Mathematica Bohemica

Mohammed Hemdaoui

A useful algebra for functional calculus

*Mathematica Bohemica*, Vol. 144 (2019), No. 1, 99–112

Persistent URL: http://dml.cz/dmlcz/147641

# A USEFUL ALGEBRA FOR FUNCTIONAL CALCULUS

Mohammed Hemdaoui, Oujda

Received June 16, 2017. Published online June 6, 2018.
Communicated by Vladimír Müller

*Remembering my Professor L. Waelbroeck 1929–2009*

*Abstract.* We show that some unital complex commutative LF-algebra of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$ (M. Hemdaoui, 2017) equipped with its natural convex vector bornology is useful for functional calculus.

*Keywords*: bornology; inductive limit; Fréchet space; functional calculus

*MSC 2010*: 46A08, 46A13, 46A17, 47A60

## 1. Introduction

Let $(\mathbf{A}, \|\cdot\|)$ be a unital commutative complex Banach algebra, $a \in \mathbf{A}$ and $\operatorname{sp}(a)$ be the spectrum of the element $a$. The positive function defined as

$$\delta(t) = \begin{cases} \|(a-t)^{-1}\|^{-1} & \text{if } t \notin \operatorname{sp}(a), \\ 0 & \text{if } t \in \operatorname{sp}(a) \end{cases}$$

is a Lipschitz function and it satisfies $\lim\limits_{|t|\to\infty} \delta(t)/|t| = 1$.

The set $\{\delta(t)(a-t)^{-1};\ t \notin \operatorname{sp}(a)\}$ is obviously bounded in $\mathbf{A}$.

More generally, Waelbroeck (see [16]) considered a unital locally convex complete commutative complex algebra $\mathbf{A}$ and a positive Lipschitz function $\delta$ on $\mathbb{C}$ satisfying as $|t| \to \infty$,

$$\varepsilon \leqslant \frac{\delta(t)}{|t|} \leqslant M, \tag{1.1}$$

where $\varepsilon > 0$, $M > 0$ are two positive real numbers independent of the variable $t$.

DOI: 10.21136/MB.2018.0063-17 

Let $a \in \mathbf{A}$ satisfy the following conditions:

- ▷ $\mathrm{sp}(a) = \delta^{-1}\{0\}$;
- ▷ The set $\{\delta(t)(a-t)^{-1};\ t \notin \delta^{-1}\{0\}\}$ is bounded in $\mathbf{A}$.

Sp$(a)$ is obviously compact. Waelbroeck in [16] showed that the closed subalgebra generated by $a$ and its resolvent function $(a-t)^{-1}$, $t \notin \delta^{-1}\{0\}$ is a unital complex Banach algebra. He gave two different descriptions of this Banach algebra and studied the extension of the holomorphic functional calculus.

In this note, let $\delta$ be a positive Lipschitz function on $\mathbb{C}$ and $a$ be an element of a unital complex complete locally convex algebra (or a unital complex complete convex bornological algebra) $\mathbf{A}$ satisfying the following conditions:

- ▷ $\mathrm{sp}(a) = \delta^{-1}\{0\}$;
- ▷ The set $\{\delta(t)(a-t)^{-1};\ t \notin \delta^{-1}\{0\}\}$ is bounded in $\mathbf{A}$.

Two questions come to mind:

(1) What can we say about the spectrum of $a$?
(2) Can we describe the closed subalgebra generated by $a$ and its resolvent function $(a-t)^{-1}$, $t \notin \delta^{-1}\{0\}$?

The spectrum of the element $a$ may be unbounded in $\mathbb{C}$. In this case, the closed subalgebra generated by $a$ and its resolvent function $(a-t)^{-1}$, $t \notin \delta^{-1}\{0\}$ need not to be a Banach algebra. My goal is to describe this closed subalgebra and see what it looks like. Fearing of falling into a purely abstract study since the spectrum is unknown in advance, I think it is wise and prudent to treat this study on a concrete example that checks the above assumptions that surely makes this subject more interesting.

In this work bornological language is used (see [3], [4], [8], [9], [10], [15], [17]).

Consider the Laplace operator $-\Delta$ acting on $\mathcal{S}'(\mathbb{R}^n)$, the vector space of tempered distributions on $\mathbb{R}^n$ (see [12]), such as

$$T \in \mathcal{S}'(\mathbb{R}^n) \to \mathfrak{F}(-\Delta T) = |\xi|^2 \mathfrak{F}(T) \in \mathcal{S}'(\mathbb{R}_n),$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}_n$, the dual space of $\mathbb{R}^n$.

For $t \in \mathbb{C} \setminus \mathbb{R}^+$ the operator $(-\Delta - t)^{-1}$ also acts on $\mathcal{S}'(\mathbb{R}^n)$ and is defined as

$$T \in \mathcal{S}'(\mathbb{R}^n) \to \mathfrak{F}((-\Delta - t)^{-1}T) = (|\xi|^2 - t)^{-1}\mathfrak{F}(T) \in \mathcal{S}'(\mathbb{R}_n),$$

where $\mathfrak{F}(T)$ is the Fourier transform of the tempered distribution $T$. The spectrum of the Laplace operator $-\Delta$ is $\mathbb{R}^+$. It is closed and unbounded in $\mathbb{C}$.

Let $\delta$ be positive Lipschitz function on $\mathbb{C}$ satisfying

$$\delta(t) = \inf\{1, d^n(t, \mathbb{R}^+), \forall\, n \in \mathbb{N}\},$$

where $d(t, \mathbb{R}^+)$ is the distance from the point $t \in \mathbb{C}$ to the set $\mathbb{R}^+$.

