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THE MULTIPLIER FOR THE WEAK MCSHANE INTEGRAL

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Abstract. The multiplier for the weak McShane integral which has been introduced by M. Saadoune and R. Sayyad (2014) is characterized.

Keywords: McShane integral; weak McShane integral; multiplier

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1. Introduction

In a previous paper [9], we defined a new method of integrability, named *weak McShane integrability*, for functions defined on a σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$ into a Banach space X. In the same paper we studied its relation with the Pettis integral, and proved that a function from S into X is weakly McShane integrable on each member of Σ if and only if it is Pettis and weakly McShane integrable on S. We also proved that if a function is weakly McShane integrable on S, then it is Pettis integrable on each member of an increasing sequence of measurable sets of finite measure with union S. Moreover, it can be seen from our methods that for weakly sequentially complete spaces or for spaces that do not contain a copy of c_0, a weakly McShane integrable function on S is always Pettis integrable. Moreover, in the same paper, a class of functions which are weakly McShane integrable on S but not McShane integrable on S is also presented.

In [1], Di Piazza and Marraffa proved the multiplier theorem for the McShane integral, that is, if $f\colon S \to X$ is McShane integrable on S and $h \in L^\infty(S, \mathbb{R})$, then the function $hf\colon S \to X$ is McShane integrable. The proof of this result uses the usual approximation techniques and the Cauchy criterion. In the spirit of these results, it is natural to address the following question:

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If $f\colon S \to X$ is a weakly McShane integrable function on S and $h \in L^{\infty}(S,\mathbb{R})$, does $hf\colon S \to X$ have to be weakly McShane integrable on S? If the answer is no, what are the conditions for hf becoming weakly McShane integrable on S?

In the present work, we give a positive answer to above question. More generally, it will be shown that if $h \in L^{\infty}(S,\mathbb{R})$, $f\colon S \to X$ is weakly McShane integrable and integrably bounded function on S and X is w^*-separable, then the function $hf\colon S \to X$ is weakly McShane integrable on S (see Theorem 3.1). Our proof makes use of the Vitali-Lebesgue convergence theorem for the Pettis integral and the diagonal process.

2. Preliminaries

In the sequel, X stands for a Banach space, whose norm is denoted by $\|\cdot\|$, and X^* stands for the topological dual of X. The closed unit ball of X^* is denoted by $\overline{B}_{X^*}$. By w, we denote the weak topology of X, and by w^* the weak topology of X^*. Let (S,Σ,μ) be a positive measure space. By Σ_f we denote the collection of all measurable sets of finite measure. By $L^1_{\mathbb{R}}(\mu)$ we denote the Banach space of all (equivalence classes of) Σ-measurable and μ-integrable real-valued functions on Ω, equipped with the classical norm $\|f\|_1 := \int_S |f|\,\mathrm{d}\mu$, and by $L^{\infty}(S,\mathbb{R})$ the set of all real-valued, bounded almost everywhere on S functions. If $h \in L^{\infty}(S,\mathbb{R})$, we denote $\|h\|_{\infty} = \inf\{M > 0\colon |h| \leqslant M\ \mu\text{-a.e}\}$. A function $f\colon S \to X$ is said to be *scalarly integrable* (or *Dunford integrable*) if for every $x^* \in X^*$, the real-valued function $\langle x^*, f\rangle$ belongs to $L^1_{\mathbb{R}}(\mu)$. In this case, for each $E \in \Sigma$ there is $x^{**}_E \in X^{**}$ such that

$$\langle x^*, x^{**}_E\rangle = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu.$$

The vector x^{**}_E is called the *Dunford integral* of f over E. In the case where $x^{**}_E \in X$ for all $E \in \Sigma$, f is called *Pettis integrable* and we write $(\mathcal{P}e)$-$\int_E f\,\mathrm{d}\mu$ instead of x^{**}_E to denote the *Pettis integral* of f over E. If $f\colon S \to X$ is a Pettis integrable function, then the set $\{\langle x^*, f\rangle\colon x^* \in \overline{B}_{X^*}\}$ is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$ (see [8], page 162). A function $f\colon S \to X$ is said to be integrably bounded if the real-valued function $\|f\|$ is a member of $L^1_{\mathbb{R}}(\mu)$.

