. Images can always be rendered $\mathcal{C}^\infty$ by convolving them with a Gaussian. Nevertheless, we emphasise that our requirements in items (2) and (5) forbid setups where the data is given on a one dimensional set Ω_K. Thus, our model deviates from the original formulation in [35], where all the information is extracted from $\partial\Omega_K$. Items (4) and (7) are necessary for technical reasons. If the Neumann and Dirichlet boundary conditions meet each other, it is possible to generate setups that lead to contradicting requirements, see [2], [36] for a more detailed discussion on the existence and regularity of solutions when the boundary conditions intersect. A more thorough discussion of intersecting boundary conditions would however be beyond the scope of this work. Finally, item (6) is necessary to assert the existence of our weighted Sobolev spaces. We remind that a weight function (i.e. a measurable and almost everywhere positive function) ω is in the $A_p(\mathbb{R}^n)$ $(1 < p < \infty)$ Muckenhoupt class if there exists a positive constant $C_{p,\omega}$ such that

$$\sup_B \left\{ \left(\frac{1}{|B|} \int_B \omega(x)\,\mathrm{d}x \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1/(1-p)}\,\mathrm{d}x \right)^{p-1} \right\} = C_{p,\omega} < \infty, \tag{2.3}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. We remark that it follows from (2.3) that $\omega^{1/(1-p)}$ will be an element of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see [39]). Therefore, item (6) implies that $1/\sqrt{1-c}$ is an element of $L^1_{\mathrm{loc}}(\Omega \setminus \Omega_K)$. Let us also remark that it follows from Theorem 2.1.4 (or Corollary 2.1.6) in [50] that $\mathcal{C}^\infty$ functions are dense in weighted Sobolev spaces with Muckenhoupt weights. In concrete applications it may be difficult to verify that $\nabla f/\sqrt{1-c}$ is in $L^2(\Omega \setminus \Omega_K)$. In many cases the function c has been determined by an optimization strategy and its properties are not completely known. Therefore, we have to require explicitly that $\nabla f/\sqrt{1-c}$ is an element of $L^2(\Omega \setminus \Omega_K)$.

Let us now rewrite (2.2) in a more suitable form (also see Figure 3). In the first step we explicitly set the regions where $c \equiv 1$ apart.

$$\begin{cases} c(u-f) + (1-c)(-\Delta)u = 0 & \text{in } \Omega \setminus \Omega_K, \\ u = f & \text{in } \partial\Omega_K, \\ \partial_n u = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.4}$$

The previous reformulation implies that $c < 1$ almost everywhere in $\Omega \setminus \Omega_K$. A small detail that will become important in the forthcoming discussions. Since $c \in H^1(\Omega, [0,1])$, we can apply the product rule and rewrite (2.4) as

$$\begin{cases} -\operatorname{div}((1-c)\nabla u) - \nabla c \cdot \nabla u + c(u-f) = 0 & \text{in } \Omega \setminus \Omega_K, \\ u = f & \text{in } \partial\Omega_K, \\ \partial_n u = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.5}$$

If u solves (2.5), it follows that $v := u - f$ also solves

$$\begin{cases} -\operatorname{div}((1-c)\nabla v) - \nabla c \cdot \nabla v + cv = g & \text{in } \Omega \setminus \Omega_K, \\ v = 0 & \text{in } \partial\Omega_K, \\ \partial_n v = h & \text{in } \partial\Omega \end{cases} \tag{2.6}$$

with $g := (1-c)\Delta f$ and $h := -\partial_n f$. For convenience of writing, we will continue calling the sought solution of (2.6) u and not v. Being able to solve (2.6) is equivalent to being able to solve (2.5). Yet, this change lets us reduce the problem to the case with homogeneous Dirichlet boundary conditions. Deriving the associated weak formulation is now straightforward. Multiplying with a suitable test function φ from some space V (with $\varphi \equiv 0$ on $\partial\Omega_K$) and integrating (2.6) by parts implies that we must seek a function $u \in V$ which solves

$$\underbrace{\int_{\Omega\setminus\Omega_K} ((1-c)\nabla u \cdot \nabla\varphi - (\nabla c \cdot \nabla u)\varphi + cu\varphi)\,\mathrm{d}x}_{=:B^c(u,\varphi)} = \underbrace{\int_{\Omega\setminus\Omega_K} g\varphi\,\mathrm{d}x + \int_{\partial\Omega} h\varphi\,\mathrm{d}\mathcal{H}^1}_{=:F(\varphi)} \tag{2.7}$$

for all $\varphi \in V$. Here, $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure. We defer the exact specification of V to the forthcoming sections.