Function $\delta$ is very flat around the positive half of the axis ($\mathbb{R}^+$), for example one considers function

$$\delta(t) = \mathrm{e}^{-1/d^2(t,\mathbb{R}^+)} \quad \forall\, t \in \mathbb{C}.$$

The function $\delta$ satisfies

$$\frac{\delta(t)}{|t|} \leqslant 1 \quad \forall\, t \in \mathbb{C}.$$

$\mathbb{C}_\delta(-\Delta)$ denotes the complex unital commutative algebra generated by $-\Delta$ and its resolvent function $(-\Delta - t)^{-1}$, $t \notin \delta^{-1}\{0\} = \mathbb{R}^+$. An element of this algebra is an operator of the form

$$\sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} C_{k_i,k'_j}(-\Delta)^{k_i}(-\Delta - t)^{-k'_j}, \quad t \in \mathbb{C}\setminus\mathbb{R}^+,\ C_{k_i,k'_j} \in \mathbb{C},\ k_i, k'_j, m, n \in \mathbb{N}.$$

Let $(M, m, n) \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$. Consider the convex and balanced subset in $\mathbb{C}_\delta(-\Delta)$ defined as

$$B_{(M,m,n)} = \left\{ \sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} C_{k_i,k'_j}\delta(t)(-\Delta)^{k_i}(-\Delta - t)^{-k'_j},\ t \notin \delta^{(-1)}\{0\},\ \sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} |C_{k_i,k'_j}| \leqslant M \right\}.$$

These subsets $(B_{(M,m,n)})_{(M,m,n)\in\mathbb{R}^+\times\mathbb{N}\times\mathbb{N}}$ satisfy

(1) $B_{(M,m,n)} + B_{(M',m',n)} \subseteq B_{(M'',m'',n'')}$, where $M'' = M + M'$, $m'' = \sup\{m, m'\}$, $n'' = \sup\{n, n'\}$,
(2) $\lambda \cdot B_{(M,m,n)} \subseteq B_{(|\lambda|M,m,n)}$ for all $\lambda \in \mathbb{C}$,
(3) $B_{(M,m,n)} \cdot B_{(M',m',n')} \subseteq B_{(M'',m'',n'')}$, where $M'' = M \cdot M'$, $m'' = m + m'$, $n'' = n + n'$.

This family of subsets $(B_{(M,m,n)})_{(M,m,n)\in\mathbb{R}^+\times\mathbb{N}\times\mathbb{N}}$ is a filter basis compatible with the algebraic structure of algebra $\mathbb{C}_\delta(-\Delta)$. It is also a convex vector bornology basis (see [9], [10]). A subset $B \subset \mathbb{C}_\delta(-\Delta)$ is said to be bounded if it is contained in some $B_{(M,m,n)}$. This natural convex vector bornology on algebra $\mathbb{C}_\delta(-\Delta)$ is separated since no one-dimensional vector subspace of algebra $\mathbb{C}_\delta(-\Delta)$ is bounded (see [9], [10]).

The positive Lipschitz function $\delta$ and the operator $-\Delta$ satisfy

(1) $(-\Delta - t)^{-1}$ exists for $t \notin \delta^{-1}\{0\}$ in $\mathbb{C}_\delta(-\Delta)$;

(2) The set $\{\delta(t)(-\Delta - t)^{-1}, t \notin \delta^{-1}\{0\}\} \subseteq B_{(1,0,1)}$ is bounded in $\mathbb{C}_\delta(-\Delta)$.

A bounded multiplicative linear functional $\varphi$ on $\mathbb{C}_\delta(-\Delta)$ maps bounded sets in $\mathbb{C}_\delta(-\Delta)$ into bounded sets in $\mathbb{C}$. Functional $\varphi$ is defined if and only if its value $\varphi(-\Delta)$ is defined. But $(-\Delta - t)^{-1}$ exists in $\mathbb{C}_\delta(-\Delta)$ if and only if $t \notin \delta^{-1}(0) = \mathbb{R}^+$. So, $\varphi((-\Delta - t)^{-1}) = (-\varphi(\Delta) - t)^{-1}$ exists in $\mathbb{C}$ if and only if $t \notin \delta^{-1}(0) = \mathbb{R}^+$. This happens if and only if $-\varphi(\Delta) \in \mathbb{R}^+$.

The set $\Phi_{\mathbb{C}_\delta(-\Delta)}$ of bounded multiplicative linear functionals on $\mathbb{C}_\delta(-\Delta)$ is identified with $\mathbb{R}^+$ in the sense

$$\varphi \in \Phi_{\mathbb{C}_\delta(-\Delta)} \Leftrightarrow \exists!\, x \in \mathbb{R}^+ \text{ such that } \varphi(-\Delta) = x.$$

The separated convex bornological algebra $\mathbb{C}_\delta(-\Delta)$ is not necessarily complete. Its completion convex bornological algebra is $\widehat{\mathbb{C}_\delta(-\Delta)}$ (see [8], [9], [10]).

Our goal is to show that the completion convex bornological algebra $\widehat{\mathbb{C}_\delta(-\Delta)}$ is bornologically isomorphic to some unital complex commutative LF-algebra of class $\mathcal{C}^{(\infty)}$ functions $\mathbb{N}$-tempered on $\mathbb{R}^+$ (see [7]) equipped with its natural convex vector bornology. This bornological isomorphism is obtained by using the functional calculus theorem (see [6]) and Gelfand transform.