Definition 2.1 ([4], Definition 246A). A subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\mu)$ is *uniformly integrable* if for every $\varepsilon > 0$ we can find a set $E \in \Sigma_f$ and an $M \geqslant 0$ such that

$$\int_S (|h| - M1_E)^+\,\mathrm{d}\mu \leqslant \varepsilon \quad \text{for every } h \in \mathcal{H},$$

where $(|h| - M1_E)^+ := \max(|h| - M1_E, 0)$.

Theorem 2.1 ([4], Theorem 246G). *A subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\mu)$ is uniformly integrable if and only if*

(1) $\sup\limits_{h\in\mathcal{H}} \left|\int_A h\,\mathrm{d}\mu\right| < \infty$ *for every μ-atom* (*in the measure space sense* (*see* [4], 211I)), *and every $A \in \Sigma$,*

(2) *for every $\varepsilon > 0$ there are $E \in \Sigma_f$ and $\eta > 0$ such that $\left|\int_F h\,\mathrm{d}\mu\right| \leqslant \varepsilon$ for every $h \in \mathcal{H}$ and for every $F \in \Sigma$ with $\mu(F \cap E) \leqslant \eta$.*

Remark 2.1 ([4], Corollary 246I). Note that when (S, Σ, μ) is a probability space, (1) and (2) may be replaced with

$$\lim_{\lambda\to\infty} \sup_{h\in\mathcal{H}} \int_{\{t\in S\colon\, |h(t)|\geqslant\lambda\}} |h|\,\mathrm{d}\mu = 0.$$

Remark 2.2. Note that if $f\colon S \to X$ is a scalarly integrable and integrably bounded function, then the set $\{\langle x^*, f\rangle\colon\, x^* \in \overline{B}_{X^*}\}$ is uniformly integrable.

The following well known result, which is the Pettis analogue of the classical Vitali convergence theorem, will play a key role in this work (see [6], [8]).

Theorem 2.2. *Let $f\colon S \to X$ be a scalarly integrable function satisfying the following two conditions:*

(i) $\{\langle x^*, f\rangle\colon\, x^* \in \overline{B}_{X^*}\}$ *is uniformly integrable,*

(ii) *there exists a sequence (f_n) of Pettis integrable functions from S into X such that*

$$\lim_{n\to\infty} \int_E \langle x^*, f_n\rangle\,\mathrm{d}\mu = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu$$

for each $x^* \in X^*$ and each $E \in \Sigma$.

Then f is Pettis integrable and

$$w\text{-}\lim_{n\to\infty} (\mathcal{P}e)\text{-}\int_E f_n\,\mathrm{d}\mu = (\mathcal{P}e)\text{-}\int_E f\,\mathrm{d}\mu$$

for all $E \in \Sigma$.

Condition (i) *may be replaced with*

(i)$'$ $\{\langle x^*, f\rangle\colon\, x^* \in \overline{B}_{X^*}\}$ *is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$* (*see* [4], *Theorem* 247C).

Let (S, Σ, μ) be a σ-finite positive measure space and $\mathcal{T} \subset \Sigma$ a topology on S making $(S, \mathcal{T}, \Sigma, \mu)$ a *quasi-Radon* measure space which is *outer regular*, that is, such that

$$\mu(E) = \inf\{\mu(G)\colon\, E \subset G,\, G \in \mathcal{T}\}, \quad E \in \Sigma.$$

For an extensive study of quasi-Radon measure spaces, the reader is referred to [5], Chapter 41. A *partial McShane partition* is a countable (may be finite) collection $\{(E_i, t_i)\}_{i\in I}$, where the E_i's are pairwise disjoint measurable subsets of S with finite measure and t_i a point of S for each $i \in I$. A *generalized McShane partition* of S is an infinite partial McShane partition $\{(E_i, t_i)\}_{i\geqslant 1}$ such that $\mu\Big(S \setminus \bigcup\limits_{i=1}^{\infty} E_i\Big) = 0$. A *gauge* on S is a function $\Delta\colon S \to \mathcal{T}$ such that $t \in \Delta(t)$ for every $t \in S$. For a given Δ on S, we say that a partial McShane partition $\{(E_i, t_i)\}_{i\in I}$ is *subordinate* to Δ if $E_i \subset \Delta(t_i)$ for every $i \in I$. A sequence $(\mathcal{P}_\infty^m)$ of a generalized McShane partitions of S is said to be adapted to the sequence of gauges (Δ_m) if $\mathcal{P}_\infty^m$ is subordinate to Δ_m for each $m \geqslant 1$. Let $f\colon S \to X$ be a function. We set

$$\sigma_n(f, \mathcal{P}_\infty) := \sum_{i=1}^{n} \mu(E_i) f(t_i)$$

for each infinite partial McShane partition $\mathcal{P}_\infty = \{(E_i, t_i)\}_{i\geqslant 1}$.