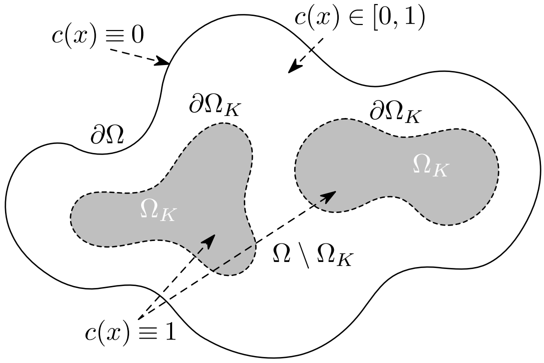


Figure 3. Visualization of the setup specified in (2.4) with a nonbinary valued mask function c.

Since c maps to the unit interval, we are in the presence of a so called degenerate elliptic equation [51], [46] or sometimes also referred to as a PDE with non-negative characteristic form [41]. Such PDEs are characterized by the fact that their highest order term is allowed to vanish. This fact, that the second order differential operator may vanish locally, requires a more sophisticated analysis. The key issue to approach such problems is to select the correct function space V and to place necessary restrictions onto c.

The canonic strategy to show the existence and uniqueness of a weak solution consists in applying the Lax-Milgram Theorem [16]. The crucial part will be the coercivity of the bilinear form B^c and the boundedness of B^c and F. Obviously, the boundedness of B^c and F depends a lot on the choice of the space V and c. To show coercivity of the bilinear form, we must study the behaviour of

$$B^c(u,u) = \int_{\Omega\setminus\Omega_K} ((1-c)|\nabla u|^2 - (\nabla c \cdot \nabla u)u + cu^2)\,\mathrm{d}x. \tag{2.8}$$

The coercivity of (2.8) is not immediately visible due to the complex interplay between u, c and their derivatives. The following section sheds light on the requirements to prove well-posedness of the considered problem.

2.1. Weighted Sobolev spaces. Weighted Sobolev spaces have been studied intensively in the past. Their uses are manifold, but they are most often found in the analysis of PDEs with vanishing or singular diffusive term. The works [51], [41], [46], [28], [33], [39] give an almost complete overview of their usefulness. For the sake of completeness, we shortly summarize how these spaces are set up.

In the following, we denote by W_Ω the set of weight functions ω, i.e. ω is a measurable and almost everywhere positive function in some domain Ω. For $1 \leqslant p < \infty$

and $\omega \in W_\Omega$ we define the corresponding weighted L^p space as

$$L^p(\Omega;\omega) := \left\{ u\colon\ \Omega \to \mathbb{R} \,\middle|\, \|u\|_{L^p(\Omega;\omega)} := \left(\int_\Omega |u(x)|^p \omega(x)\,\mathrm{d}x \right)^{1/p} < \infty \right\}. \tag{2.9}$$

In a similar way as Sobolev spaces refine the Lebesgue spaces we can also refine our weighted L^p spaces by including the weak derivatives into the norm. Here, weak derivatives $D^\alpha u$ of a function u are to be understood as (see also [20])

$$\int_\Omega u(x)(D^\alpha \eta(x))\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega (D^\alpha u(x))\eta(x)\,\mathrm{d}x \quad \forall\, \eta \in C_0^\infty(\Omega). \tag{2.10}$$

Different weights for different derivatives are also possible. For a given collection $S_k := \{\omega_\alpha \in W_\Omega \mid |\alpha| \leqslant k\}$ of weight functions we denote by $W^{k,p}(\Omega; S_k)$ the set of all functions u defined on Ω and whose (weak) derivatives $D^\alpha u$ of order $|\alpha| \leqslant k$ (α being a multi-index) belong to $L^p(\Omega;\omega_\alpha)$. We can equip this vector space $W^{k,p}(\Omega; S_k)$ with the norm

$$\begin{aligned} \|u\|_{W^{k,p}(\Omega;S_k)} &:= \left(\sum_{|\alpha|\leqslant k} \int_\Omega \|D^\alpha u(x)\|^p \omega_\alpha(x)\,\mathrm{d}x \right)^{1/p} \\ &= \left(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|^p_{L^p(\Omega;\omega_\alpha)} \right)^{1/p}. \end{aligned} \tag{2.11}$$

