Yansheng Wu; Gaohua Tang; Guixin Deng; Yiqiang Zhou

Nil-clean and unit-regular elements in certain subrings of $\mathbb{M}_2(\mathbb{Z})$

Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 1, 197–205

Persistent URL: http://dml.cz/dmlcz/147627

Czechoslovak Mathematical Journal, 69 (144) (2019), 197–205

NIL-CLEAN AND UNIT-REGULAR ELEMENTS IN CERTAIN SUBRINGS OF $\mathbb{M}_2(\mathbb{Z})$

Yansheng Wu, Nanjing, Gaohua Tang, Guixin Deng, Nanning, Yiqiang Zhou, St. John's

Received May 25, 2017. Published online July 23, 2018.

In memory of the birth of Wu's nephew Zirui Wu

Abstract. An element in a ring is clean (or, unit-regular) if it is the sum (or, the product) of an idempotent and a unit, and is nil-clean if it is the sum of an idempotent and a nilpotent. Firstly, we show that Jacobson's lemma does not hold for nil-clean elements in a ring, answering a question posed by Koşan, Wang and Zhou (2016). Secondly, we present new counter-examples to Diesl's question whether a nil-clean element is clean in a ring. Lastly, we give new examples of unit-regular elements that are not clean in a ring. The rings under consideration in our examples are particular subrings of $\mathbb{M}_2(\mathbb{Z})$.

Keywords: clean element; nil-clean element; unit-regular element; Jacobson's lemma for nil-clean elements

MSC 2010: 16U60, 16S50, 11D09

1. Introduction

In 2013, Diesl in [5] introduced the notion of a nil-clean element (ring), as a variant of the much-studied notion of a clean element (ring) due to Nicholson. An element in a ring is called nil-clean (clean) if it is a sum of an idempotent and a nilpotent (unit), and the ring is nil-clean (clean) if its every element is nil-clean (clean). Nil-clean rings have attracted much attention recently and have been shown to have

The research of Tang was supported by National Natural Science Foundation of China (11661014, 11661013, 11461010), the Guangxi Science Research and Technology Development Project (1599005-2-13), the Guangxi Natural Science Foundation (2016GXNSFDA380017), and the Scientific Research Fund of Guangxi Education Department (KY2015ZD075). Zhou's research was supported by a Discovery Grant from NSERC of Canada.

DOI: 10.21136/CMJ.2018.0256-17

close connections with clean rings, strongly π-regular rings, Boolean rings, and Köthe conjecture.

For any two elements $a, b \in R$, $1 - ab$ is a unit if and only if $1 - ba$ is a unit. This result is known as Jacobson's lemma for units. There are several analogous results in the literature. It is known that Jacobson's lemma holds for Drazin invertible elements (see [4]) and for generalized Drazin invertible elements (see [10]). In [8], the authors proved that Jacobson's lemma holds for π-regular elements and unit π-regular elements, but fails for clean elements. In [7], it is proved that Jacobson's lemma holds for strongly nil-clean elements and a question left open in [7] asks whether Jacobson's lemma holds for nil-clean elements. Here we give a negative answer to this question.

In [5], Diesl proved, among others, that a nil-clean ring is clean, and asked whether a nil-clean element is clean. In [1], Andrica and Călugăreanu found a nil-clean but not clean element in the matrix ring $\mathbb{M}_2(\mathbb{Z})$ by a long, fairly difficult process, involving solving Pell equations. Here we reconsider Diesl's question by working on the subring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ of $\mathbb{M}_2(\mathbb{Z})$ instead of $\mathbb{M}_2(\mathbb{Z})$. Because the subring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ contains much less clean elements than $\mathbb{M}_2(\mathbb{Z})$, there is a huge advantage to working in $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ for constructing counter-examples to Diesl's question. Here we present a simple and direct way to construct a nil-clean but not clean element in the ring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ for every positive integer $s \geqslant 3$. We also find a nil-clean but not clean element in the ring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$, but our handling of this case needs the help of a result of Andrica and Călugăreanu in [1]. Thus, not every nil-clean element is clean in the ring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ for every $0 \neq s \in \mathbb{Z}$.