From now on, $(S, \mathcal{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space.

Definition 2.2 ([3]). A function $f\colon S \to X$ is *McShane integrable* ($\mathcal{M}$-*integrable* for short) with McShane integral ϖ if for every $\varepsilon > 0$ there is a gauge $\Delta\colon S \to \mathcal{T}$ such that

$$\limsup_{n\to\infty} \|\sigma_n(f, \mathcal{P}_\infty) - \varpi\| \leqslant \varepsilon$$

for every generalized McShane partition $\mathcal{P}_\infty$ of S subordinate to Δ. We set $\varpi := (\mathcal{M})\text{-}\int_S f \, \mathrm{d}\mu$.

For the properties of McShane integrable functions on a quasi-Radon measure space we refer to [2], [3], [4].

Remark 2.3. For the sake of comparison with the weak McShane integral it is interesting to observe the two following sequential formulations of the preceding definition.

A function $f\colon S \to X$ is $\mathcal{M}$-integrable with McShane integral ϖ if and only if there is a sequence of gauges (Δ_m) from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \|\sigma_n(f, \mathcal{P}_\infty) - \varpi\| = 0,$$

where $\Pi_\infty(\Delta_m)$ denotes the collection of all generalized McShane partitions of S subordinate to Δ_m.

Equivalently,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|\sigma_n(f, \mathcal{P}_\infty^m) - \varpi\| = 0$$

for every sequence $(\mathcal{P}_\infty^m)$ of generalized McShane partitions of S adapted to (Δ_m).

Before proceeding further, we list below some basic properties of the McShane integral that will be needed in this work. They are borrowed from [3].

Theorem 2.3. *Let $f\colon S \to X$ be a function.*

(1) *If f is $\mathcal{M}$-integrable, then the restriction $f_{|A}$ is $\mathcal{M}$-integrable (with respect to the σ-finite outer regular quasi-Radon measure space $(A, A\cap\mathcal{T}, A\cap\Sigma, \mu_{|A})$) for every $A \subset S$.*

(2) *Let $E \in \Sigma$. Then f is $\mathcal{M}$-integrable on E if and only if $f_{|E}$ is $\mathcal{M}$-integrable, and in this case the integrals are equal.*

(3) *Suppose $X = \mathbb{R}$. Then f is $\mathcal{M}$-integrable if and only if it is Lebesgue integrable, and the two integrals are equal.*

Definition 2.3 ([9])**.** We say that a function $f\colon S \to X$ is *weakly McShane integrable* ($\mathcal{WM}$-integrable for short) on S with weak McShane integral ϖ if there is a sequence of gauges (Δ_m) from S into $\mathcal{T}$ such that the property

$$(*)\qquad \lim_{m\to\infty}\limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}^m_\infty)\rangle - \langle x^*, \varpi\rangle| = 0$$

holds for every $x^* \in X^*$ and for every sequence $(\mathcal{P}^m_\infty)$ of generalized McShane partitions of S adapted to (Δ_m), that is, $\mathcal{P}^m_\infty$ is subordinate to Δ_m for each $m \geqslant 1$.

We set $\varpi = (\mathcal{WM})\text{-}\int_S f \,\mathrm{d}\mu$.

▷ f is said to be $\mathcal{WM}$-integrable on a measurable subset E of S if the function $1_E f$ is $\mathcal{WM}$-integrable on S. We set $(\mathcal{WM})\text{-}\int_E f\,\mathrm{d}\mu := (\mathcal{WM})\text{-}\int_S 1_E f\,\mathrm{d}\mu$.

▷ f is said to be $\mathcal{WM}$-integrable on Σ if it is $\mathcal{WM}$-integrable on every measurable subset of S.

According to [9], Proposition 3.2, $(*)$ may be replaced with

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty\in\Pi_\infty(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| = 0 \quad \forall\, x^* \in X^*,$$

where $\Pi_\infty(\Delta_m)$ denotes the collection of all generalized McShane partitions of S subordinate to Δ_m.

In the next theorems we list basic properties of the weak McShane integral that will be needed in this paper. They are borrowed from [9].