## 2. Functional calculus theorem

For $\varepsilon \in \,]\frac{1}{2}, 1[$ consider the closed sector in the complex plan defined by

$$T_\varepsilon = \{z \in \mathbb{C}\colon\ \Re e(z) \geqslant -\varepsilon \text{ and } |\arg(z+\varepsilon)| \leqslant \varepsilon\}.$$

$\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ denotes the unital commutative algebra of complex continuous functions $f$ on the closed sector $T_\varepsilon$ and of class $\mathcal{C}^{(1)}$ in $\mathring{T}_\varepsilon$ satisfying for a certain positive integer $N \in \mathbb{N}$

$$\|\!|f|\!\|_N = \sup_{t \in T_\varepsilon} \delta_0^N(t)|f(t)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^N(t)\delta^{-1}(t)|\overline{\partial} f(t)\,\mathrm{d}t| < \infty, \tag{2.1}$$

where $\delta_0(t) = 1/\sqrt{1+|t|^2}$, $\overline{\partial} f(t) = \partial(f(t)\,\mathrm{d}\overline{t})/\partial\overline{t}$ and $\delta$ is the positive Lipschitz function defined in the first paragraph.

$\mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ denotes the vector subspace of $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ of functions $f$ satisfying equation (2.1) for the same positive integer $N$, which normed by (2.1) is a complex Banach space.

For any pair $(N, N') \in \mathbb{N} \times \mathbb{N}$ such that $N \leqslant N'$ the canonical injection

$$i_{N,N'}\colon\ \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \hookrightarrow \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$$

is continuous since it satisfies

$$\| i_{N,N'}(f)\|_{N'} = \|f\|_{N'} \leqslant \|f\|_N.$$

For any pair $(N, N') \in \mathbb{N} \times \mathbb{N}$ the bilinear mapping

$$(f,g) \in \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \times \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to f \cdot g \in \mathcal{C}_{N+N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$$

is continuous since it satisfies

$$\|f \cdot g\|_{N+N'} \leqslant \|f\|_N \|g\|_{N'}.$$

The unital commutative algebra $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) = \bigcup_{N\in\mathbb{N}} \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is equipped with the inductive limit topology of Banach spaces $\mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$.

A set $B \subset \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is bounded if there exists a pair $(N', M') \in \mathbb{N} \times \mathbb{R}^+$ such that $B \subset \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ and

$$\|f\|_{N'} = \sup_{t\in T_\varepsilon} \delta_0^{N'}(t)|f(t)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^{N'}(t)\delta^{-1}(t)\big|\overline{\partial} f(t)\,\mathrm{d}t\big| \leqslant M' \quad \forall\, f \in B.$$

The family of convex and balanced sets $(B_{(N',M')})_{(N',M')\in\times\mathbb{R}^+}$, where

$$B_{(N',M')} = \{f \in \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})\colon\ \|f\|_{N'} \leqslant M'\},$$

is a natural convex vector bornology basis (see [9] and [10]) on the algebra $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$.

The complete convex bornological algebra $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) = \varinjlim_{N} \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is a $b$-algebra (see [15]).

**Theorem 2.1** ([6]). *There exists a bounded algebra homomorphism*

$$T\colon\ f \in \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to T(f) = f[-\Delta] \in \widehat{\mathbb{C}_\delta(-\Delta)}.$$

The proof of Theorem 2.1 is very technical, so we omit it (see [5], [11], [13], [14] and [18]).

**Corollary 2.2** ([6]). *The kernel of the bounded algebra homomorphism*

$$T\colon\ f \in \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to T(f) = f[-\Delta] \in \widehat{\mathbb{C}_\delta(-\Delta)}$$

*is the closed ideal* $\{f \in \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})\colon\ f(x) = 0$ *for all* $x \in \mathbb{R}^+\}$.

**Theorem 2.3.** *Let $f \in \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$. Then its restriction $f|_{\mathbb{R}^+}$ to $\mathbb{R}^+$ satisfies*

(1) *$f|_{\mathbb{R}^+}$ is of class $\mathcal{C}^{(\infty)}$ on $\mathbb{R}^+$;*

(2) *for all $n \in \mathbb{N}$, $\sup\limits_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| < \infty$,*

*where $\delta_0(x) = 1/\sqrt{1+x^2}$ and $f^{(n)}$ is the $n$th derivative of $f$.*

Proof. For $f \in \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ we have

$$\|f\|_N = \sup_{t\in T_\varepsilon} \delta_0^N(t)|f(t)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^N(t)\delta^{-1}(t)\big|\overline{\partial} f(t)\,\mathrm{d}t\big| < \infty.$$

Fix $\varepsilon \in \,]0,1[$ and $x$ in $\mathbb{R}^+$. Put $\varrho = (x+\varepsilon)\sin\varepsilon > 0$. Consider the closed discus centered at $x$ with radius $\varrho > 0$:

$$D(x,\varrho) = \{z \in T_\varepsilon\colon\ |z-x| \leqslant \varrho\}.$$

Function $f$ is continuous on the sector $T_\varepsilon$ and of class $\mathcal{C}^{(1)}$ in $\mathring{T}_\varepsilon$, using Pompieu formula (generalized Cauchy formula) on the closed discus $D(x,\varrho)$. We have

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)} \frac{f(t)}{t-x}\,\mathrm{d}t + \frac{1}{2\pi\mathrm{i}}\int_{D(x,\varrho)} \frac{1}{t-x}\overline{\partial} f(t)\,\mathrm{d}t,$$

where $\overline{\partial} f(t) = \partial(f(t)\,\mathrm{d}\overline{t})/\partial\overline{t}$.

For all $n \in \mathbb{N}$ function $t \in \partial D(x,\varrho) \to f(t)(t-x)^{-n}$ is continuous on the compact boundary. So the integral $\int_{\partial D(x,\varrho)} f(t)(t-x)^{-n}\,\mathrm{d}t$ exists for all $n \in \mathbb{N}$.