An element in a ring is unit-regular if it is a product of an idempotent and a unit, and a ring is unit-regular if its every element is unit-regular. By Camillo and Khurana in [2], every unit-regular ring is clean. This motivated Khurana and Lam in [6] to consider whether a single unit-regular element in a ring is clean. In [6], a criterion is given for a matrix $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ to be clean in the ring $\mathbb{M}_2(K)$ over a commutative ring K. When it is applied to $K = \mathbb{Z}$, the authors of [6] are able to give many examples of unit-regular matrices that are not clean in $\mathbb{M}_2(\mathbb{Z})$. Here as a supplement to Khurana and Lam's work, we give more examples of unit-regular elements that are not clean in the ring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ and our argument is fairly simple.

Throughout the paper, $\mathbb{Z}$ is the ring of integers, $\mathbb{M}_2(\mathbb{Z})$ is the 2×2 matrix ring over $\mathbb{Z}$ whose identity is denoted by I_2.

2. Jacobson's lemma for nil-clean elements

Our first needed lemma is [3], Lemma 1.5 (or see [1], Lemma 1).

Lemma 2.1. *Let $s \in \mathbb{Z}$. A matrix A in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ is a nontrivial idempotent if and only if $A = \begin{pmatrix} a+1 & u \\ vs & -a \end{pmatrix}$ with $a^2 + a + suv = 0$.*

As our first result, the following theorem shows that Jacobson's lemma does not hold for nil-clean elements.

Theorem 2.2. *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ -28 & -3 \end{pmatrix} \in R$. Then $I_2 - AB$ is nil-clean but $I_2 - BA$ is not nil-clean in R.*

Proof. We see that $I_2 - AB = \begin{pmatrix} 3 & 12 \\ -28 & -2 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ -24 & -8 \end{pmatrix} + \begin{pmatrix} -6 & 9 \\ -4 & 6 \end{pmatrix}$ is a sum of an idempotent and a nilpotent in R. Assume on the contrary that $I_2 - BA = \begin{pmatrix} 3 & 6 \\ -56 & -2 \end{pmatrix}$ is nil-clean in R. Then there exists an idempotent C in R such that $I_2 - BA - C$ is a nilpotent in R. It can be seen that $C \neq 0$ and $C \neq I_2$. So, by Lemma 2.1, $C = \begin{pmatrix} a+1 & u \\ 4v & -a \end{pmatrix}$ where $a^2 + a + 4uv = 0$. Moreover, by [1], Lemma 2, $I_2 - BA - C = \begin{pmatrix} b & x \\ 4y & -b \end{pmatrix}$ where $b^2 + 4xy = 0$. Thus, $\begin{pmatrix} 3 & 6 \\ -56 & -2 \end{pmatrix} = \begin{pmatrix} a+b+1 & u+x \\ 4v+4y & -a-b \end{pmatrix}$. Therefore, we have

$$a + b = 2, \quad u + x = 6, \quad v + y = -14, \quad a^2 + a + 4uv = 0 = b^2 + 4xy,$$

and we deduce $5a = 56u - 24v - 332$. Then $u = \frac{5a+24v+332}{56}$. From $a^2 + a + 4uv = 0$, it follows that

$$a(a+1) + \frac{5a + 24v + 332}{14} v = 0.$$

That is,

$$14a^2 + (14 + 5v)a + (24v + 332)v = 0. \tag{2.1}$$

The discriminant of (2.1), considered as a quadratic equation in a, is $\Delta = (14+5v)^2 - 56v(24v + 332) = -1319v^2 - 18452v + 196$. In order to have integer solutions for equation (2.1), it is necessary that $\Delta \geqslant 0$ and Δ is a perfect square. The quadratic function $f(v) = -1319v^2 - 18452v + 196$ concaves down and has two zeros at -14 and $\frac{14}{1319}$. So $f(v) \geqslant 0$ if and only if $-14 \leqslant v \leqslant \frac{14}{1319}$. Hence, if equation (2.1) has an integer solution, then v must be an integer between -14 and 0. Now we can proceed with the following cases.