Theorem 2.4. *Let f, $g\colon S \to X$ be two functions and $E \in \Sigma$.*

(1) *If f and g are $\mathcal{WM}$-integrable on S and α is a real number, then $\alpha f + g$ is $\mathcal{WM}$-integrable on S and*

$$(\mathcal{WM})\text{-}\int_S \alpha f + g \,\mathrm{d}\mu = \alpha(\mathcal{WM})\text{-}\int_S f\,\mathrm{d}\mu + (\mathcal{WM})\text{-}\int_S g\,\mathrm{d}\mu.$$

(2) *If f is $\mathcal{WM}$-integrable on S and if $f = g$ μ-a.e., then the function g is $\mathcal{WM}$-integrable on S and*

$$(\mathcal{WM})\text{-}\int_S g\,\mathrm{d}\mu = (\mathcal{WM})\text{-}\int_S f\,\mathrm{d}\mu.$$

(3) *The function $1_E f$ is $\mathcal{WM}$-integrable on S if and only if the restriction $f_{|E}$ is $\mathcal{WM}$-integrable on E, and the two integrals are equal.*

(4) *If f is $\mathcal{WM}$-integrable on E, then it is scalarly integrable on E (that is, $\langle x^*, f\rangle$ is Lebesgue integrable on E for all $x^* \in X^*$), and we have*

$$\int_E \langle x^*, f\rangle\,\mathrm{d}\mu = \left\langle x^*, (\mathcal{WM})\text{-}\int_E f\,\mathrm{d}\mu\right\rangle \quad \forall\, x^* \in X^*.$$

As a consequence of Corollary 4.3 of [9], note that a function f which is $\mathcal{WM}$-integrable on S need not to be Pettis integrable; therefore not $\mathcal{WM}$-integrable on Σ. However, we have:

Theorem 2.5 ([9], Theorem 4.2 and Corollary 4.1)**.** *Let $f\colon S \to X$ be a function. Then the following statements are equivalent:*

(i) *f is $\mathcal{WM}$-integrable on Σ.*

(ii) *f is $\mathcal{WM}$-integrable on S and the set $\{\langle x^*, f\rangle\colon x^* \in \overline{B}_{X^*}\}$ is uniformly integrable.*

(iii) *f is $\mathcal{WM}$-integrable on S and Pettis integrable.*

3. The multiplier for the weak McShane integral

In this section we present our principal result in which we characterize the multiplier of the weak McShane integral:

Theorem 3.1. *Let $f\colon S \to X$ be a $\mathcal{WM}$-integrable function on S and $h \in L^\infty(S, \mathbb{R})$. If*

(i) *f is integrably bounded, and*

(ii) *X is w^*-separable,*

then hf is $\mathcal{WM}$-integrable on S.

P r o o f. The proof of Theorem 3.1 involves the following lemma.

Lemma 3.1. *Let $f\colon S \to X$ be a function. If*

(i) *there exists an increasing sequence (S_k) in Σ_f with union S such that $1_{S_k} f$ is $\mathcal{WM}$-integrable on S for each $k \geqslant 1$,*

(ii) *f is integrably bounded,*

then f is $\mathcal{WM}$-integrable on Σ.

P r o o f. As each function $1_{S_k} f$ is $\mathcal{WM}$-integrable on S, then by Theorem 2.4 (4) it is scalarly integrable on S. Therefore by Remark 2.2 the set $\{\langle x^*, 1_{S_k} f\rangle\colon\ x^* \in B_{X^*}\}$ is uniformly integrable. It follows from Theorem 2.5 that each $1_{S_k} f$ is $\mathcal{WM}$-integrable on Σ, therefore Pettis integrable. Condition (ii) also gives that the set $\{\langle x^*, f\rangle\colon\ x^* \in B_{X^*}\}$ is uniformly integrable. On the other hand,

$$\lim_{k\to\infty} \int_E \langle x^*, 1_{S_k} f\rangle \,\mathrm{d}\mu = \lim_{k\to\infty} \int_{E\cap S_k} \langle x^*, f\rangle \,\mathrm{d}\mu = \int_E \langle x^*, f\rangle \,\mathrm{d}\mu$$