The differential form $t \in D(x,\varrho) \to (t-x)^{-n}\overline{\partial} f(t)$ is integrable on the compact discus $D(x,\varrho)$ since

$$\frac{\delta(t)}{|t-x|^n} \leqslant 1 \quad \forall n \in \mathbb{N}\ \forall t \in D(x,\varrho)$$

and

$$\int_{\partial D(x,\varrho)} \delta_0^N(t)\delta^{-1}(t)\frac{\delta(t)}{|t-x|^n}|\overline{\partial} f(t)\,\mathrm{d}t| \leqslant \int_{T_\varepsilon} \delta_0^N(t)\delta^{-1}(t)|\overline{\partial} f(t)\,\mathrm{d}t| < \infty.$$

Thus, for every positive integer $n \in \mathbb{N}$ we have

$$f^{(n)}(x) = \frac{n!}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)} \frac{f(t)}{(t-x)^{n+1}}\,\mathrm{d}t + \frac{n!}{2\pi\mathrm{i}}\int_{D(x,\varrho)} \frac{1}{(t-x)^{n+1}}\overline{\partial} f(t)\,\mathrm{d}t.$$

So $f|_{\mathbb{R}^+}$ is of class $\mathcal{C}^{(\infty)}$ on $\mathbb{R}^+$.

Now, we show for all $n \in \mathbb{N}$, $\sup\limits_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| < \infty$. In fact:

$$\delta_0^N(x) f^{(n)}(x) = \frac{n!}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)} \delta_0^N(x)\frac{f(t)}{(t-x)^{n+1}}\,\mathrm{d}t + \frac{n!}{2\pi\mathrm{i}}\int_{D(x,\varrho)} \delta_0^N(x)\frac{\overline{\partial} f(t)}{(t-x)^{n+1}}\,\mathrm{d}t.$$

For the first integral we have

$$\left|\frac{n!}{2\pi i}\int_{\partial D(x,\varrho)}\delta_0^N(x)\frac{f(t)}{(t-x)^{n+1}}\,dt\right| \leqslant \frac{n!}{2\pi\varrho^n}\sup_{t\in\partial B(x,\varrho)}\left(\frac{\delta_0(x)}{\delta_0(t)}\right)^N \sup_{t\in T_\varepsilon}\delta_0^N(t)|f(t)|.$$

For the second integral, since $\delta(t)/|t-x|^n \leqslant 1$ for all $n\in\mathbb{N}$, we get

$$\left|\frac{n!}{2\pi i}\int_{D(x,\varrho)}\delta_0^N(x)\frac{\overline{\partial}f(t)}{(t-x)^{n+1}}\,dt\right|$$
$$\leqslant \frac{n!}{2\pi}\sup_{t\in B(x,\varrho)}\left(\frac{\delta_0(x)}{\delta_0(t)}\right)^N \int_{T_\varepsilon}\delta_0^N(t)\delta^{-1}(t)|\overline{\partial}f(t)\,dt| < \infty.$$

Since $|t-x|=\varrho$, we have $t = x+\varrho e^{\theta i} = x+\varrho\cos\theta+\varrho\sin\theta i$. So

$$|t|^2 = (x+\varrho\cos\theta)^2+(\varrho\sin\theta)^2 \leqslant (x+\varrho)^2.$$

But $\varrho=(x+\varepsilon)\sin\varepsilon$, where $\varrho\in\,]0,1[$. We have $|t|^2\leqslant(2x+1)^2$. So

$$\left(\frac{\delta_0(x)}{\delta_0(t)}\right)^N = \left(\frac{1+|t|^2}{1+x^2}\right)^{N/2} \leqslant \left(\frac{1+(2x+1)^2}{1+x^2}\right)^{N/2} = \sqrt[N]{\frac{4x^2+4x+2}{1+x^2}}.$$

The positive continuous function of one real variable

$$x\in[0,\infty[\,\to \sqrt[N]{\frac{4x^2+4x+2}{1+x^2}}$$

is bounded on $\mathbb{R}^+$. There exists a positive real number $C_N>0$ such that

$$\sqrt[N]{\frac{4x^2+4x+2}{1+x^2}} \leqslant C_N \quad \forall\, x\geqslant 0.$$

So, for every positive integer $n$ and $\varrho$ fixed in $]0,1[$ we get

$$\left|\frac{n!}{2\pi i}\int_{\partial D(x,\varrho)}\delta_0^N(x)\frac{f(t)}{(t-x)^{n+1}}\,dt\right| \leqslant \frac{n!C_N}{2\pi(\varepsilon\sin\varepsilon)^n}\sup_{t\in T_\varepsilon}\delta_0^N(t)|f(t)|<\infty$$

and

$$\left|\frac{n!}{2\pi i}\int_{D(x,\varrho)}\delta_0^N(x)\frac{\overline{\partial}f(t)}{(t-x)^{n+1}}\,dt\right| \leqslant \frac{n!C_N}{2\pi(\varepsilon\sin\varepsilon)^n}\int_{T_\varepsilon}\delta_0^N(t)\delta^{-1}(t)|\overline{\partial}f(t)\,dt|<\infty.$$

Finally, for all $n\in\mathbb{N}\Rightarrow\exists\,M_{(\varepsilon,n,N)}>0$ such that $\delta_0^N(x)|f^{(n)}(x)|\leqslant M_{(\varepsilon,n,N)}$,

$$\sup_{t\in T_\varepsilon}\delta_0^N(t)|f(t)| + \frac{1}{2\pi}\int_{T_\varepsilon}\delta_0^N(t)\delta^{-1}(t)|\overline{\partial}f(t)\,dt|<\infty.$$

So

$$\forall\, n\in\mathbb{N},\quad \sup_{x\in\mathbb{R}^+}\delta_0^N(x)|f^{(n)}(x)|\leqslant M_{(\varepsilon,n,N)}|\!|\!|f|\!|\!|_N<\infty,$$

where $M_{(\varepsilon,n,N)} = n!\,C_N/2\pi(\varepsilon\sin\varepsilon)^n$. □

**Definition 2.4** ([7]). Let $N$ be a positive integer. A $\mathcal{C}^{(\infty)}$ complex function $f$ on $\mathbb{R}^+$ is $N$-tempered if for any positive integer $n \in \mathbb{N}$ the $n$th derivative $f^{(n)}$ of $f$ satisfies

$$\sup_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| < \infty,$$

where $\delta_0(x) = 1/\sqrt{1+x^2}$.