Case 1. If v is any of the values $-11, -10, -9, -8, -7, -6, -5, -2$ and -1, then $f(v)$ is not a perfect square.

Case 2. For $v = 0$, we have $y = -14$ and $a^2 + a = 0$. So $(2-a)^2 - 56x = 0$. As $a = 0$ or -1, such an integer x does not exist.

Case 3. If $v = -14$, then $y = 0$, $b = 0$ and $a = 2$. So $6 - 56u = 0$. But such an integer u does not exist.

Case 4. If $v = -13$, then $y = -1$. Thus, $a^2 + a + 4uv = 0$ gives $a^2 + a - 52u = 0$ and $b^2 + 4xy = 0$ gives $a^2 - 4a - 20 + 4u = 0$. We deduce $14a^2 - 51a - 260 = 0$, so $a = \frac{91}{14}$ or $-\frac{20}{7}$, a contradiction.

Case 5. If $v = -3$, then $y = -11$. As argued in case 4, we obtain $14a^2 - a - 780 = 0$, which gives $a = \frac{15}{2}$ or $-\frac{52}{7}$, a contradiction.

Case 6. If $v = -12$, then $y = -2$. As argued in case 4, we get $7a^2 - 23a - 264 = 0$, which gives $a = 8$ or $-\frac{33}{7}$. But if $a = 8$, then $a^2 + a + 4uv = 0$ gives $72 - 48u = 0$ and so $u = \frac{3}{2}$, a contradiction.

Case 7. If $v = -4$, then $y = -10$. As argued in case 4, we obtain $7a^2 - 3a - 472 = 0$, which gives $a = -8$ or $\frac{59}{7}$. But if $a = -8$, then $a^2 + a + 4uv = 0$ gives $56 - 16u = 0$ and so $u = \frac{7}{2}$, a contradiction.

Therefore, we have proved that $I_2 - BA$ is not nil-clean in R. □

3. Nil-clean elements need not be clean: more counter-examples

By [1], not every nil-clean matrix is clean in $\mathbb{M}_2(\mathbb{Z})$. We next prove that, for any positive integer $s \geqslant 2$, not every nil-clean element is clean in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. In contrast to the difficult search of the counter-example in [1], our construction in Theorem 3.1 below is direct and fairly simple.

Theorem 3.1. *If $s \geqslant 3$, then $\begin{pmatrix} 1+s & 1 \\ s^2 & -s \end{pmatrix}$ is nil-clean, but not clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.*

Proof. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. We see that

$$A := \begin{pmatrix} 1+s & 1 \\ s^2 & -s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2s^2 & 0 \end{pmatrix} + \begin{pmatrix} s & 1 \\ -s^2 & -s \end{pmatrix}$$

is a sum of an idempotent and a nilpotent in R. Assume on the contrary that $A = E + (A-E)$ where $E^2 = E \in R$ and $A - E$ is invertible in R. Then one can easily see that $E \neq 0$ and $E \neq I_2$. So we can write $E = \begin{pmatrix} r+1 & p \\ s^2q & -r \end{pmatrix}$ with $r^2 + r + s^2pq = 0$ by Lemma 2.1. We have $\pm 1 = \det(A - E) = r - 2s^2 + 2rs + s^2(p+q)$. It follows that $\gcd(r, s) = 1$. As $r(1+r) + s^2pq = 0$, we deduce that $s^2 \mid 1 + r$.

If $\det(A-E) = 1$, then $1 = r - 2s^2 + 2rs + s^2(p+q)$, so $1 + r = -2s^2 + 2r(1+s) + s^2(p+q)$. It follows that s^2 divides $2(1+s)r$. But $\gcd(r, s) = 1$ and $\gcd(s, s+1) = 1$, we infer $s^2 \mid 2$, a contradiction.