for all $x^* \in X^*$ and $E \in \Sigma$. Then we can invoke Theorem 2.2, which shows that f is Pettis integrable. By virtue of Theorem 2.5, it is enough to prove that f is $\mathcal{WM}$-integrable on S. Using again the $\mathcal{WM}$-integrability of $1_{S_k} f$ on S and the fact that each real-valued function $1_{S\setminus S_k}\|f\|$ is Lebesgue integrable (i.e. McShane integrable, see Theorem 2.3 (3)), we obtain the existence of a sequence $(\Delta_m^k)_{m\geqslant 1}$ of gauges from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left|\langle x^*, \sigma_n(1_{S_k} f, \mathcal{P}_\infty^m)\rangle - \langle x^*, (\mathcal{WM})\text{-}\int_{S_k} f \,\mathrm{d}\mu\rangle\right| = 0, \tag{3.1}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left|\sigma_n(1_{S\setminus S_k}\|f\|, \mathcal{P}_\infty^m) - \int_{S\setminus S_k} \|f\| \,\mathrm{d}\mu\right| = 0 \tag{3.2}$$

for every $x^* \in X^*$ and for every sequence $(\mathcal{P}_\infty^m)$ of McShane partitions of S adapted to (Δ_m^k). For each $m \geqslant 1$ define the gauge Δ_m from S into $\mathcal{T}$ by

$$\Delta_m(t) := \bigcap_{k=1}^m \Delta_m^k(t), \quad t \in S$$

and let $(\mathcal{P}_\infty^m)$ be a sequence of generalized McShane partitions of S adapted to (Δ_m). Let $x^* \in X^*$ be arbitrary fixed. Then by the triangle inequality we have

$$\begin{aligned}
&\left|\langle x^*, \sigma_n(1_{S_k} f, \mathcal{P}_\infty^m)\rangle - \int_{S\setminus S_k} \langle x^*, f\rangle \,\mathrm{d}\mu\right| \\
&\qquad\leqslant \sigma_n(1_{S\setminus S_k}\|f\|, \mathcal{P}_\infty^m) + \left|\int_{S\setminus S_k} \langle x^*, f\rangle \,\mathrm{d}\mu\right| \\
&\qquad\leqslant \left|\sigma_n(1_{S\setminus S_k}\|f\|, \mathcal{P}_\infty^m) - \int_{S\setminus S_k} \|f\| \,\mathrm{d}\mu\right| + 2\int_{S\setminus S_k} \|f\| \,\mathrm{d}\mu,
\end{aligned}$$

and hence, by (3.2),

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \left|\langle x^*, \sigma_n(1_{S_k} f, \mathcal{P}_\infty^m)\rangle - \int_{S\setminus S_k} \langle x^*, f\rangle \,\mathrm{d}\mu\right| \leqslant 2\int_{S\setminus S_k} \|f\| \,\mathrm{d}\mu.$$

This inequality together with (3.1) entails

$$\begin{aligned}
\limsup_{m\to\infty}\limsup_{n\to\infty}&\left|\langle x^*,\sigma_n(f,\mathcal{P}_\infty^m)\rangle-\left\langle x^*,(\mathcal{P}e)\text{-}\int_S f\,\mathrm{d}\mu\right\rangle\right|\\
&\leqslant \limsup_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(1_{S_k}f,\mathcal{P}_\infty^m)\rangle-\int_{S_k}\langle x^*,f\rangle\,\mathrm{d}\mu\right|\\
&\quad+\limsup_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(1_{S\setminus S_k}f,\mathcal{P}_\infty^m)\rangle-\int_{S\setminus S_k}\langle x^*,f\rangle\,\mathrm{d}\mu\right|\\
&\leqslant 2\int_{S\setminus S_k}\|f\|\,\mathrm{d}\mu
\end{aligned}$$

for every $k\geqslant 1$. This yields, by letting $k\to\infty$ in the above inequality, to

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(f,\mathcal{P}_\infty^m)\rangle-\left\langle x^*,(\mathcal{P}e)\text{-}\int_S f\,\mathrm{d}\mu\right\rangle\right|=0,$$

because (S_k) is an increasing sequence with union S and $\|f\|$ is positive and integrable (condition (ii)). Then f is $\mathcal{WM}$-integrable on S and

$$(\mathcal{WM})\text{-}\int_S f\,\mathrm{d}\mu=(\mathcal{P}e)\text{-}\int_S f\,\mathrm{d}\mu.$$

□

P r o o f of Theorem 3.1.