For every positive integer $N \in \mathbb{N}$, $\mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ denotes the vector space of functions of class $\mathcal{C}^{(\infty)}$ $N$-tempered on $\mathbb{R}^+$ equipped with the family of norms

$$f \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to \|f\|_{N,n} = \sup_{0\leqslant k\leqslant n} \sup_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(k)}(x)|, \quad n \in \mathbb{N}.$$

$(\mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}), (\|\cdot\|_{N,n})_{n\in\mathbb{N}})$ is a Fréchet space (see [12]).

For any pair $(N, N')$ of positive integers such that $N \leqslant N'$ the canonical injection

$$i_{N,N'}\colon\ \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \hookrightarrow \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$$

is obviously continuous since

$$\|i_{N,N'}(f)\|_{N',n} = \|f\|_{N',n} \leqslant \|f\|_{N,n} \quad \forall\, f \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}).$$

Consider the bilinear mapping

$$(f,g) \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \times \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to f\cdot g \in \mathcal{C}_{N+N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}).$$

Let $n \in \mathbb{N}$, the $n$th derivative $(f\cdot g)^{(n)}$ of the function $f\cdot g$ at any point $x \in \mathbb{R}^+$ by Leibniz formula is

$$(f(x)g(x))^{(n)} = \sum_{p=0}^{n} \binom{n}{p} f^{(n-p)}(x)\cdot g^{(p)}(x),$$

where $\binom{n}{p} = n!/(n-p)!\,p!$ is the binomial coefficient.

So

$$\delta_0^{N+N'}(x)|(f(x)g(x))^{(n)}| \leqslant 2^n \sup_{x\in\mathbb{R}^+} \sup_{0\leqslant k\leqslant n} \delta_0^N(x)|f^{(k)}(x)| \sup_{x\in\mathbb{R}^+} \sup_{0\leqslant p\leqslant n} \delta_0^{N'}(x)|g^{(p)}(x)|.$$

Then

$$\forall\, n \in \mathbb{N}, \quad \|f\cdot g\|_{N+N',n} \leqslant 2^n \|f\|_{N,n}\|g\|_{N',n}.$$

This inequality shows the continuity of the bilinear mapping

$$(f,g) \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \times \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to f\cdot g \in \mathcal{C}_{N+N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}).$$

The vector space $\mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C}) = \varinjlim_{N} \mathcal{C}^{(\infty)}_{N}(\mathbb{R}^+,\delta_0,\mathbb{C}) = \bigcup_{N\in\mathbb{N}} \mathcal{C}^{(\infty)}_{N}(\mathbb{R}^+,\delta_0,\mathbb{C})$ of functions of class $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered on $\mathbb{R}^+$ equipped with the inductive limit topology of Fréchet spaces $\mathcal{C}^{(\infty)}_{N}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is a unital complex commutative topological algebra since the bilinear mapping

$$(f,g)\in \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C})\times \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C}) \to f\cdot g\in \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

is continuous.

Let $(N,(M_{N,n})_{n\in\mathbb{N}})\in\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}$ and the convex and balanced subset be defined as

$$B_{(N,(M_{N,n})_{n\in\mathbb{N}})} = \{f\in \mathcal{C}^{(\infty)}_{N}(\mathbb{R}^+,\delta_0,\mathbb{C})\colon\ \forall\, n\in\mathbb{N},\ \|f\|_{N,n}\leqslant M_{N,n}\}.$$

The family $(B_{(N,(M_{N,n})_{n\in\mathbb{N}})})_{(N,(M_{N,n})_{n\in\mathbb{N}})\in\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}}$ of subsets satisfies

(1) $B_{(N,(M_{N},n)_{n\in\mathbb{N}})} + B_{(N',(M'_{N'},n)_{n\in\mathbb{N}})} \subseteq B_{(N'',(M''_{N''},n)_{n\in\mathbb{N}})}$, where for all $n\in\mathbb{N}$, $M''_{N'',n} = M_{N,n} + M'_{N',n}$ and $N'' = \sup(N,N')$,

(2) $\lambda\cdot B_{(N,(M_{N},n)_{n\in\mathbb{N}})} \subseteq B_{(N,(|\lambda|\cdot M_{N},n)_{n\in\mathbb{N}})}$ for all $\lambda\in\mathbb{C}$,

(3) $B_{(N,(M_{N},n)_{n\in\mathbb{N}})} \cdot B_{(N',(M'_{N'},n)_{n\in\mathbb{N}})} \subseteq B_{(N'',(M''_{N''},n)_{n\in\mathbb{N}})}$, where for all $n\in\mathbb{N}$, $M''_{N'',n} = \sum\limits_{p=0}^{n} M_{N,n-p}\cdot M'_{N',p}$.

This family $(B_{(N,(M_{N,n})_{n\in\mathbb{N}})})_{(N,(M_{N,n})_{n\in\mathbb{N}})\in\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}}$ of subsets is a natural convex vector bornology basis (see [9], [10]) compatible with the vector structure of the algebra $\mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C})$.