If $\det(A-E)=-1$, then $-(1+r)=-2s^2+2rs+s^2(p+q)$. It follows that $s^2\mid 2rs$, so $s\mid 2$, a contradiction. $\square$

Remark 3.2. By Theorem 3.1, for $s\geqslant 3$ the matrix $\begin{pmatrix}1+s & 1\\ s^2 & -s\end{pmatrix}$ is nil-clean but not clean in $\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ s^2\mathbb{Z} & \mathbb{Z}\end{pmatrix}$. However, $\begin{pmatrix}1+s & 1\\ s^2 & -s\end{pmatrix}$ is clean in $\mathbb{M}_2(\mathbb{Z})$, because $\begin{pmatrix}1+s & 1\\ s^2 & -s\end{pmatrix}=\begin{pmatrix}s & 1\\ s-s^2 & 1-s\end{pmatrix}+\begin{pmatrix}1 & 0\\ -s+2s^2 & -1\end{pmatrix}$ is a sum of an idempotent and a unit in $\mathbb{M}_2(\mathbb{Z})$. This computation shows that there is a huge advantage to working in $\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ s^2\mathbb{Z} & \mathbb{Z}\end{pmatrix}$ instead of $\mathbb{M}_2(\mathbb{Z})$ for constructing counter-examples to Diesl's question.

Theorem 3.3. *Not every nil-clean matrix is clean in* $\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 4\mathbb{Z} & \mathbb{Z}\end{pmatrix}$.

Proof. Let $R=\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 4\mathbb{Z} & \mathbb{Z}\end{pmatrix}$. As seen in Theorem 2.2, $C=\begin{pmatrix}3 & 12\\ -28 & -2\end{pmatrix}$ is nil-clean in R. Next we show that C is not clean in R. Assume on the contrary that $C=E+(C-E)$ where $E^2=E\in R$ and $C-E$ is invertible in R. One easily sees that E must be a nontrivial idempotent. So, we can write $E=\begin{pmatrix}\gamma+1 & p\\ 4q & -\gamma\end{pmatrix}$ with $\gamma^2+\gamma+4pq=0$ by Lemma 2.1. To get a contradiction, we use a result of Andrica and Călugăreanu in [1].

We write $C=\begin{pmatrix}\alpha+\beta+1 & u+x\\ 4v+4y & -\alpha-\beta\end{pmatrix}$ in R, where $\alpha=8$, $\beta=-6$, $u=3$, $v=-6$, $x=9$ and $y=-1$, and where $C=\begin{pmatrix}3 & 12\\ -28 & -2\end{pmatrix}=\begin{pmatrix}\alpha+1 & u\\ 4v & -\alpha\end{pmatrix}+\begin{pmatrix}\beta & x\\ 4y & -\beta\end{pmatrix}$ is a sum of an idempotent and a nilpotent in R and hence in $\mathbb{M}_2(\mathbb{Z})$. Moreover, it is clear that neither C nor I_2-C is a nilpotent. Let $r:=\alpha+\beta=2$ and $\delta:=-\det(C)=-330$.

Case 1: $\det(C-E)=1$. By Andrica and Călugăreanu [1], Theorem 4, we have the (elliptic) Pell equation

$$X^2-(1+4\delta)Y^2=4(4v+4y)^2(2r+1)^2(\delta^2+2\delta+2)$$

with

$$X=(2r+1)[-(1+4\delta)4q+(2\delta+3)(4v+4y)],$$
$$Y=2(4v+4y)^2p+(2r^2+2r+1+2\delta)4q-(2\delta+3)(4v+4y).$$

That is,

$$X^2+1\,319Y^2=8\,486\,172\,800$$

with

$$X=26\,380q+91\,980,$$
$$Y=1\,568p-2\,588q-18\,396.$$

As 20 divides X, 20 divides Y. As $Y\neq 0$, $20^2\leqslant Y^2$. Thus, $X^2=8\,486\,172\,800-1\,319Y^2\leqslant 8\,486\,172\,800-1\,319\cdot 20^2=8\,485\,645\,200$, so $-92\,117.5<X<92\,117.5$, i.e., $-92\,117.5 < 26\,380q + 91\,980 < 92\,117.5$. It follows that $-7 < q < 1$. A case-by-case checking shows that only when $q = 0$ the Pell equation has integer solutions, which are $X = 91\,980$ and $Y = \pm 140$. But this would yield that $p = \frac{2\,317}{196}$ or $p = \frac{1\,141}{98}$, a contradiction.