Case 1: μ is finite. Without loss of generality we can assume that h is bounded, see Theorem 2.4 (2). Then by [7], Theorem 11.35, there is a sequence (h_k) of real-valued simple functions such that

$$\lim_{k\to\infty}\|h_k-h\|_\infty=0.$$

Let $M:=\sup\limits_{k\geqslant 1}\|h_k-h\|_\infty+\|h\|_\infty$. Then

$$\|h_k(t)f(t)\|\leqslant M\|f(t)\|, \tag{3.3}$$

$$\|h_k(t)f(t)-h(t)f(t)\|\leqslant\|h_k-h\|_\infty\|f(t)\| \tag{3.4}$$

for all $k\geqslant 1$ and for all $t\in S$. Let $p\geqslant 1$. For each $k\geqslant 1$ we consider the set A_k defined as

$$A_k:=\Big\{t\in S\colon \sup_{i\geqslant k}\|h(t)f(t)-h_i(t)f(t)\|\leqslant\frac{1}{p}\Big\}.$$

By virtue of inequality (3.4), it is clear that $\mu^*\Big(S\setminus\bigcup\limits_{k\geqslant 1}A_k\Big)=0$. Let $B_k\in\Sigma$ be such that $A_k\subset B_k$ and $\mu(B_k)=\mu^*(A_k)$ (B_k is a measurable envelope of A_k, that is, $A_k\subset B_k$ and $\mu(E\cap B_k)=\mu^*(E\cap A_k)$ for every $E\in\Sigma$). So we have

$$1_{B_k}|\langle x^*,h(t)f(t)-h_k(t)f(t)\rangle|\leqslant\frac{1}{p}\quad\text{a.e.}$$

(the exceptional set depends on x^*). Since X is w^*-separable (condition (ii)), we must have

$$1_{B_k}\|h(t)f(t)-h_k(t)f(t)\| \leqslant \frac{1}{p} \quad \forall\, t\in S\setminus N \text{ with } \mu(N)=0. \tag{3.5}$$

Since by (ii) f is integrably bounded, the same is true by (3.3) for $h_k f$. Moreover, Theorem 2.4 (1) shows that $h_k f$ is $\mathcal{WM}$-integrable on S, therefore by Remark 2.2 and Theorem 2.5 it follows that the set $\{\langle x^*, h_k f\rangle\colon\, x^*\in \overline{B}_{X^*}\}$ is uniformly integrable for each $k\geqslant 1$. Using again Theorem 2.5 we obtain that each function $h_k f$ is $\mathcal{WM}$-integrable on Σ and so Pettis integrable. On the other hand, inequality (3.4) implies

$$\begin{aligned}\left|\int_E \langle x^*, h_k f\rangle\,\mathrm{d}\mu - \int_E \langle x^*, hf\rangle\,\mathrm{d}\mu\right| &\leqslant \int_S |\langle x^*, h_k f\rangle\,\mathrm{d}\mu - \langle x^*, hf\rangle|\,\mathrm{d}\mu\\ &\leqslant \|h_k-h\|_\infty \int_S \|f\|\,\mathrm{d}\mu\end{aligned}$$

for all $x^*\in X^*$ and $E\in\Sigma$. Then

$$\lim_{k\to\infty}\int_E \langle x^*, h_k f\rangle\,\mathrm{d}\mu = \int_E \langle x^*, hf\rangle\,\mathrm{d}\mu$$

for all $x^*\in X^*$ and $E\in\Sigma$. Since the set $\{\langle x^*, hf\rangle\colon\, x^*\in\overline{B}_{X^*}\}$ is uniformly integrable by inequality (3.3), then, by virtue of the Theorem 2.2, we have that hf is Pettis integrable and

$$(\mathcal{P}e)\text{-}\int_E hf\,\mathrm{d}\mu = w\text{-}\lim_{k\to\infty}(\mathcal{P}e)\text{-}\int_E h_k f\,\mathrm{d}\mu \quad \forall\, E\in\Sigma. \tag{3.6}$$