One can show that the space $W^{k,p}(\Omega; S_k)$ is a Banach space if $\omega_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ and $\omega_\alpha^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$ for all $|\alpha| \leqslant k$, see [30], [31]. Note that this requires that all derivatives up to the order k must be attributed to such a weight ω_α. However, one can also show that $W^{k,p}(\Omega; \widetilde{S}_k)$ is still complete if $\widetilde{S}_k \subsetneqq S_k$ contains at least one weight ω_α with $|\alpha| = k$ and a weight for $|\alpha| = 0$, see [29], [32].

We note that for $p = 2$ there is a canonical choice for a scalar product:

$$\langle u, v\rangle_{W^{k,2}(\Omega;S_k)} := \sum_{|\alpha|\leqslant k} \int_\Omega D^\alpha u(x) D^\alpha v(x) \omega_\alpha(x)\,\mathrm{d}x. \tag{2.12}$$

Thus, with a suitable choice of weights we obtain a Hilbert space. If all the weight functions are constant and equal to one, then our weighted spaces coincide with the usual definition of Sobolev spaces. We refer to [28], [33] for a more complete listing of possible weighted Sobolev space constructions.

By looking at (2.7) it becomes apparent why these weighted Sobolev spaces are useful. The function c (or $1 - c$) can be considered as a weight function and simply be integrated into the space definition. This simplifies the proofs to show existence and uniqueness, since boundedness and coercivity are easier to show and theorems such as Lax-Milgram can be applied in any Hilbert space.

Our goal now will be to consider the corresponding weak formulation of (2.5) in a suitable weighted Sobolev space V. By applying the Theorem of Lax-Milgram in these spaces we will show the existence and uniqueness of a weak solution of (2.5). Let us also note that alternative approaches may be derived from the works [10], [14].

The weights for our space definition should be chosen so that the bilinear form is equivalent to the norm of our space. Often the multiplicative factors of the individual derivatives in the bilinear form offer themselves as viable choices for this task. In our case however, the function c may vanish locally. This prevents us from using $1-c$ and c as weights to define a norm. They only give us a seminorm structure. Such a situation is briefly described in [29]. We mostly follow that presentation and we propose the following correspondence between multi-indices $\alpha \in \mathbb{N}_0^2$ and weights ω_α

$$\omega_{\binom{0}{0}} := 1, \quad \omega_{\binom{1}{0}} := 1 - c(x), \quad \omega_{\binom{0}{1}} := 1 - c(x). \tag{2.13}$$

This yields the scalar product and norm

$$\langle u, v\rangle_V := \int_{\Omega\setminus\Omega_K} ((1-c)\nabla u\cdot\nabla v + uv)\,\mathrm{d}x, \tag{2.14a}$$

$$\|u\|_V := \left(\int_{\Omega\setminus\Omega_K} ((1-c)|\nabla u|^2 + u^2)\,\mathrm{d}x\right)^{1/2}, \tag{2.14b}$$

as well as the following definition for our space V:

$$V := \{\phi \in W^{1,2}(\Omega\setminus\Omega_K; S_c) \mid \phi \equiv 0 \quad \text{on } \partial\Omega_K\}, \tag{2.15}$$

where S_c is our set of weights given in (2.13). In addition, we define the seminorm

$$|\!|\!| u |\!|\!|_V := \left(\int_{\Omega\setminus\Omega_K} (1-c)|\nabla u|^2\,\mathrm{d}x\right)^{1/2}. \tag{2.16}$$

Finally, following the presentation in [33], we note that the bilinear form B^c in (2.7) can be written compactly as a ternary quadratic form

$$B^c(u,\varphi) = \sum_{|\alpha|,|\beta|\leqslant 1} \int_{\Omega\setminus\Omega_K} a_{\alpha,\beta} D^\beta u D^\alpha \varphi\,\mathrm{d}x, \tag{2.17}$$

where α, β are multi-indices in $\mathbb{N}_0^2$. The weights $a_{\alpha,\beta}$ must be set as follows to yield our model:

$$a_{\binom{1}{0},\binom{1}{0}} = a_{\binom{0}{1},\binom{0}{1}} = 1 - c(x), \quad a_{\binom{0}{0},\binom{0}{0}} = c(x), \tag{2.18a}$$

$$a_{\binom{0}{0},\binom{1}{0}} = -\partial_x c(x), \quad a_{\binom{0}{0},\binom{0}{1}} = \partial_y c(x), \tag{2.18b}$$

and $a_{\alpha,\beta} = 0$ for any other combination of multi-indices. In addition to the previous assumptions, we now assume further:

(8) There exists a constant $\kappa > 0$ such that for all $|\alpha|, |\beta| \leqslant 1$, $\alpha \neq \beta$,

$$|a_{\alpha,\beta}| \leqslant \kappa \sqrt{a_{\alpha,\alpha} a_{\beta,\beta}} \tag{2.19}$$

almost everywhere in $\Omega \setminus \Omega_K$. For our choice in (2.18), this reduces to

$$|\partial_x c| \leqslant \kappa \sqrt{c(1-c)}, \quad |\partial_y c| \leqslant \kappa \sqrt{c(1-c)} \tag{2.20}$$

almost everywhere in $\Omega \setminus \Omega_K$.

(9) There exists a constant $\kappa' > 0$ such that for all real vectors $\xi \in \mathbb{R}^3$ with entries ξ_γ (γ being a multi-index in $\mathbb{N}_0^2$ such that $|\gamma| \leqslant 1$) we have

$$\sum_{|\alpha|,|\beta| \leqslant 1} a_{\alpha,\beta} \xi_\alpha \xi_\beta \geqslant \kappa' \sum_{|\gamma| \leqslant 1} a_{\gamma,\gamma} \xi_\gamma^2 \tag{2.21}$$

almost everywhere in $\Omega \setminus \Omega_K$. For our choice in (2.18), this reduces to

$$c\xi_1^2 + (1-c)\xi_2^2 + (1-c)\xi_3^2 - \partial_x c \xi_1 \xi_3 + \partial_y c \xi_1 \xi_2 \geqslant \kappa'((1-c)\xi_3^2 + (1-c)\xi_2^2 + c\xi_1^2) \tag{2.22a}$$

$$\Leftrightarrow (\partial_y c)\xi_1\xi_2 - (\partial_x c)\xi_1\xi_3 \geqslant (\kappa' - 1)((1-c)\xi_3^2 + (1-c)\xi_2^2 + c\xi_1^2) \tag{2.22b}$$

almost everywhere in $\Omega \setminus \Omega_K$.

Items (8) and (9) are technical requirements that are necessary for the coercivity and the boundedness of B^c. They cannot be avoided without substantial changes to the forthcoming proofs. Let us point out that (2.21) can be deduced from (2.19) provided that $\kappa < \frac{1}{2}$ holds. We refer to [33] for a detailed proof. Equations (2.20) and (2.22b) enforce a certain well-behaviour on c by restricting for example the growth speed.

The following findings are a direct consequence of the foregoing results.

Proposition 2.1. *The bilinear form B^c from* (2.17) *is continuous.*

Proof. By using (2.19) and the Hölder inequality we obtain

$$\begin{aligned} |B^c(u,\varphi)| &\leqslant \sum_{|\alpha|,|\beta| \leqslant 1} \int_{\Omega \setminus \Omega_K} |a_{\alpha,\beta}||D^\beta u||D^\alpha \varphi| \,\mathrm{d}x \\ &\leqslant \max\{\kappa, 1\} \sum_{|\alpha|,|\beta| \leqslant 1} \int_{\Omega \setminus \Omega_K} |D^\beta u| \sqrt{|a_{\beta,\beta}|}\, |D^\alpha \varphi| \sqrt{|a_{\alpha,\alpha}|} \,\mathrm{d}x \\ &\leqslant K \|u\|_V \|\varphi\|_V, \end{aligned} \tag{2.23}$$

where K is a positive constant. We emphasise that the last estimate requires $c \leqslant 1$ almost everywhere to be valid. □

Proposition 2.2. *There exists a constant $\kappa' > 0$ such that the bilinear form B^c from* (2.17) *satisfies the estimate $B^c(u,u) \geqslant \kappa' |\!|\!| u |\!|\!|_V^2$.*

P r o o f. We replace ξ_α by $D^\alpha u$ and ξ_β by $D^\beta u$ in (2.21). Integrating the resulting inequality over $\Omega \setminus \Omega_K$ yields

$$\begin{aligned} B^c(u,u) &= \sum_{|\alpha|,|\beta| \leqslant 1} \int_{\Omega \setminus \Omega_K} a_{\alpha,\beta} D^\alpha u D^\beta u \,\mathrm{d}x \\ &\geqslant \kappa' \sum_{|\gamma| \leqslant 1} \int_{\Omega \setminus \Omega_K} a_{\gamma,\gamma} (D^\gamma u)^2 \,\mathrm{d}x \geqslant \kappa' |\!|\!| u |\!|\!|_V^2. \end{aligned} \tag{2.24}$$

□

To complete the proof of the coercivity of the bilinear form B^c we need a Friedrichs-like estimate of the form $\|u\|_V \leqslant K |\!|\!| u |\!|\!|_V$ with a positive constant K. The particular formulation and preliminaries that we need can be found in [52] as Theorem 2.3. We repeat it here verbatim for the sake of completeness but refer to its source for a detailed proof.

In the following theorem we denote by $W_c(X)$ the subset of weights on the space X which are bounded from above and below by positive constants on each compact subset $Q \subset X$, i.e. we only allow our weights to degenerate at the boundary of the domain. The next theorem also considers a constant A which is defined as follows. For an arbitrary domain X we assume that we can write

$$X = \bigcup_{k=1}^{\infty} X_k, \tag{2.25}$$

where $(X_k)_k$ is a sequence of bounded domains whose boundary can be locally described by functions satisfying the Lipschitz condition and where $X_k \subset \overline{X}_k \subset X_{k+1}$ holds for each k. Finally, let $X^k := X \setminus X_k$ and define

$$A_k = \sup_{\|u\|_{W^{m,p}(X;S_m)} \leqslant 1} \|u\|_{L^p(X^k;w_0)}, \tag{2.26}$$

where $w_0 \in S_m$ is the weight that corresponds to $|\alpha| = 0$. We define additionally $A := \lim\limits_{k\to\infty} A_k$. Obviously $A \in [0,1]$ always holds. This number A is also the ball measure of noncompactness of the embedding $W^{m,p}(X;S_m) \to L^p(X;w_0)$, see [52], [15]. One can interpret the number A as the distance from the embedding operator to the next closest compact operator from $W^{m,p}(X;S_m)$ into $L^p(X;w_0)$. Also, the numbers A_k can be understood as indicators on how much "weight" is put onto the function along the boundary. $A_k < 1$ means that there is at least some weight on the derivatives or inside the domain. Note that in our setup (2.26) simplifies to

$$A_k = \sup_{\|u\|_{W^{1,2}(\Omega\setminus\Omega_K;S_c)}\leqslant 1} \|u\|_{L^2(X^k)}, \tag{2.27}$$

where X^k is the complement of a set $X_k \subset \Omega\setminus\Omega_K$ and where S_c is the set of weights from (2.13).

For the following theorem it is important that $A < 1$, i.e. the weight is not completely concentrated on the boundary. Let us remark that this requirement is in accordance with the discrete theory established in [34], [22]. In the discrete setting, there should be at least one position with positive weight in the interior of the domain.

Let us also emphasise that for our task at hand, such a construction with the requirement that $A < 1$ is an additional regularity assumption on our image data f and the mask function c. Indeed, part of the boundary of the domain that we consider is fixed where $c \equiv 1$. Since Ω_k need boundaries that can be described locally by functions that fulfil the Lipschitz condition, this requirement carries over to the function c.

As already mentioned, the next theorem is a almost verbatim copy of Theorem 2.3 in [52].

Theorem 2.1. *Suppose $1 \leqslant p < \infty$ and $S_k \subset W_c(X)$. Let l be a functional on $W^{k,p}(X;S_k)$ with the following properties.*

(1) *l is continuous on $W^{k,p}(X;S_k)$.*
(2) *$l(\lambda u) = \lambda l(u)$ for all $\lambda > 0$ and all $u \in W^{k,p}(X,S_k)$.*
(3) *If $u \in P_{k-1} \cap W^{k,p}(X;S_k)$ (P_{k-1} being the set of all polynomials on $\mathbb{R}^n$ of degree less than k) and $l(u) = 0$, then $u = 0$.*

Let $A < 1$. Then there is a constant κ_0 such that

$$\int_X |u|^p w_0 \,\mathrm{d}x \leqslant \kappa_0\Bigl(|l(u)|^p + \sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(X;w_\alpha)}\Bigr). \tag{2.28}$$

Here, w_0 is the weight that corresponds to $|\alpha| = 0$.

The previous theorem can be seen as a generalisation to weighted spaces of a well-known theorem for constructing equivalent norms out of seminorms in regular Sobolev spaces. See Theorem 7.3.12 in [1]. Equation (2.28) can also be considered as a higher dimensional generalisation of the Hardy inequality. We refer to [42] for an extensive treatise on this inequality.

We now use Theorem 2.1 with $p = 2$, $k = 1$, $n = 2$, $w_0 \equiv 1$, $w_\alpha = 1 - c$ for all α, and

$$l(u) = \int_{\partial\Omega_K} u \, \mathrm{d}\mathcal{H}^1. \tag{2.29}$$

With these choices we obtain Friedrichs' inequality in our space V:

$$\|u\|^2_{L^2(\Omega\setminus\Omega_K)} \leqslant \kappa_0 |||u|||^2_V. \tag{2.30}$$

Equation (2.30) is the final key building block in showing the existence and uniqueness of a solution of our PDE. It allows us to show the coercivity of our bilinear form.

Proposition 2.3. *If* (2.30) *holds, i.e. the requirements of Theorem* 2.1 *are fulfilled for the choice of* l *from* (2.29) *and for our selection of weights for our space* V*, then the bilinear form* B^c *from* (2.17) *is coercive.*

P r o o f. Equation (2.30) immediately implies that $\|u\|^2_V \leqslant (1 + \kappa_0)|||u|||^2_V$. In combination with (2.24) it follows that

$$B^c(u, u) \geqslant \kappa' |||u|||^2_V \geqslant \frac{\kappa'}{1 + \kappa_0} \|u\|^2_V. \tag{2.31}$$

□

Proposition 2.3 completes the analysis of our bilinear form B^c. It remains to show that the right-hand side of our weak formulation is continuous if we want to apply the Theorem of Lax-Milgram. This final step is done in the following proposition.

Proposition 2.4. *The linear operator* F *from* (2.7) *is continuous provided that* g*,* Δf *and* $\nabla f/\sqrt{1-c}$ *are in* $L^2(\Omega \setminus \Omega_K)$.

P r o o f. We note that $\varphi \in V$ is 0 along $\partial\Omega_K$, and thus we can extend the boundary integral over that part. Using the Hölder inequality and Green's first identity, we obtain

$$\begin{aligned} |F(\varphi)| &\leqslant \int_{\Omega\setminus\Omega_K} |g|\,|\varphi| \,\mathrm{d}x + \left| \int_{\partial\Omega} h\varphi \,\mathrm{d}\mathcal{H}^1 \right| \\ &\leqslant \|g\|_{L^2(\Omega\setminus\Omega_K)} \|\varphi\|_{L^2(\Omega\setminus\Omega_K)} + \left| \int_{\Omega\setminus\Omega_K} (\Delta f \varphi + \nabla f \cdot \nabla\varphi) \,\mathrm{d}x \right| \\ &\leqslant \|g\|_{L^2(\Omega\setminus\Omega_K)} \|\varphi\|_V + \|\Delta f\|_{L^2(\Omega\setminus\Omega_K)} \|\varphi\|_V + \left| \int_{\Omega\setminus\Omega_K} \nabla f \cdot \nabla\varphi \,\mathrm{d}x \right|. \end{aligned} \tag{2.32}$$

The last integral can be estimated as

$$\left|\int_{\Omega\setminus\Omega_K} \nabla f\cdot\nabla\varphi\,\mathrm{d}x\right| = \left|\int_{\Omega\setminus\Omega_K} \frac{\nabla f}{\sqrt{1-c}}\sqrt{1-c}\,\nabla\varphi\,\mathrm{d}x\right| \leqslant \left\|\frac{\nabla f}{\sqrt{1-c}}\right\|_{L^2(\Omega\setminus\Omega_K)}\|\nabla\varphi\|_{L^2(\Omega\setminus\Omega_K;1-c)} \leqslant \left\|\frac{\nabla f}{\sqrt{1-c}}\right\|_{L^2(\Omega\setminus\Omega_K)}\|\varphi\|_V. \tag{2.33}$$

Therefore, it follows that

$$|F(\varphi)| \leqslant \Big(\|g\|_{L^2(\Omega\setminus\Omega_K)} + \|\Delta f\|_{L^2(\Omega\setminus\Omega_K)} + \left\|\frac{\nabla f}{\sqrt{1-c}}\right\|_{L^2(\Omega\setminus\Omega_K)}\Big)\|\varphi\|_V. \tag{2.34}$$

Thus, F is a bounded linear functional. □

The authors are not aware of a proof that shows that the requirement $\nabla f/\sqrt{1-c}\in L^2(\Omega\setminus\Omega_K)$ follows from the requirement

$$\omega_{\binom{1}{0}}^{-1/2} = \frac{1}{\sqrt{1-c}} \in L^1_{\mathrm{loc}}(\Omega).$$

As a consequence, both assumptions need to be stated separately. We can now combine our results to prove our main result.

Theorem 2.2. *The weak formulation* (2.7) *of the mixed boundary value problem* (2.6) *has a unique solution in the space V. In addition, we know that*

$$\|u\|_V \leqslant \frac{1+\kappa_0}{\kappa'}\|F\|_{V^*}, \tag{2.35}$$

where $\kappa'/(1+\kappa_0)$ is the constant from (2.31)*. Here, V^* denotes the dual space of V.*

P r o o f. From Proposition 2.1 and Proposition 2.3 it follows that our bilinear form B^c is bounded and coercive. Proposition 2.4 shows that the corresponding right-hand side F is bounded, too. Therefore, from the Theorem of Lax-Milgram (see [16]) it follows that there exists a unique $u\in V$ such that $B^c(u,\varphi)=F(\varphi)$ holds for all $\varphi\in V$. In addition, this u fulfils $\|u\|\leqslant(1+\kappa_0)/\kappa'\|F\|_{V^*}$. □

Theorem 2.2 shows that a unique solution exists in the space V, which is a subspace of $W^{1,2}(\Omega\setminus\Omega_K;S_c)$. We now use the following Proposition from [50], where it is stated as Proposition 2.1.3.

Proposition 2.5. *Let $D\subset\mathbb{R}^n$ be open, $1\leqslant p<\infty$ and m a non-negative integer. Suppose $\omega\in A_p(\mathbb{R}^n)$. Then $W^{m,p}_\omega(D)\subset W^{m,1}_{\mathrm{loc}}(D)$ and if D is bounded, $W^{m,p}_\omega(D)\subset W^{m,1}(D)$.*

It follows from Proposition 2.5 that $V \subset W^{1,1}(\Omega \setminus \Omega_K)$. We remark that Proposition 2.1.3 in [50] is stated for a single weight. However, it carries over to multiple weight functions since it relies only on the inclusions $L^p(D, \omega) \subset L^1_{\mathrm{loc}}(D)$.

2.2. What happens if $c \geqslant 1$? Let us shortly discuss the consequences of c exceeding its upper limit 1. Similar conclusions can also be drawn for the case $c \leqslant 0$, however, this latter situation usually does not occur in practice.

There are no restrictions on c when establishing the weak formulation. Applying $c \geqslant 1$, the main difference would be that $1 - c$ and c would have different signs. In order to follow the same strategy as in this paper, one would have to find suitable weights for the space definition. In [29] the authors discuss the situation when one of the weights in the weak formulation is negative and they suggest to multiply the negative weight with another negative constant to render it positive. Afterwards, a similar approach as in this paper could be possible.

In our situation there exists a second issue that may be harder to resolve. We require certain restrictions on the growth of the function c, which are of the form

$$|\partial_z c| \leqslant \kappa \sqrt{c(1 - c)} \tag{2.36}$$

for z being either x or y. The left-hand side of this inequality is always a non-negative real number. However, the right-hand side becomes complex-valued once c exceeds 1. These growth restrictions are important to show the coercivity of the bilinear form.

To conclude this section we remark that an alternative approach by means of the Helmholtz equation already exists for the case $c > 1$, see [24]. However, this approach uses different assumptions and yields a well-posedness theory in different spaces.

3. Conclusion

We have shown that a solution to the inpainting problem with the weighted Laplacian exists if the weight is a function that maps into the interval $[0, 1]$. The well-posedness of the task can be asserted if certain regularity conditions on the weight function c are met. These requirements are similar to what is needed to show existence and uniqueness of a solution in a discrete setting. The results in this manuscript complete the analysis of the inpainting problem with the Laplacian. While the theory for the discrete setup was complete for any choice of $c \geqslant 0$, the continuous theory only covered the setup where $c > 1$. This work complements the setup where c maps to $[0, 1]$.

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