Case 2: $\det(C - E) = -1$. By Andrica and Călugăreanu [1], Theorem 4, we have the (elliptic) Pell equation

$$X^2 - (1+4\delta)Y^2 = 4(4v+4y)^2(2r+1)^2\delta(\delta-2)$$

with

$$X = (2r+1)[-(1+4\delta)4q + (2\delta-1)(4v+4y)],$$
$$Y = 2(4v+4y)^2p + (2r^2+2r+1+2\delta)4q - (2\delta-1)(4v+4y).$$

That is,

$$X^2 + 1\,319Y^2 = 8\,589\,504\,000$$

with

$$X = 26\,380q + 92\,540,$$
$$Y = 1\,568p - 2\,588q - 18\,508.$$

As 20 divides X, 20 divides Y. As $Y \neq 0$, $20^2 \leqslant Y^2$. Thus, $X^2 = 8\,589\,504\,000 - 1\,319Y^2 \leqslant 8\,589\,504\,000 - 1\,319 \cdot 20^2 = 8\,588\,976\,400$, so $-92\,677 < X < 92\,677$, i.e., $-92\,677 < 26\,380q + 92\,540 < 92\,677$. It follows that $-7 \leqslant q \leqslant 0$. A case-by-case checking shows that the Pell equation has integer solutions only when $q = 0$ or $q = -7$. When $q = 0$, the solutions are $X = 92\,540$ and $Y = \pm 140$, which implies $p = \frac{333}{28}$ or $p = \frac{82}{7}$, a contradiction. When $q = -7$, the solutions are $X = -92\,120$ and $Y = \pm 280$, which implies $p = \frac{3}{7}$ or $p = \frac{1}{14}$, a contradiction.

Hence, we have proved that C is not clean in R. □

To sum up we can conclude the following:

Theorem 3.4. *If $s \geqslant 1$, then not every nil-clean element is clean in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Remark 3.5. We point out that, for two distinct positive integers s and t, the two rings $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ t^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ are not isomorphic. To see this, we note that $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cong \begin{pmatrix} \mathbb{Z} & s\mathbb{Z} \\ s\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ via $\begin{pmatrix} a & x \\ s^2y & b \end{pmatrix} \leftrightarrow \begin{pmatrix} a & sx \\ sy & b \end{pmatrix}$, and that $\begin{pmatrix} \mathbb{Z} & s\mathbb{Z} \\ s\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cong \mathbb{M}_2(\mathbb{Z}; s)$, the formal matrix ring defined in [9] (see [9], Proposition 4 (3)). Hence, by [9], Example 23, $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cong \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ t^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ if and only if $s = t$.

4. Unit-regular elements need not be clean: more counter-examples

Every unit-regular ring is clean by Camillo and Khurana in [2]. By Khurana and Lam in [6], a single unit-regular element in a ring need not be clean. Indeed, a criterion is given in [6] for a matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ in $\mathbb{M}_2(K)$ over a commutative ring K to be clean, and this enables the authors of [6] to give many examples of unit-regular matrices in $\mathbb{M}_2(\mathbb{Z})$ that are not clean.

Next we give more examples of unit-regular elements that are not clean in some subrings of $\mathbb{M}_2(\mathbb{Z})$ and our argument is fairly simple.

Theorem 4.1. *If $s \geqslant 3$, then $\begin{pmatrix} s+1 & s \\ 0 & 0 \end{pmatrix}$ is unit-regular, but not clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.*

Proof. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. We see that

$$A := \begin{pmatrix} s+1 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s+1 & 1 \\ -s^2 & -s+1 \end{pmatrix}$$

is a product of an idempotent and a unit in R. Assume on the contrary that $A = E + (A - E)$ where $E^2 = E \in R$ and $A - E$ is invertible in R. Then one can easily see that $E \neq 0$ and $E \neq I_2$. So we can write $E = \begin{pmatrix} r+1 & p \\ s^2q & -r \end{pmatrix}$ with $r^2 + r + s^2pq = 0$ by Lemma 2.1. In view of $r^2+r+s^2pq = 0$, we have $\pm 1 = \det(A-E) = rs+r+s^2q$, and hence $\gcd(r, s) = 1$. Thus, it follows from $r^2 + r + s^2pq = 0$ that $s \mid 1 + r$.