Using again the $\mathcal{WM}$-integrability of $h_k f$ on Σ and the fact that each real-valued function $1_{S\setminus B_k}\|f\|$ is Lebesgue integrable (i.e. McShane integrable, see Theorem 2.3 (3)), we obtain the existence of a sequence $(\Delta_m^k)_{m\geqslant 1}$ of gauges from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*, \sigma_n(1_{B_k}h_k f, \mathcal{P}_\infty^m)\rangle - \left\langle x^*, (\mathcal{WM})\text{-}\int_{B_k} h_k f\,\mathrm{d}\mu\right\rangle\right| = 0, \tag{3.7}$$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|\sigma_n(1_{S\setminus B_k}\|f\|, \mathcal{P}_\infty^m) - \int_{S\setminus B_k}\|f\|\,\mathrm{d}\mu\right| = 0 \tag{3.8}$$

for every $x^*\in X^*$ and for every sequence $(\mathcal{P}_\infty^m)$ of McShane partitions of S adapted to (Δ_m^k). For each $m\geqslant 1$ we define the gauge Δ_m from S into $\mathcal{T}$ by

$$\Delta_m(t) := \bigcap_{k=1}^m \Delta_m^k(t), \quad t\in S.$$

Let $(\mathcal{P}_\infty^m) = (\{(E_i^m, t_i^m)\}_{i\geqslant 1})_{m\geqslant 1}$ be a sequence of generalized McShane partitions of S adapted to (Δ_m). Let $x^* \in X^*$ be arbitrary fixed. Then, by the triangle inequality, we have

$$\begin{aligned}
&\left|\langle x^*, \sigma_n(1_{B_k} hf, \mathcal{P}_\infty^m)\rangle - \int_{B_k} \langle x^*, hf\rangle \,\mathrm{d}\mu\right| \\
&\qquad \leqslant |\langle x^*, \sigma_n(1_{B_k} hf, \mathcal{P}_\infty^m)\rangle - \langle x^*, \sigma_n(1_{B_k} h_k f, \mathcal{P}_\infty^m)\rangle| \\
&\qquad\quad + \left|\langle x^*, \sigma_n(1_{B_k} h_k f, \mathcal{P}_\infty^m)\rangle - \int_{B_k} \langle x^*, h_k f\rangle \,\mathrm{d}\mu\right| \\
&\qquad\quad + \left|\int_{B_k} \langle x^*, h_k f\rangle \,\mathrm{d}\mu - \int_{B_k} \langle x^*, hf\rangle \,\mathrm{d}\mu\right| \\
&\qquad \leqslant \sum_{i=1}^{i=n} \mu(E_i^m) 1_{B_k}(t_i^m) |\langle x^*, h(t_i^m) f(t_i^m)\rangle - \langle x^*, h_k(t_i^m) f(t_i^m)\rangle| \\
&\qquad\quad + \left|\langle x^*, \sigma_n(1_{B_k} h_k f, \mathcal{P}_\infty^m)\rangle - \left\langle x^*, (\mathcal{WM})\text{-}\int_{B_k} h_k f \,\mathrm{d}\mu\right\rangle\right| \\
&\qquad\quad + \|h - h_k\|_\infty \int_S \|f\| \,\mathrm{d}\mu \\
&\qquad \leqslant \frac{\mu(S)}{p} + \left|\langle x^*, \sigma_n(1_{B_k} h_k f, \mathcal{P}_\infty^m)\rangle - \int_{B_k} \langle x^*, h_k f\rangle \,\mathrm{d}\mu\right| \\
&\qquad\quad + \|h - h_k\|_\infty \int_S \|f\| \,\mathrm{d}\mu \qquad\qquad \text{by (3.4) and (3.5),}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\langle x^*, \sigma_n(1_{S\setminus B_k} hf, \mathcal{P}_\infty^m)\rangle - \int_{S\setminus B_k} \langle x^*, hf\rangle \,\mathrm{d}\mu\right| \\
&\qquad \leqslant \|h\|_\infty \sigma_n(1_{S\setminus B_k}\|f\|, \mathcal{P}_\infty^m) + \left|\int_{S\setminus B_k} \langle x^*, hf\rangle \,\mathrm{d}\mu\right| \\
&\qquad \leqslant \|h\|_\infty \left|\sigma_n(1_{S\setminus B_k}\|f\|, \mathcal{P}_\infty^m) - \int_{S\setminus B_k} \|f\| \,\mathrm{d}\mu\right| + 2\|h\|_\infty \int_{S\setminus B_k} \|f\| \,\mathrm{d}\mu.
\end{aligned}$$