A set $B\subset \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is bounded if there exists a pair $(N,(M_{N,n})_{n\in\mathbb{N}})$ in $\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}$ such that $B\subset B_{(N,(M_{N,n})_{n\in\mathbb{N}})}\subset \mathcal{C}^{(\infty)}_{N}(\mathbb{R}^+,\delta_0,\mathbb{C})$.

$\mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C})$ equipped with this family of bounded sets becomes a unital complex complete commutative convex bornological algebra (see [9], [10]). $\mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is a unital complex commutative LF-algebra.

Theorem 2.3 ensures that the restriction mapping

$$\mathcal{R}_N\colon\ \mathcal{C}^{(1)}_{N}(T_\varepsilon,\delta,\delta_0,\mathbb{C}) \to \mathcal{C}^{(\infty)}_{N}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

is continuous. Since

$$\mathcal{C}^{(1)}(T_\varepsilon,\delta,\delta_0,\mathbb{C}) = \varinjlim_{N} \mathcal{C}^{(1)}_{N}(T_\varepsilon,\delta,\delta_0,\mathbb{C}) \text{ and } \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C}) = \varinjlim_{N} \mathcal{C}^{(\infty)}_{N}(\mathbb{R}^+,\delta_0,\mathbb{C}),$$

(see [1], [2], [12]) there exists a unique continuous restriction mapping

$$\mathcal{R}\colon\ f\in \mathcal{C}^{(1)}(T_\varepsilon,\delta,\delta_0,\mathbb{C}) \to \mathcal{R}(f) = f|_{\mathbb{R}^+}\in \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

satisfying for every positive integer $N \in \mathbb{N}$ the commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) & \xrightarrow{\mathcal{R}_N} & \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \\
{\scriptstyle j_N}\big\downarrow & \searrow^{\mathcal{P}_N} & \big\downarrow{\scriptstyle i_N} \\
\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) & \xrightarrow{\mathcal{R}} & \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}),
\end{array}
\tag{2.2}
$$

where

$$\mathcal{P}_N = i_N \circ \mathcal{R}_N = \mathcal{R} \circ j_N \quad \forall\, N \in \mathbb{N}$$

and

$i_N \colon \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ is the continuous canonical injection,
$j_N \colon \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is the continuous canonical injection,
$\mathcal{R}_N \colon \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ is the continuous restriction mapping.

## 3. Bornological isomorphism between $\widehat{\mathbb{C}_\delta(-\Delta)}$ and $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$

In Section 1, we considered the Laplace operator acting on $\mathcal{S}'(\mathbb{R}^n)$ (the space of tempered distributions on $\mathbb{R}^n$) and $\delta$ being the Lipschitz function considered in the first paragraph.

The spectrum of the Laplace operator is $\mathbb{R}^+$.

$\mathbb{C}_\delta(-\Delta)$ denotes the unital commutative complex algebra generated by the Laplace operator $-\Delta$ and the resolvent function $(-\Delta - t)^{-1}$, $t \notin \delta^{-1}(0) = \mathbb{R}^+$ equipped with the described convex vector bornology. Its completion convex bornological algebra is $\widehat{\mathbb{C}_\delta(-\Delta)}$ (see [8], [9], [10]).

$\Phi_{\mathbb{C}_\delta(-\Delta)} = (\varphi_x)_{x \in \mathbb{R}^+}$ denotes the set of all bounded multiplicative linear functionals on the convex bornological algebra $\mathbb{C}_\delta(-\Delta)$ identified with $\mathbb{R}^+$.

The Gelfand transform is the algebra homomorphism defined by

$$u \in \mathbb{C}_\delta(-\Delta) \to \widehat{G}(u) = \widehat{u} \colon \mathbb{R}^+ \to \mathbb{C},$$
$$\widehat{u}(\varphi_x) = \varphi_x(u) = u(x) \in \mathbb{C}, \quad x \in \mathbb{R}^+.$$

But $u \in \mathbb{C}_\delta(-\Delta)$, so it is an operator of the form

$$u = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} C_{k_i, k'_j} (-\Delta)^{k_i} (-\Delta - t)^{-k'_j}, \quad t \in \mathbb{C} \setminus \mathbb{R}^+,\ C_{k_i, k'_j} \in \mathbb{C},\ k_i, k'_j, m, n \in \mathbb{N}.$$

So,

$$u(x) = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} C_{k_i, k'_j} \frac{x^{k_i}}{(x - t)^{k'_j}} \quad \forall\, t \in \mathbb{C} \setminus \mathbb{R}^+.$$

For a fixed $t$ the function of variable $x \in \mathbb{R}^+$

$$\delta(t)u(x) = \sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} C_{k_i,k'_j}\delta(t)\frac{x^{k_i}}{(x-t)^{k'_j}} \quad \forall\, t \in \mathbb{C}\setminus\mathbb{R}^+$$

is of class $\mathcal{C}^{(\infty)}$ and $k$-tempered on $\mathbb{R}^+$ since by Leibniz formula, the derivative of order $n$ of the function $u$ with respect to the variable $x$ is

$$\delta(t)u^{(n)}(x) = \sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} C_{k_i,k'_j}\delta(t)\sum_{p=0}^{n}\binom{n}{p}(x^{k_i})^{(n-p)}\Big(\frac{1}{(x-t)^{k'_j}}\Big)^{(p)}.$$

It is a finite sum of terms each one of which is at most $k$-tempered on $\mathbb{R}^+$ since we have

$$\frac{\delta(t)}{|x-t|^l} \leqslant 1 \quad \forall\, l \in \mathbb{N},\ \forall\, x \in \mathbb{R}^+,\ \forall\, t \in \mathbb{C}\setminus\mathbb{R}^+.$$

Thus, we have

$$\forall\, n \in \mathbb{N}, \quad \sup_{x\in\mathbb{R}^+} \delta_0^k(x)\delta(t)|u^{(n)}(x)| < \infty,$$

where $k = \sup_{0\leqslant i\leqslant m} k_i \in \mathbb{N}$.

Obviously, the Gelfand transform $\widehat{G}\colon \mathbb{C}_\delta(-\Delta) \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is injective and bounded algebra homomorphism. It can be extended (see [8], [9], [10]) as a unique bounded algebra homomorphism $\widetilde{\widehat{G}}\colon \widehat{\mathbb{C}_\delta(-\Delta)} \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ satisfying the diagram

(3.1)
$$\begin{array}{ccc} \mathbb{C}_\delta(-\Delta) & \xrightarrow{\ \mathcal{J}\ } & \widehat{\mathbb{C}_\delta(-\Delta)} \\ & \searrow^{\widehat{G}} & \downarrow{\scriptstyle \widetilde{\widehat{G}}} \\ & & \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}), \end{array}$$

where $\widetilde{\widehat{G}} \circ \mathcal{J} = \widehat{G}$ and $\mathcal{J}\colon \mathbb{C}_\delta(-\Delta) \to \widehat{\mathbb{C}_\delta(-\Delta)}$ is the canonical bounded algebra homomorphism.

**Theorem 3.1.** *The continuous restriction mapping*

$$\mathcal{R}\colon \mathcal{C}^{(1)}(T_\varepsilon,\delta,\delta_0,\mathbb{C}) \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

*is surjective.*

P r o o f. Let $f \in \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+, \delta_0, \mathbb{C}) \Rightarrow \exists\, (N, (M_N, n)_{n\in\mathbb{N}}) \in \mathbb{N} \times {\mathbb{R}^+}^{\mathbb{N}}$ such that

$$\forall\, n \in \mathbb{N}, \quad \sup_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| = M_{N,n} < \infty.$$

Applying Borel theorem and Lemma 3.1 of [7] for all $\varepsilon > 0$, $f$ can be extended as a $\mathcal{C}^{(1)}$ function $\widetilde{f}_\varepsilon$ defined in $[-\varepsilon, \infty[ + \mathbb{R}\mathrm{i}$ by a locally finite series

$$\widetilde{f}_\varepsilon(x + y\mathrm{i}) = \sum_{n=0}^{\infty} \frac{1}{n!} f_\varepsilon^{(n)}(x)\varphi_n(y)(y\mathrm{i})^n$$

such that

$$\frac{\partial}{\partial \overline{z}} \widetilde{f}_\varepsilon(z)|_{\{z=x\}} = 0 \quad \forall\, x \in [-\varepsilon, \infty[,$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}^{(\infty)}$ positive functions with compact support in $]-1, 1[$ satisfying

(1) $\varphi_n(y) = 1$ if $|y| \leqslant \frac{1}{2}\alpha_n^{-1}$ for all $n \in \mathbb{N}$,
(2) $\operatorname{supp}(\varphi_{n+1}) \subset \operatorname{supp}(\varphi_n)$ for all $n \in \mathbb{N}$,

where

$$\forall\, n \in \mathbb{N}, \quad M_{N,n} = \sup_{x\in[-\varepsilon,\infty[} \delta_0^N(x)|f_\varepsilon^{(n)}(x)| \quad \text{and} \quad \alpha_n = \sum_{p=0}^{n+1}(M_{N,p} + 1).$$

$\widetilde{f}_\varepsilon$ is defined on the sector $T_\varepsilon$ since $T_\varepsilon \subset [-\varepsilon, \infty[ + \mathbb{R}\mathrm{i}$.

A calculation leads to

$$\|\widetilde{f}_\varepsilon\|_{N+2} = \sup_{z\in T_\varepsilon} \delta_0^{N+2}(z)|\widetilde{f}_\varepsilon(z)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^{N+2}(t)\delta^{-1}(t)\Big|\frac{\partial}{\partial \overline{t}}\widetilde{f}_\varepsilon(t)\,\mathrm{d}\overline{t}\,\mathrm{d}t\Big| < \infty,$$

where $\delta_0(z) = 1/\sqrt{1 + |z|^2}$ and $\delta(z) = \inf\{1, d^n(z, \mathbb{R}^+)$ for all $n \in \mathbb{N}\}$.

Thus $\widetilde{f}_\varepsilon \in \mathcal{C}^{(1)}_{N+2}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \subset \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$. □

Since the bounded restriction algebra homomorphism $\mathcal{R}$ is surjective (Theorem 3.1) there exists a bounded algebra homomorphism

$$\widetilde{T}\colon\ \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to \widehat{\mathbb{C}_\delta(-\Delta)}$$

satisfying the commutative diagram

(3.2)
$$\begin{array}{ccc} \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) & \xrightarrow{\ T\ } & \widehat{\mathbb{C}_\delta(-\Delta)} \\ & \searrow^{\mathcal{R}} & \uparrow{\scriptstyle \widetilde{T}} \\ & & \mathcal{C}^{(\infty)}_{\mathbb{N}}(\mathbb{R}^+, \delta_0, \mathbb{C}), \end{array}$$

where $\widetilde{T} \circ \mathcal{R} = T$, and $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ is the algebra of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$.

$\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is the algebra of functions defined in Section 2.

From diagrams 3.1 and 3.2 we get the following equations

$$\widetilde{\widetilde{\widehat{G}}} \circ \mathcal{J} = \widehat{G} \quad \text{and} \quad \widetilde{T} \circ \mathcal{R} = T, \tag{3.3}$$

$$\widetilde{T} \circ \widehat{G} = \mathcal{J} \quad \text{and} \quad \widetilde{\widetilde{\widehat{G}}} \circ T = \mathcal{R}. \tag{3.4}$$

From equations (3.3) and (3.4) we get the following new equations

$$\widetilde{\widetilde{\widehat{G}}} \circ \widetilde{T} = 1_{\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+ \delta_0, \mathbb{C})} \quad \text{and} \quad \widetilde{T} \circ \widetilde{\widetilde{\widehat{G}}} = 1_{\widehat{\mathbb{C}_\delta(-\Delta)}}, \tag{3.5}$$

where $1_{\widehat{\mathbb{C}_\delta(-\Delta)}}$ and $1_{\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})}$ are, respectively, the identity operators on $\widehat{\mathbb{C}_\delta(-\Delta)}$ and $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$.

The algebras $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ and $\widehat{\mathbb{C}_\delta(-\Delta)}$ are bornologically isomorphic.

The complete convex bornological algebra $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$ is then useful for functional calculus.

A c k n o w l e d g e m e n t s. The author would like to express his thanks to the referee for his suggestions and remarks concerning this paper.

*References*


[1] *N. Bourbaki*: Éléments de Mathématique. Fasc. XXVII: Algèbre Commutative. Hermann, Paris, 1961. (In French.) zbl MR

[2] *N. Bourbaki*: Éléments de Mathématique. Fasc. XXXII: Théories spectrales. Hermann, Paris, 1967. (In French.) zbl MR

[3] *H. Buchwalter*: Espaces vectoriels bornologiques. Publ. Dép. Math., Lyon *2* (1965), 2–53. (In French.) zbl MR

[4] *J. P. Ferrier*: Séminaire sur les Algèbres complètes. Lectures Notes in Mathematics 164. Berlin, Springer, 1970. (In French.) zbl MR doi

[5] *M. Hemdaoui*: Approximations analytique sur les compacts de $\mathbb{C}^n$. Thèse 3ème cycle. Année universitaire (Février) 1982. Université des Sciences et Technologies. Lille 1, France (In French.); Avaible at `https://ori-nuxeo.univ-lille1.fr/nuxeo/site/esupversions/1b9d01d3-5d59-496d-877d-9ca2a836b129`.

[6] *M. Hemdaoui*: Calcul symbolique et l'opérateur de Laplace. Thèse Année Académique 1986–1987: ULB. Bruxelles, Belgique (In French.); Avaible at `https://dipot.ulb.ac.be/dspace/bitstream/2013/213480/1/680effa8-3d2a-4225-bcb6-c0c6e156cac8.txt`.

[7] *M. Hemdaoui*: An application of Mittag-Leffler lemma to the L.F algebra of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$. Sarajevo J. Math. *13* (2017), 61–70. MR doi

[8] *H. Hogbe-Nlend*: Complétion, tenseurs et nucléarité en bornologie. J. Math. Pures Appl., IX. Sér. *49* (1970), 193–288. (In French.) zbl MR

[9] *H. Hogbe-Nlend*: Theory of Bornologies and Application. Lecture Notes 213. Springer, Berlin, 1971. (In French.) zbl MR doi

[10] *H. Hogbe-Nlend*: Bornologies and Functional Analysis. Introductory Course on the Theory of Duality Topology-Bornology and Its Use in Functional Analysis. North-Holland Mathematics Studies 26; Notas de Matematica 62. North Holland Publishing Company, Amsterdam, 1977. zbl MR doi

[11] *Nguyen The Hoc*: Croissance des coefficients spectraux et calcul fonctionnel. Thèse Année Académique 1975–1976. ULB. Bruxelles, Belgique (In French.); Avaible at `https://dipot.ulb.ac.be/dspace/bitstream/2013/214383/1/62487895-bcef-4c2e-b604-8eab9d4023a7.txt`.

[12] *J. Horvarth*: Topological Vector Spaces and Distributions. Vol. 1. Addison-Wesley Series in Mathematics. Addison-Wesley, Ontario, 1966. zbl MR

[13] *F. H. Vasilescu*: Analytic Functional Calculus and Spectral Decompositions. Mathematics and Its Applications 1 (East European Series). D. Reidel Publishing compagny, Dordrecht, 1982. zbl MR

[14] *L. Waelbroeck*: Le calcul symbolique dans les algèbres commutatives. C. R. Acad. Sci., Paris *238* (1954), 556–558. (In French.) zbl MR

[15] *L. Waelbroeck*: Études spectrales des algèbres complètes. Mém. Cl. Sci., Collect. Octavo, Acad. R. Belg. *31* (1960), 142 pages. (In French.) zbl MR

[16] *L. Waelbroeck*: Calcul symbolique lié à la croissance de la résolvante. Rend. Sem. Mat. Fis. Milano *34* (1964), 51–72. (In French.) zbl MR doi

[17] *L. Waelbroeck*: The holomorphic functional calculus and non-Banach algebras. Algebras in Analysis. Proc. Instr. Conf. and NATO Advanced Study Inst., Birmingham 1973. Academic Press, London, 1975, pp. 187–251. zbl MR

[18] *W. Wrobel*: Extension du calcul fonctionnel holomorphe et applications à l'approximation. C. R. Acad. Sci. Paris, Sér. A *275* (1972), 175–177. (In French.) zbl MR



*Author's address*: *Mohammed Hemdaoui*, Laboratoire Analyse Géométrie et Applications (LAGA), Université Mohamed Premier, Faculté des Sciences, Département de Mathématiques, Oujda, Maroc, e-mail: `mhemdaoui@yahoo.fr`.