If $\det(A-E) = 1$, then $rs+r+s^2q = 1$ and so $r(s+2) = (1+r) - s^2q$. It follows that $s^2 \mid r(s + 2)$. Hence $s^2 \mid s + 2$, and so $s \mid 2$, a contradiction.

If $\det(A-E) = -1$, then $rs+r+s^2q = -1$ and so $rs = -(1+r) - s^2q$. It follows that $s^2 \mid rs$, so $s \mid r$, a contradiction. $\square$

Remark 4.2. The matrix $\begin{pmatrix} s+1 & s \\ 0 & 0 \end{pmatrix}$ in Theorem 4.1 is clean in $\mathbb{M}_2(\mathbb{Z})$, because $\begin{pmatrix} s+1 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} s+1 & s \\ -1 & -1 \end{pmatrix}$ is a sum of an idempotent and a unit in $\mathbb{M}_2(\mathbb{Z})$.

Example 4.3. The matrix $\begin{pmatrix} 11 & 1 \\ 0 & 0 \end{pmatrix}$ is unit-regular, but not clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Proof. As $A := \begin{pmatrix} 11 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 11 & 1 \\ 32 & 3 \end{pmatrix}$, A is unit-regular in $R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Assume on the contrary that $A = E + (A - E)$ where $E^2 = E \in R$ and $A - E$ is invertible in R. Then one easily sees that $E \neq 0$ and $E \neq I_2$, so $E = \begin{pmatrix} r+1 & p \\ 4q & -r \end{pmatrix}$ with $r^2 + r + 4pq = 0$ by Lemma 2.1. It follows that $\pm 1 = \det(A - E) = 11r + 4q$.

If $11r+4q = 1$, then we have an equation $q(121p+4q-13) = -3$, which has no integer solutions for p, q. If $11r+4q = -1$, then we have an equation $q(242p+8q-18) = 5$, which has no integer solutions for p, q. Hence, A is not clean in R. $\square$

Thus, we can conclude the following:

Theorem 4.4. *If $s \geqslant 1$, then not every unit-regular element is clean in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

By Khurana and Lam in [6], the matrix $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ is unit-regular but not clean in $\mathbb{M}_2(\mathbb{Z})$, and this is the "smallest" such example one can find in $\mathbb{M}_2(\mathbb{Z})$. But "smaller" such examples can be found in some subrings of $\mathbb{M}_2(\mathbb{Z})$.

Example 4.5. The matrix $\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ is unit-regular, but not clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Proof. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. We see that

$$A := \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}$$

is unit-regular in R. Assume on the contrary that $A = E + (A - E)$ where $E^2 = E \in R$ and $A - E$ is invertible in R. Then one easily sees that $E \neq 0$ and $E \neq I_2$, so $E = \begin{pmatrix} r+1 & p \\ 8q & -r \end{pmatrix}$ with $r^2 + r + 8pq = 0$ by Lemma 2.1. As $r^2 + r + 8pq = 0$, we have $\pm 1 = \det(A - E) = 3r + 8q$, and hence $\gcd(2, r) = 1$. Thus, it follows from $r^2 + r + 8pq = 0$ that $8 \mid 1 + r$.

If $\det(A - E) = 1$, then $3r + 8q = 1$ and so $4r = (1 + r) - 8q$. It follows that $8 \mid 4r$, so $2 \mid r$, a contradiction.