By letting, respectively, $n \to \infty$ and $m \to \infty$ in the above two inequalities, and together with (3.7) and (3.8), we get

$$\begin{aligned}
\limsup_{m\to\infty} \limsup_{n\to\infty} &\left|\langle x^*, \sigma_n(1_{B_k} hf, \mathcal{P}_\infty^m)\rangle - \int_{B_k} \langle x^*, hf\rangle \,\mathrm{d}\mu\right| \\
&\qquad \leqslant \frac{\mu(S)}{p} + \|h - h_k\|_\infty \int_S \|f\| \,\mathrm{d}\mu
\end{aligned}$$

and

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(1_{S\setminus B_k}hf,\mathcal{P}_\infty^m)\rangle-\int_{S\setminus B_k}\langle x^*,hf\rangle\,\mathrm{d}\mu\right| \leqslant 2\|h\|_\infty\int_{S\setminus B_k}\|f\|\,\mathrm{d}\mu,$$

consequently,

$$\begin{aligned}\limsup_{m\to\infty}\limsup_{n\to\infty}&\left|\langle x^*,\sigma_n(hf,\mathcal{P}_\infty^m)\rangle-\int_S\langle x^*,hf\rangle\,\mathrm{d}\mu\right|\\ &\leqslant \limsup_{k\to\infty}\limsup_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(1_{B_k}hf,\mathcal{P}_\infty^m)\rangle-\int_{B_k}\langle x^*,hf\rangle\,\mathrm{d}\mu\right|\\ &\quad+\limsup_{k\to\infty}\limsup_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(1_{S\setminus B_k}hf,\mathcal{P}_\infty^m)\rangle-\int_{S\setminus B_k}\langle x^*,hf\rangle\,\mathrm{d}\mu\right|\\ &\leqslant \frac{\mu(S)}{p}.\end{aligned}$$

Since $\lim_{k\to\infty}\mu(S\setminus B_k)=0$ and the function $\|f\|$ is positive and integrable, by the arbitrariness of $p\geqslant 1$ we get

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(hf,\mathcal{P}_\infty^m)\rangle-\int_S\langle x^*,hf\rangle\,\mathrm{d}\mu\right|=0$$

for all $x^*\in X^*$. Thus, hf is $\mathcal{WM}$-integrable on S and

$$(\mathcal{WM})\text{-}\int_S hf\,\mathrm{d}\mu=(\mathcal{P}e)\text{-}\int_S hf\,\mathrm{d}\mu.$$

Case 2: μ is σ-finite. Let $(S_i)_{i\geqslant 1}$ be a sequence in Σ_f such that $S=\bigcup_{i=1}^{\infty}S_i$. Set

$$S'_k:=\bigcup_{i=1}^{k}S_i,\quad k\geqslant 1.$$

Clearly, $(S'_k)_{k\geqslant 1}$ is a non-decreasing sequence in Σ_f with union S. By condition (i), we can invoke Theorem 2.5, which shows that each function $1_{S'_k}f$ is $\mathcal{WM}$-integrable on S. Equivalently, Theorem 2.4 (3) shows that $f_{|S'_k}$ is $\mathcal{WM}$-integrable on S'_k, then by case 1, the restriction $(hf)_{|S'_k}=h_{|S'_k}f_{|S'_k}$ is $\mathcal{WM}$-integrable on S'_k, and equivalently by Theorem 2.4 (3), $1_{S'_k}hf$ is $\mathcal{WM}$-integrable on S and by remarking that hf is integrably bounded (since $\|h(t)f(t)\|=|h(t)|\|f(t)\|\leqslant\|h\|_\infty\|f(t)\|$), we show that hf is $\mathcal{WM}$-integrable on S, in view of Lemma 3.1. $\square$

To close this section we would like to mention the following problem:

Problem. Let $f\colon S \to X$ be a McShane integrable in the limit function on S and $h \in L^{\infty}(S, \mathbb{R})$. Does $hf\colon S \to X$ have to be McShane integrable in the limit on S? If the answer is no, what are the conditions for hf becoming McShane integrable in the limit on S?

Recall that a function $f\colon S \to X$ is said to be McShane integrable in the limit on S with McShane integral in the limit ϖ if for every $\varepsilon > 0$ there is a gauge $\Delta\colon S \to \mathcal{T}$ such that

$$\limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| \leqslant \varepsilon$$

for all $x^* \in \overline{B}_{X^*}$ and for every generalized McShane partition $\mathcal{P}_\infty$ of S subordinate to Δ. We set $\varpi := (\mathcal{ML})\text{-}\int_S f \, \mathrm{d}\mu$ (see [10]).

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