If $\det(A - E) = -1$, then $3r + 8q = -1$ and so $2r = -(1 + r) - 8q$. It follows that $8 \mid 2r$, so $4 \mid r$, a contradiction. □

Example 4.6. If $n \geqslant 3$, then the matrix $\begin{pmatrix} 2^{n-1}-1 & 1 \\ 0 & 0 \end{pmatrix}$ is unit-regular, but not clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Proof. The matrix $A := \begin{pmatrix} 2^{n-1}-1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2^{n-1}-1 & 1 \\ 2^{2n-2}-2^n & 2^{n-1}-1 \end{pmatrix}$ is unit-regular in $R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Assume on the contrary that $A = E + (A - E)$ where $E^2 = E \in R$ and $A - E$ is invertible in R. Then one sees that $E \neq 0$ and $E \neq I_2$, so $E = \begin{pmatrix} r+1 & p \\ 2^n q & -r \end{pmatrix}$ with $r^2 + r + 2^n pq = 0$ by Lemma 2.1. As $r^2 + r + 2^n pq = 0$, we have $\pm 1 = \det(A - E) = (2^{n-1} - 1)r + 2^n q$. It follows that $\gcd(2, r) = 1$ and $2^n \mid 1 + r$.

If $(2^{n-1} - 1)r + 2^n q = 1$, then $(2^{n-1})r = (1 + r) - 2^n q$. So $2^n \mid 2^{n-1}r$ and thus $2 \mid r$, a contradiction.

If $(2^{n-1} - 1)r + 2^n q = -1$, then $(2^{n-1} - 2)r = -(1 + r) - 2^n q$. So $2^n \mid (2^{n-1} - 2)r$, and hence $2 \mid 1$, a contradiction.

So, A is not clean in R. □

Acknowledgment. The authors are grateful to the reviewer and the editor for their valuable comments and suggestions to improve this paper.

References

[1] *D. Andrica, G. Călugăreanu*: A nil-clean 2×2 matrix over the integers which is not clean. J. Algebra Appl. *13* (2014), Article ID 1450009, 9 pages. zbl MR doi

[2] *V. P. Camillo, D. Khurana*: A characterization of unit regular rings. Commun. Algebra *29* (2001), 2293–2295. zbl MR doi

[3] *J. Chen, X. Yang, Y. Zhou*: On strongly clean matrix and triangular matrix rings. Commun. Algebra *34* (2006), 3659–3674. zbl MR doi

[4] *D. Cvetkovic-Ilic, R. Harte*: On Jacobson's lemma and Drazin invertibility. Appl. Math. Lett. *23* (2010), 417–420. zbl MR doi

[5] *A. J. Diesl*: Nil clean rings. J. Algebra *383* (2013), 197–211. zbl MR doi

[6] *D. Khurana, T. Y. Lam*: Clean matrices and unit-regular matrices. J. Algebra *280* (2004), 683–698. zbl MR doi

[7] *T. Koşan, Z. Wang, Y. Zhou*: Nil-clean and strongly nil-clean rings. J. Pure Appl. Algebra *220* (2016), 633–646. zbl MR doi

[8] *T. Y. Lam, P. P. Nielsen*: Jacobson's lemma for Drazin inverses. Ring Theory and Its Applications (D. V. Huynh et al., eds.). Contemporary Mathematics 609, American Mathematical Society, Providence, 2014, pp. 185–195. zbl MR doi

[9] *G. Tang, Y. Zhou*: A class of formal matrix rings. Linear Algebra Appl. *438* (2013), 4672–4688. zbl MR doi

[10] *G. Zhuang, J. Chen, J. Cui*: Jacobson's lemma for the generalized Drazin inverse. Linear Algebra Appl. *436* (2012), 742–746. zbl MR doi

Authors' addresses: Yansheng Wu, Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu, 211100, P. R. China, e-mail: `wysasd@163.com`; Gaohua Tang, Key Laboratory of Environment Change and Resources Use in Beibu Gulf (Guangxi Teachers Education University), Ministry of Education, P. R. China; School of Mathematics and Statistics, Guangxi Teachers Education University, Nanning, Guangxi, 530023, P. R. China; School of Sciences, Qinzhou University, Qinzhou, Guangxi 535011, P. R. China; Guixin Deng, School of Mathematics and Statistics, Guangxi Teachers Education University, Nanning, Guangxi, 530023, P. R. China; Yiqiang Zhou, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada.