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*Czechoslovak Mathematical Journal*, Vol. 69 (2019), No. 1, 117–129

Persistent URL: <http://dml.cz/dmlcz/147622>

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GORENSTEIN PROJECTIVE COMPLEXES WITH RESPECT TO  
COTORSION PAIRS

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Received April 25, 2017. Published online July 10, 2018.

*Abstract.* Let  $(\mathcal{A}, \mathcal{B})$  be a complete and hereditary cotorsion pair in the category of left  $R$ -modules. In this paper, the so-called Gorenstein projective complexes with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  are introduced. We show that these complexes are just the complexes of Gorenstein projective modules with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$ . As an application, we prove that both the Gorenstein projective modules with respect to cotorsion pairs and the Gorenstein projective complexes with respect to cotorsion pairs possess stability.

*Keywords:* cotorsion pair; Gorenstein projective complex with respect to cotorsion pairs; stability of Gorenstein categories

*MSC 2010:* 18G25, 18G35

## 1. INTRODUCTION

Let  $(\mathcal{A}, \mathcal{B})$  be a complete and hereditary cotorsion pair in the category of left  $R$ -modules. Then there are two induced cotorsion pairs  $(\tilde{\mathcal{A}}, \text{dg } \tilde{\mathcal{B}})$  and  $(\text{dg } \tilde{\mathcal{A}}, \tilde{\mathcal{B}})$  ([8]), and both of them are complete and hereditary ([17], [14]). Recently, among others, the Gorenstein category  $\mathcal{G}(\mathcal{A})$  with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  was introduced and studied by Yang and Chen in [16], see Definition 2.3. In this paper, we generalize this notion to the category of complexes of left  $R$ -modules, namely, we introduce the Gorenstein projective complexes with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$ , see Definition 3.1. The class of these complexes will be denoted by  $\mathcal{G}(\tilde{\mathcal{A}})$ . It contains Gorenstein projective complexes [4],  $\mathbf{F}$ -Gorenstein flat complexes, see [10], and Gorenstein flat complexes [7] over right coherent rings as its special cases. By us-

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The research was partially supported by the National Natural Science Foundation of China (Nos. 11361052, 11561039, 11761060).

ing the techniques of Bravo and Gillespie in [3], we prove the following result, see Theorem 3.5.

**Theorem 1.1.** *A complex  $C$  of left  $R$ -modules belongs to  $\mathcal{G}(\tilde{\mathcal{A}})$  if and only if each  $C_n$  belongs to  $\mathcal{G}(\mathcal{A})$ .*

Theorem 1.1 generalizes [10], Theorem 4.7. and [18], Theorems 2.2, 3.1, and it provides interesting relationships between the Gorenstein projectivity with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  of a complex and that of all its terms. By this connection, we prove the following results, see Theorem 4.1 and Theorem 4.2, respectively.

**Theorem 1.2.** *A left  $R$ -module  $M$  belongs to  $\mathcal{G}(\mathcal{A})$  if and only if there exists a  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact exact sequence  $\dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow \dots$  in  $\mathcal{G}(\mathcal{A})$  such that  $M \cong \text{Im}(G_0 \rightarrow G_{-1})$ .*

**Theorem 1.3.** *A complex  $C$  of left  $R$ -modules belongs to  $\mathcal{G}(\tilde{\mathcal{A}})$  if and only if there exists a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}})$ -exact exact sequence  $\dots \rightarrow G^1 \rightarrow G^0 \rightarrow G^{-1} \rightarrow \dots$  in  $\mathcal{G}(\tilde{\mathcal{A}})$  such that  $C \cong \text{Im}(G^0 \rightarrow G^{-1})$ .*

These two results imply that the categories  $\mathcal{G}(\mathcal{A})$  and  $\mathcal{G}(\tilde{\mathcal{A}})$  possess stability, respectively.

The contents of this paper are summarized as follows. In Section 2, we review some basic notation and notions for use throughout the paper. Section 3 is devoted to introducing the notion of Gorenstein projective complexes with respect to cotorsion pairs and giving the proof of Theorem 1.1. By using Theorem 1.1, in Section 4 we give the proof of Theorem 1.2 and Theorem 1.3.

## 2. PRELIMINARIES

Throughout this article,  $R$  denotes an associative ring with identity, modules are assumed to be unitary, and the default action of the ring is on the left. Right modules over  $R$  are hence treated as (left) modules over the opposite ring  $R^\circ$ . We use  $R\text{-Mod}$  to denote the category of  $R$ -modules,  $C(R)$  to denote the category of complexes of  $R$ -modules and  $\mathcal{P}, \mathcal{F}, \mathcal{C}$  to denote the class of projective, flat, cotorsion  $R$ -modules, respectively.

A complex

$$\dots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \dots$$

will be denoted by  $(C, \delta)$  or simply  $C$ . The  $n$ th cycle, boundary, homology of  $C$  is denoted by  $Z_n(C), B_n(C), H_n(C)$ , respectively. We will use superscripts to distinguish

complexes. So if  $\{C^i\}_{i \in I}$  is a family of complexes,  $C^i$  will be complex

$$\cdots \longrightarrow C_{n+1}^i \xrightarrow{\delta_{n+1}^i} C_n^i \xrightarrow{\delta_n^i} C_{n-1}^i \longrightarrow \cdots$$

Given an  $R$ -module  $M$ , we will denote by  $\overline{M}$  the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with  $M$  in 1st and 0th degrees. Given a  $C \in C(R)$  and an integer  $m$ ,  $C[m]$  denotes the complex such that  $C[m]_n = C_{n-m}$  and whose boundary operators are  $(-1)^m \delta_{n-m}$ . Given  $C, D \in C(R)$ , we use  $\text{Hom}_{C(R)}(C, D)$  to present the group of all morphisms from  $C$  to  $D$ , and  $\text{Ext}_{C(R)}^i(C, D)$  denotes the groups one gets from the right derived functor of  $\text{Hom}$  for  $i \geq 0$ .

Let  $C \in C(R^\circ)$  and  $D \in C(R)$ . The tensor product  $C \otimes_R D$  is the  $\mathbb{Z}$ -complex whose underlying graded module is given by  $(C \otimes_R D)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j$ , and whose differential is defined by specifying its action on an elementary tensor of homogeneous elements as  $\delta^{C \otimes_R D}(x \otimes y) = \delta^C(x) \otimes y + (-1)^{|x|} x \otimes \delta^D(y)$ , where  $|x|$  is the degree of  $x$  in  $C$ . Let  $C \overline{\otimes}_R D = C \otimes_R D / B(C \otimes_R D)$ , that is,  $C \overline{\otimes}_R D$  is the complex of abelian groups with  $n$ th entry  $(C \overline{\otimes}_R D)_n = (C \otimes_R D)_n / B_n(C \otimes_R D)$  and boundary map  $\delta^{C \overline{\otimes}_R D}(\overline{x \otimes y}) = \overline{\delta^C(x) \otimes y}$ , where  $\overline{x \otimes y}$  is used to denote the coset in  $(C \otimes_R D)_n / B_n(C \otimes_R D)$ . This gives us a new right exact bifunctor  $-\overline{\otimes}_R-$  which has left derived functor  $\overline{\text{Tor}}_i(-, -)$ .

For  $C, D \in C(R)$ ,  $\text{Hom}_R(C, D)$  is the complex of abelian groups with the degree- $n$  term  $\text{Hom}_R(C, D)_n = \prod_{i \in \mathbb{Z}} \text{Hom}_R(C_i, D_{n+i})$ , and its boundary operators are  $\delta_n^{\text{Hom}_R(C, D)}((f_i)_{i \in \mathbb{Z}}) = (\delta_{n+i}^D f_i - (-1)^n f_{i-1} \delta_i^C)_{i \in \mathbb{Z}}$  for any  $(f_i)_{i \in \mathbb{Z}} \in \text{Hom}_R(C, D)_n$ . Let  $\underline{\text{Hom}}_R(C, D) = Z(\text{Hom}_R(C, D))$ , that is,  $\underline{\text{Hom}}_R(C, D)$  is the complex of abelian groups with  $n$ th entry  $\underline{\text{Hom}}_R(C, D)_n = Z_n(\text{Hom}_R(C, D)) = \text{Hom}_{C(R)}(C, D[-n])$  and boundary map  $\delta_n^{\underline{\text{Hom}}_R(C, D)}((f_i)_{i \in \mathbb{Z}}) = ((-1)^n \delta_{n+i}^D f_i)_{i \in \mathbb{Z}}$  for any  $(f_i)_{i \in \mathbb{Z}} \in \text{Hom}_R(C, D)_n$ . Then we get new functors  $\underline{\text{Hom}}_R(C, -)$  and  $\underline{\text{Hom}}_R(-, D)$  which are both left exact. The book [7] is a standard reference for complexes.

Let  $\mathcal{D}$  be an abelian category. A pair  $(\mathcal{A}, \mathcal{B})$  of classes of objects of  $\mathcal{D}$  is called a cotorsion pair if  $\mathcal{A}^\perp = \mathcal{B}$  and  $\mathcal{A} = {}^\perp \mathcal{B}$ , where  $\mathcal{A}^\perp = \{D \in \mathcal{D} : \text{Ext}_{\mathcal{D}}^1(A, D) = 0 \text{ for all } A \in \mathcal{A}\}$  and  ${}^\perp \mathcal{B} = \{D \in \mathcal{D} : \text{Ext}_{\mathcal{D}}^1(D, B) = 0 \text{ for all } B \in \mathcal{B}\}$ . A special  $\mathcal{A}$ -precover or special  $\mathcal{B}$ -preenvelope of an object  $D \in \mathcal{D}$  is a short exact sequence  $0 \rightarrow B \rightarrow A \rightarrow D \rightarrow 0$  or  $0 \rightarrow D \rightarrow B' \rightarrow A' \rightarrow 0$ , where  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  or  $A' \in \mathcal{A}$  and  $B' \in \mathcal{B}$ , respectively. A cotorsion pair  $(\mathcal{A}, \mathcal{B})$  is said to be complete if every object  $D \in \mathcal{D}$  has a special  $\mathcal{A}$ -precover and a special  $\mathcal{B}$ -preenvelope. A cotorsion pair  $(\mathcal{A}, \mathcal{B})$  in  $\mathcal{D}$  is said to be hereditary if  $\text{Ext}_{\mathcal{D}}^i(A, B) = 0$  for all  $A \in \mathcal{A}, B \in \mathcal{B}$  and all  $i \geq 1$ .

If we choose  $\mathcal{D} = R\text{-Mod}$  for some ring  $R$ , the most obvious example of a complete hereditary cotorsion pair is  $(\mathcal{P}, R\text{-Mod})$ . Perhaps one of the most useful complete hereditary cotorsion pairs is the flat cotorsion pair  $(\mathcal{F}, \mathcal{C})$ . For a good reference on cotorsion pairs see [6].

**Definition 2.1** ([8], Definition 3.3). Let  $(\mathcal{A}, \mathcal{B})$  be a cotorsion pair in  $R\text{-Mod}$  and  $X$  an  $R$ -complex.

- (1)  $X$  is called an  $\mathcal{A}$  complex if it is exact and  $Z_n(X) \in \mathcal{A}$  for all  $n \in \mathbb{Z}$ .
- (2)  $X$  is called a  $\mathcal{B}$  complex if it is exact and  $Z_n(X) \in \mathcal{B}$  for all  $n \in \mathbb{Z}$ .
- (3)  $X$  is called a dg- $\mathcal{A}$  complex if each  $X_n \in \mathcal{A}$  and  $\text{Hom}_R(X, B)$  is exact whenever  $B$  is a  $\mathcal{B}$  complex.
- (4)  $X$  is called a dg- $\mathcal{B}$  complex if each  $X_n \in \mathcal{B}$  and  $\text{Hom}_R(A, X)$  is exact whenever  $A$  is an  $\mathcal{A}$  complex.

We denote the class of  $\mathcal{A}$  complexes by  $\tilde{\mathcal{A}}$  and the class of dg- $\mathcal{A}$  complexes by  $\text{dg}\tilde{\mathcal{A}}$ . Similarly, the  $\mathcal{B}$  complexes are denoted by  $\tilde{\mathcal{B}}$  and the class of dg- $\mathcal{B}$  complexes is denoted by  $\text{dg}\tilde{\mathcal{B}}$ . Sometimes  $\mathcal{A}$  complexes and dg- $\mathcal{A}$  complexes are called complexes by the name of the class  $\mathcal{A}$ . For example, the projective and dg-projective complexes are actually the  $\mathcal{P}$  and dg- $\mathcal{P}$  complexes, respectively. It follows from [8], Proposition 3.6, that  $(\tilde{\mathcal{A}}, \text{dg}\tilde{\mathcal{B}})$  and  $(\text{dg}\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$  are cotorsion pairs in  $C(R)$ . Moreover, by [8], Corollary 3.13, [17], Theorem 2.4 and Corollary 2.7, or [14], Theorem 3.5, we have the following facts.

**Lemma 2.2.** *Let  $(\mathcal{A}, \mathcal{B})$  be a complete hereditary cotorsion pair in  $R\text{-Mod}$ . Then the induced cotorsion pairs  $(\tilde{\mathcal{A}}, \text{dg}\tilde{\mathcal{B}})$  and  $(\text{dg}\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$  in  $C(R)$  are both complete and hereditary. Furthermore,  $\text{dg}\tilde{\mathcal{A}} \cap \mathcal{E} = \tilde{\mathcal{A}}$  and  $\text{dg}\tilde{\mathcal{B}} \cap \mathcal{E} = \tilde{\mathcal{B}}$ , where  $\mathcal{E}$  is the class of exact complexes.*

Let  $\mathcal{D}$  be an abelian category and  $\mathcal{H}$  a full subcategory of  $\mathcal{D}$ . Recall that a sequence  $\mathbf{S}$  in  $\mathcal{D}$  is  $\text{Hom}_{\mathcal{D}}(-, \mathcal{H})$ -exact (resp.,  $\text{Hom}_{\mathcal{D}}(\mathcal{H}, -)$ -exact) if the sequence  $\text{Hom}_{\mathcal{D}}(\mathbf{S}, H)$  (resp.,  $\text{Hom}_{\mathcal{D}}(H, \mathbf{S})$ ) is exact for any  $H \in \mathcal{H}$ .

**Definition 2.3** ([16], Definition 3.1). Let  $(\mathcal{A}, \mathcal{B})$  be a complete and hereditary cotorsion pair in  $R\text{-Mod}$ . An  $R$ -module  $M$  is called Gorenstein projective with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  if there exists a  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact exact sequence  $\dots \rightarrow A_1 \rightarrow A_0 \rightarrow A_{-1} \rightarrow \dots$  with each  $A_i \in \mathcal{A}$ , such that  $M \cong \text{Im}(A_0 \rightarrow A_{-1})$ . We let  $\mathcal{G}(\mathcal{A})$  be the class of Gorenstein projective  $R$ -modules with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$ .

**Remark 2.4.** (1) By completeness of the cotorsion pair  $(\mathcal{A}, \mathcal{B})$ , an  $R$ -module  $M$  is in  $\mathcal{G}(\mathcal{A})$  if and only if  $\text{Ext}_R^{i \geq 1}(M, N) = 0$  for any  $N \in \mathcal{A} \cap \mathcal{B}$  and there exists

a  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact exact sequence  $0 \rightarrow M \rightarrow A_0 \rightarrow A_{-1} \rightarrow \dots$  with each  $A_i \in \mathcal{A}$ .

(2) This definition unifies the following notions: Gorenstein projective modules [5], [9] (in the case  $(\mathcal{A}, \mathcal{B}) = (\mathcal{P}, R\text{-Mod})$ );  $\mathbf{F}$ -Gorenstein flat modules [10] (when  $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$ ); and Gorenstein flat modules [7] (when  $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$  and  $R$  is a right coherent ring), see [10], Lemma 3.2.

In what follows, we always assume that  $(\mathcal{A}, \mathcal{B})$  is a complete and hereditary cotorsion pair in  $R\text{-Mod}$ .

### 3. GORENSTEIN PROJECTIVE COMPLEXES WITH RESPECT TO COTORSION PAIRS

**Definition 3.1.** An  $R$ -complex  $C$  is called Gorenstein projective with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  if there exists a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}})$ -exact exact sequence  $\dots \rightarrow A^1 \rightarrow A^0 \rightarrow A^{-1} \rightarrow \dots$  with each  $A^i \in \tilde{\mathcal{A}}$  such that  $C \cong \text{Im}(A^0 \rightarrow A^{-1})$ .

We denote the class of Gorenstein projective  $R$ -complexes with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  by  $\mathcal{G}(\tilde{\mathcal{A}})$ .

**Remark 3.2.** (1) It is clear that  $\tilde{\mathcal{A}} \subseteq \mathcal{G}(\tilde{\mathcal{A}})$ . If  $\mathbf{A} = \dots \rightarrow A^1 \rightarrow A^0 \rightarrow A^{-1} \rightarrow \dots$  is a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}})$ -exact exact sequence of complexes in  $\tilde{\mathcal{A}}$ , then by symmetry, all the images, the kernels and the cokernels of  $\mathbf{A}$  are in  $\mathcal{G}(\tilde{\mathcal{A}})$ .

(2) If  $(\mathcal{A}, \mathcal{B}) = (\mathcal{P}, R\text{-Mod})$ , then Gorenstein projective complexes with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  are exactly the Gorenstein projective complexes in [4].

(3) If  $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$ , the flat cotorsion pair, then Gorenstein projective complexes with respect to the cotorsion pair  $(\mathcal{A}, \mathcal{B})$  are just  $\mathbf{F}$ -Gorenstein flat complexes in [10].

Recall from [7] that a short exact sequence  $0 \rightarrow S \rightarrow C \rightarrow C/S \rightarrow 0$  in  $C(R)$  is pure if the sequence  $0 \rightarrow D \overline{\otimes}_R S \rightarrow D \overline{\otimes}_R C \rightarrow D \overline{\otimes}_R C/S \rightarrow 0$  is exact for any  $D \in C(R^\circ)$ . According to [7], an  $R$ -complex  $C$  is called Gorenstein flat if there exists an exact sequence of flat complexes  $\dots \rightarrow F^1 \rightarrow F^0 \rightarrow F^{-1} \rightarrow \dots$  with  $C \cong \text{Im}(F^0 \rightarrow F^{-1})$  which remains exact after applying  $I \overline{\otimes}_R -$  for any injective  $R^\circ$ -complex  $I$ . The next result shows that Gorenstein projective complexes with respect to the cotorsion pair  $(\mathcal{F}, \mathcal{C})$  over right coherent rings are just Gorenstein flat complexes.

**Proposition 3.3.** *If  $R$  is a right coherent ring, then  $C$  is an  $\mathbf{F}$ -Gorenstein flat complex if and only if  $C$  is Gorenstein flat.*

**Proof.**  $\Rightarrow$ : Assume that  $C$  is an  $\mathbf{F}$ -Gorenstein flat complex. Then there exists a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{F}} \cap \text{dg } \tilde{\mathcal{C}})$ -exact exact sequence  $\dots \rightarrow F^1 \rightarrow F^0 \rightarrow F^{-1} \rightarrow \dots$  of flat

complexes such that  $C \cong \text{Im}(F^0 \rightarrow F^{-1})$ . Let  $I$  be an injective complex of right  $R$ -modules. Then  $I^+[-n] \in \tilde{\mathcal{F}} \cap \text{dg}\tilde{\mathcal{C}}$  for any  $n \in \mathbb{Z}$  since  $R$  is right coherent, where  $I^+ = \underline{\text{Hom}}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ . Thus the sequence

$$\begin{aligned} \dots &\rightarrow \text{Hom}_{C(R)}(F^{-1}, I^+[-n]) \rightarrow \\ &\rightarrow \text{Hom}_{C(R)}(F^0, I^+[-n]) \rightarrow \text{Hom}_{C(R)}(F^1, I^+[-n]) \rightarrow \dots \end{aligned}$$

is exact for any  $n \in \mathbb{Z}$ , and so

$$\dots \rightarrow \underline{\text{Hom}}_R(F^{-1}, I^+) \rightarrow \underline{\text{Hom}}_R(F^0, I^+) \rightarrow \underline{\text{Hom}}_R(F^1, I^+) \rightarrow \dots$$

is exact. Hence the sequence

$$\dots \rightarrow I \otimes_R F^1 \rightarrow I \otimes_R F^0 \rightarrow I \otimes_R F^{-1} \rightarrow \dots$$

is exact by [7], Proposition 4.2.1 (1). Therefore  $C$  is Gorenstein flat.

$\Leftarrow$ : Suppose that  $C$  is a Gorenstein flat complex. Then there is an exact sequence  $\dots \rightarrow F^1 \rightarrow F^0 \rightarrow F^{-1} \rightarrow \dots$  of flat complexes with  $C \cong \text{Im}(F^0 \rightarrow F^{-1})$  which remains exact after applying  $I \otimes_R -$  for any injective  $R^\circ$ -complex  $I$ . Let  $K \in \tilde{\mathcal{F}} \cap \text{dg}\tilde{\mathcal{C}}$ . Then we have a pure exact sequence  $0 \rightarrow K \rightarrow K^{++} \rightarrow K^{++}/K \rightarrow 0$  by [7], Proposition 5.1.4 (4). Since  $K \in \tilde{\mathcal{F}}$ , we get  $K^{++} \in \tilde{\mathcal{F}}$ . So  $K^{++}/K \in \tilde{\mathcal{F}}$  by [8], Lemma 4.7. Thus the sequence  $0 \rightarrow K \rightarrow K^{++} \rightarrow K^{++}/K \rightarrow 0$  is split. By [7], Proposition 4.2.1 (1), we have the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & (K^+ \otimes_R F^{-1})^+ & \longrightarrow & (K^+ \otimes_R F^0)^+ & \longrightarrow & (K^+ \otimes_R F^1)^+ & \longrightarrow & \dots \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \\ \dots & \longrightarrow & \underline{\text{Hom}}_R(F^{-1}, K^{++}) & \longrightarrow & \underline{\text{Hom}}_R(F^0, K^{++}) & \longrightarrow & \underline{\text{Hom}}_R(F^1, K^{++}) & \longrightarrow & \dots \end{array}$$

where the top row is exact since  $K^+$  is injective. So the lower row is exact. Hence the sequence

$$\dots \rightarrow \underline{\text{Hom}}_R(F^{-1}, K) \rightarrow \underline{\text{Hom}}_R(F^0, K) \rightarrow \underline{\text{Hom}}_R(F^1, K) \rightarrow \dots$$

is exact. In particular, the sequence

$$\dots \rightarrow \text{Hom}_{C(R)}(F^{-1}, K) \rightarrow \text{Hom}_{C(R)}(F^0, K) \rightarrow \text{Hom}_{C(R)}(F^1, K) \rightarrow \dots$$

is exact. So  $C$  is an  $\mathbf{F}$ -Gorenstein flat complex.  $\square$

The following result will be used in the sequel.

**Lemma 3.4.** *Let  $\dots \rightarrow X^1 \rightarrow X^0 \rightarrow X^{-1} \rightarrow \dots$  be a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}})$ -exact sequence of complexes, then the sequence  $\dots \rightarrow X_n^1 \rightarrow X_n^0 \rightarrow X_n^{-1} \rightarrow \dots$  is  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact for any  $n \in \mathbb{Z}$ .*

*Proof.* Let  $K \in \mathcal{A} \cap \mathcal{B}$  and  $n \in \mathbb{Z}$ . Then  $\overline{K}[n] \in \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}}$  by [8], Lemma 3.4. So we have the exact sequence

$$\dots \rightarrow \text{Hom}_{C(R)}(X^{-1}, \overline{K}[n]) \rightarrow \text{Hom}_{C(R)}(X^0, \overline{K}[n]) \rightarrow \text{Hom}_{C(R)}(X_1, \overline{K}[n]) \rightarrow \dots$$

Using the standard adjunction of [8], Lemma 3.1 (2), we get the exact sequence

$$\dots \rightarrow \text{Hom}_R(X_n^{-1}, K) \rightarrow \text{Hom}_R(X_n^0, K) \rightarrow \text{Hom}_R(X_n^1, K) \rightarrow \dots$$

This completes the proof.  $\square$

Now, we are in position to prove our main result, which gives a characterization of complexes in  $\mathcal{G}(\tilde{\mathcal{A}})$  and unifies [10], Theorem 4.7.

**Theorem 3.5.** *Let  $C$  be an  $R$ -complex. Then  $C \in \mathcal{G}(\tilde{\mathcal{A}})$  if and only if  $C_n \in \mathcal{G}(\mathcal{A})$  for any  $n \in \mathbb{Z}$ .*

*Proof.*  $\Rightarrow$ : Assume that  $C \in \mathcal{G}(\tilde{\mathcal{A}})$ . Then there exists a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}})$ -exact exact sequence  $\dots \rightarrow A^1 \rightarrow A^0 \rightarrow A^{-1} \rightarrow \dots$  with each  $A^i \in \tilde{\mathcal{A}}$  such that  $C \cong \text{Im}(A^0 \rightarrow A^{-1})$ . Now for any but fixed  $n \in \mathbb{Z}$ , by Lemma 3.4, we have the  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact exact sequence of modules in  $\mathcal{A}$

$$\dots \rightarrow A_n^1 \rightarrow A_n^0 \rightarrow A_n^{-1} \rightarrow A_n^{-2} \rightarrow \dots$$

such that  $C_n = \text{Im}(A_n^0 \rightarrow A_n^{-1})$ . Hence  $C_n \in \mathcal{G}(\mathcal{A})$ .

$\Leftarrow$ : Suppose that  $C_n \in \mathcal{G}(\mathcal{A})$  for all  $n \in \mathbb{Z}$ . Then for any  $n \in \mathbb{Z}$  there exists an exact sequence

$$0 \rightarrow C_n \rightarrow A_n \rightarrow L_n \rightarrow 0,$$

where  $A_n \in \mathcal{A}$  and  $L_n \in \mathcal{G}(\mathcal{A})$ . These exact sequences induce a short exact sequence of complexes

$$0 \rightarrow \bigoplus_{n \in \mathbb{Z}} \overline{C_n}[n] \rightarrow \bigoplus_{n \in \mathbb{Z}} \overline{A_n}[n] \rightarrow \bigoplus_{n \in \mathbb{Z}} \overline{L_n}[n] \rightarrow 0.$$

Put  $A^{-1} = \bigoplus_{n \in \mathbb{Z}} \overline{A_n}[n]$ . It is easy to see that  $A^{-1} \in \tilde{\mathcal{A}}$ . On the other hand, there is an obvious (degreewise split) short exact sequence

$$0 \longrightarrow C \xrightarrow{\begin{pmatrix} 1 \\ \delta \end{pmatrix}} \bigoplus_{n \in \mathbb{Z}} \overline{C_n}[n] \xrightarrow{(-\delta, 1)} C[1] \longrightarrow 0,$$



where  $\delta$  is the differential of  $C$ . Now let  $\alpha: C \rightarrow A^{-1}$  be the composite

$$C \rightarrow \bigoplus_{n \in \mathbb{Z}} \overline{C_n}[n] \rightarrow \bigoplus_{n \in \mathbb{Z}} \overline{A_n}[n].$$

Then  $\alpha$  is monoic since it is the composite of two monomorphisms. Denote Coker  $\alpha$  by  $C^{-1}$ . Then by the Snake lemma, we have a short exact sequence

$$0 \rightarrow C[1] \rightarrow C^{-1} \rightarrow \bigoplus_{n \in \mathbb{Z}} \overline{L_n}[n] \rightarrow 0.$$

Since each degree of  $\bigoplus_{n \in \mathbb{Z}} \overline{L_n}[n]$  and  $C[1]$  is in  $\mathcal{G}(\mathcal{A})$ , each degree of  $C^{-1}$  belongs to  $\mathcal{G}(\mathcal{A})$  by [16], Proposition 3.3 (1). Let  $K \in \widetilde{\mathcal{A}} \cap \text{dg } \widetilde{\mathcal{B}}$ . Then  $K \in \widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}} = \widetilde{\mathcal{A} \cap \mathcal{B}}$  by [8], Theorem 3.12. Thus  $K \cong \prod_{n \in \mathbb{Z}} \overline{Z_n(K)}[n]$  by [11], Lemma 4.1. Hence

$$\text{Ext}_{C(R)}^1(C^{-1}, K) \cong \prod_{n \in \mathbb{Z}} \text{Ext}_{C(R)}^1(C^{-1}, \overline{Z_n(K)}[n]) \cong \prod_{n \in \mathbb{Z}} \text{Ext}_R^1(C_n^{-1}, Z_n(K)) = 0,$$

where the second isomorphism follows from [8], Lemma 3.1 (2), and the last equality follows from Remark 2.4 (1). This implies that  $0 \rightarrow C \rightarrow A^{-1} \rightarrow C^{-1} \rightarrow 0$  is  $\text{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \text{dg } \widetilde{\mathcal{B}})$ -exact. Notice that  $C^{-1}$  has the same property as  $C$ , so we can use the same procedure to construct a  $\text{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \text{dg } \widetilde{\mathcal{B}})$ -exact exact sequence of complexes

$$(\dagger) \quad 0 \rightarrow C \rightarrow A^{-1} \rightarrow A^{-2} \rightarrow \dots,$$

where each  $A^i$  is an  $\mathcal{A}$ -complex.

Since  $(\widetilde{\mathcal{A}}, \text{dg } \widetilde{\mathcal{B}})$  is a complete cotorsion pair, we have a short exact sequence  $0 \rightarrow C^1 \rightarrow A^0 \rightarrow C \rightarrow 0$ , where  $A^0 \in \widetilde{\mathcal{A}}$  and  $C^1 \in \text{dg } \widetilde{\mathcal{B}}$ . Note that  $C_n \in \mathcal{G}(\mathcal{A})$  for any  $n \in \mathbb{Z}$ , this sequence is  $\text{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \text{dg } \widetilde{\mathcal{B}})$ -exact by a discussion similar to the above. Also, it follows from the exact sequence and [16], Proposition 3.3 (1), that each  $C_n^1 \in \mathcal{G}(\mathcal{A})$  for any  $n \in \mathbb{Z}$ . Thus we can continuously use the same method to construct a  $\text{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \text{dg } \widetilde{\mathcal{B}})$ -exact exact sequence

$$(\ddagger) \quad \dots \rightarrow A^1 \rightarrow A^0 \rightarrow C \rightarrow 0,$$

where each  $A^i$  is an  $\mathcal{A}$ -complex.

Finally, gluing the sequences  $(\dagger)$  and  $(\ddagger)$  together, one has a  $\text{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \text{dg } \widetilde{\mathcal{B}})$ -exact exact sequence of complexes

$$\dots \rightarrow A^1 \rightarrow A^0 \rightarrow A^{-1} \rightarrow A^{-2} \rightarrow \dots$$

with all  $A^i \in \widetilde{\mathcal{A}}$  such that  $C \cong \text{Im}(A^0 \rightarrow A^{-1})$ . Hence  $C \in \mathcal{G}(\widetilde{\mathcal{A}})$ .  $\square$

Let  $\mathcal{D}$  be an abelian category with enough projective objects and injective objects. Recall that a class  $\mathcal{X}$  of objects of  $\mathcal{D}$  is said to be projectively resolving or injectively resolving if it is closed under extensions and kernels of surjections or cokernels of injections, and contains all projective or injective objects of  $\mathcal{D}$ , respectively.

**Corollary 3.6.**  $\mathcal{G}(\tilde{\mathcal{A}})$  is projectively resolving.

*Proof.* Clearly,  $\tilde{\mathcal{P}} \subseteq \tilde{\mathcal{A}} \subseteq \mathcal{G}(\tilde{\mathcal{A}})$ . Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  be a short exact sequence of complexes with  $C'' \in \mathcal{G}(\tilde{\mathcal{A}})$ . Then for any  $n \in \mathbb{Z}$ , in the exact sequence  $0 \rightarrow C'_n \rightarrow C_n \rightarrow C''_n \rightarrow 0$ ,  $C''_n \in \mathcal{G}(\mathcal{A})$  by Theorem 3.5. So  $C'_n \in \mathcal{G}(\mathcal{A})$  if and only if  $C_n \in \mathcal{G}(\mathcal{A})$  by [16], Proposition 3.3 (1). Hence  $C' \in \mathcal{G}(\tilde{\mathcal{A}})$  if and only if  $C \in \mathcal{G}(\tilde{\mathcal{A}})$  by Theorem 3.5. Now the result follows.  $\square$

**Corollary 3.7.** Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  be an exact sequence of complexes. If  $C', C$  belong to  $\mathcal{G}(\tilde{\mathcal{A}})$ , then  $C'' \in \mathcal{G}(\tilde{\mathcal{A}})$  if and only if  $\text{Ext}_{C(R)}^1(C'', K) = 0$  for any  $K \in \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}}$ .

*Proof.*  $\Rightarrow$ : It is obvious.

$\Leftarrow$ : Let  $n \in \mathbb{Z}$ . Consider the exact sequence of  $R$ -modules

$$0 \rightarrow C'_n \rightarrow C_n \rightarrow C''_n \rightarrow 0.$$

By Theorem 3.5,  $C'_n, C_n$  belong to  $\mathcal{G}(\mathcal{A})$ . Let  $K \in \mathcal{A} \cap \mathcal{B}$ . Then  $\overline{K}[n] \in \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}}$ . Thus  $\text{Ext}_R^1(C''_n, K) \cong \text{Ext}_{C(R)}^1(C'', \overline{K}[n]) = 0$  by [8], Lemma 3.1 (2), and the hypothesis. Hence  $C''_n \in \mathcal{G}(\mathcal{A})$  by [16], Proposition 3.3 (2). Therefore  $C'' \in \mathcal{G}(\tilde{\mathcal{A}})$  by Theorem 3.5.  $\square$

By Proposition 3.3, Theorem 3.5 and [10], Lemma 3.2, we immediately get:

**Corollary 3.8** ([18], Theorem 3.1). Let  $C$  be an  $R$ -complex. If  $R$  is a right coherent ring, then  $C$  is Gorenstein flat if and only if  $C_n$  is a Gorenstein flat  $R$ -module for any  $n \in \mathbb{Z}$ .

#### 4. STABILITY OF GORENSTEIN CATEGORIES WITH RESPECT TO COTORSION PAIRS

The stability of Gorenstein categories was initiated by Sather-Wagstaff, Sharif and White [12]. They proved that if  $R$  is a commutative ring, then an  $R$ -module  $M$

is a Gorenstein projective or injective module if and only if there exists an exact sequence of Gorenstein projective or injective  $R$ -modules  $G = \dots \rightarrow G_1 \xrightarrow{\delta_1} G_0 \xrightarrow{\delta_0} G_{-1} \rightarrow \dots$  such that the complexes  $\text{Hom}_R(H, G)$  and  $\text{Hom}_R(G, H)$  are exact for each Gorenstein projective or injective  $R$ -module  $H$ , respectively, and  $M = \text{Im} \delta_0$ . This was developed by Bouchiba [1], Xu and Ding [13], respectively. They showed, via different methods, that over any ring  $R$ , an  $R$ -module  $M$  is Gorenstein projective or injective if and only if there exists an exact sequence of Gorenstein projective or injective  $R$ -modules  $G = \dots \rightarrow G_1 \xrightarrow{\delta_1} G_0 \xrightarrow{\delta_0} G_{-1} \rightarrow \dots$  such that the complex  $\text{Hom}_R(G, H)$  or  $\text{Hom}_R(H, G)$  is exact for any projective or injective  $R$ -module  $H$ , respectively, and  $M = \text{Im} \delta_0$ . For more details, see [1]. The stability of Gorenstein flat  $R$ -module has been treated by Bouchiba and Khaloui [2], Xu and Ding [13], Yang and Liu [15], respectively. By using totally different techniques, they showed that over a left GF-closed ring  $R$  (a ring  $R$  over which the class of the Gorenstein flat  $R$ -modules is closed under extensions), an  $R$ -module  $M$  is Gorenstein flat if and only if there exists an exact sequence of Gorenstein flat  $R$ -modules  $G = \dots \rightarrow G^1 \xrightarrow{\delta^1} G^0 \xrightarrow{\delta^0} G^{-1} \rightarrow \dots$  such that the complex  $I \otimes_R G$  is exact for each Gorenstein injective (or injective)  $R^\circ$ -module  $I$  and  $M = \text{Im} \delta^0$ . By using Theorem 3.5, in this section we investigate the stability of  $\mathcal{G}(\mathcal{A})$  and  $\mathcal{G}(\tilde{\mathcal{A}})$ .

The next result shows that the category  $\mathcal{G}(\mathcal{A})$  possesses stability, which is a generalization of [12], Theorem A, [13], Theorem A, and [10], Theorem 3.8.

**Theorem 4.1.** *Let  $M$  be an  $R$ -module. Then the following statements are equivalent:*

- (1)  $M \in \mathcal{G}(\mathcal{A})$ .
- (2) *There exists a both  $\text{Hom}_R(\mathcal{G}(\mathcal{A}), -)$ -exact and  $\text{Hom}_R(-, \mathcal{G}(\mathcal{A}))$ -exact exact sequence*

$$\dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow \dots$$

*in  $\mathcal{G}(\mathcal{A})$  so that  $M \cong \text{Im}(G_0 \rightarrow G_{-1})$ .*

- (3) *There exists a  $\text{Hom}_R(-, \mathcal{G}(\mathcal{A}))$ -exact exact sequence*

$$\dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow \dots$$

*in  $\mathcal{G}(\mathcal{A})$  such that  $M \cong \text{Im}(G_0 \rightarrow G_{-1})$ .*

- (4) *There exists a  $\text{Hom}_R(-, \mathcal{A})$ -exact exact sequence*

$$\dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow \dots$$

*in  $\mathcal{G}(\mathcal{A})$  such that  $M \cong \text{Im} G_0 \rightarrow G_{-1}$ .*

(5) There exists a  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact exact sequence

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow \cdots$$

in  $\mathcal{G}(\mathcal{A})$  such that  $M \cong \text{Im}(G_0 \rightarrow G_{-1})$ .

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1): Assume that there is a  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact exact sequence

$$G = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$

in  $\mathcal{G}(\mathcal{A})$  such that  $M \cong Z_{-1}(G)$ . Then  $G \in \mathcal{G}(\tilde{\mathcal{A}})$  by Theorem 3.5. Thus there exists a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg } \tilde{\mathcal{B}})$ -exact exact sequence

$$\cdots \longrightarrow A^1 \xrightarrow{\sigma^1} A^0 \xrightarrow{\sigma^0} A^{-1} \xrightarrow{\sigma^{-1}} A^{-2} \longrightarrow \cdots$$

with each  $A^i \in \tilde{\mathcal{A}}$  such that  $G \cong \text{Ker } \sigma^{-1}$ . Set  $K^i = \text{Ker } \sigma^i$  for  $i \in \mathbb{Z}$ . Then  $K^i \in \mathcal{G}(\tilde{\mathcal{A}})$  and  $K^i$  is exact for any  $i \in \mathbb{Z}$  since  $K^{-1} = G$  and all  $A^i$  are exact. So, by [11], Lemma 4.15 (1), we have the exact sequence

$$(h) \quad \cdots \longrightarrow Z_{-1}(A^1) \xrightarrow{Z_{-1}(\sigma^1)} Z_{-1}(A^0) \xrightarrow{Z_{-1}(\sigma^0)} Z_{-1}(A^{-1}) \longrightarrow \cdots$$

with each  $Z_{-1}(A^i) \in \mathcal{A}$ , such that  $M \cong Z_{-1}(G) = \text{Ker}(Z_{-1}(\sigma^{-1}))$ . To show  $M \in \mathcal{G}(\mathcal{A})$ , we need only to show that the sequence (h) is  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact.

Let  $H \in \mathcal{A} \cap \mathcal{B}$ , it suffices to show that  $\text{Ext}_R^1(Z_{-1}(K^i), H) = 0$  for all  $i \in \mathbb{Z}$ . Since each  $K^i \in \mathcal{G}(\tilde{\mathcal{A}})$ , all  $K_n^i \in \mathcal{G}(\mathcal{A})$  by Theorem 3.5. Thus, for any  $i \in \mathbb{Z}$ , the sequence

$$0 \longrightarrow \text{Hom}_R(K^{i-1}, H) \longrightarrow \text{Hom}_R(A^i, H) \longrightarrow \text{Hom}_R(K^i, H) \longrightarrow 0$$

is exact. By the hypothesis,  $\text{Hom}_R(K^{-1}, H)$  is exact. Note that  $\text{Hom}_R(A^i, H)$  is exact for each  $i \in \mathbb{Z}$ , then  $\text{Hom}_R(K^i, H)$  is exact for any  $i \in \mathbb{Z}$ . Hence  $\text{Ext}_R^1(Z_{-1}(K^i), H) = 0$  since each  $K_0^i \in \mathcal{G}(\mathcal{A})$ . Thus the sequence (h) is  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact, as desired.  $\square$

Finally, by applying Theorem 3.5 and Theorem 4.1, we can achieve the following stability result for  $\mathcal{G}(\tilde{\mathcal{A}})$ , which is a unification of [13], Theorem 3.1, and [10], Theorem 4.11.

**Theorem 4.2.** *Let  $C$  be a complex of  $R$ -modules. Then the following statements are equivalent:*

- (1)  $C \in \mathcal{G}(\tilde{\mathcal{A}})$ .  
(2) There exists a both  $\text{Hom}_{C(R)}(\mathcal{G}(\tilde{\mathcal{A}}), -)$ -exact and  $\text{Hom}_{C(R)}(-, \mathcal{G}(\tilde{\mathcal{A}}))$ -exact exact sequence

$$\dots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \dots$$

in  $\mathcal{G}(\tilde{\mathcal{A}})$  such that  $C \cong \text{Im}(G^0 \rightarrow G^{-1})$ .

- (3) There is a  $\text{Hom}_{C(R)}(-, \mathcal{G}(\tilde{\mathcal{A}}))$ -exact exact sequence

$$\dots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \dots$$

in  $\mathcal{G}(\tilde{\mathcal{A}})$  such that  $C \cong \text{Im}(G^0 \rightarrow G^{-1})$ .

- (4) There is a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}})$ -exact exact sequence

$$\dots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \dots$$

in  $\mathcal{G}(\tilde{\mathcal{A}})$  such that  $C \cong \text{Im}(G^0 \rightarrow G^{-1})$ .

- (5) There is a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg}\tilde{\mathcal{B}})$ -exact exact sequence

$$\dots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \dots$$

in  $\mathcal{G}(\tilde{\mathcal{A}})$  such that  $C \cong \text{Im}(G^0 \rightarrow G^{-1})$ .

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1): Suppose that there exists a  $\text{Hom}_{C(R)}(-, \tilde{\mathcal{A}} \cap \text{dg}\tilde{\mathcal{B}})$ -exact exact sequence

$$\dots \longrightarrow G^1 \xrightarrow{\sigma^1} G^0 \xrightarrow{\sigma^0} G^{-1} \xrightarrow{\sigma^{-1}} G^{-2} \longrightarrow \dots$$

in  $\mathcal{G}(\tilde{\mathcal{A}})$  such that  $C \cong \text{Im} \sigma^0$ . Then for any  $n \in \mathbb{Z}$ , by Lemma 3.4 we have the  $\text{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$ -exact exact sequence of modules

$$\dots \longrightarrow G_n^1 \xrightarrow{\sigma_n^1} G_n^0 \xrightarrow{\sigma_n^0} G_n^{-1} \xrightarrow{\sigma_n^{-1}} G_n^{-2} \longrightarrow \dots$$

such that  $C_n \cong \text{Im} \sigma_n^0$ . By Theorem 3.5,  $G_n^i \in \mathcal{G}(\mathcal{A})$  for each  $i \in \mathbb{Z}$ . Thus  $C_n \in \mathcal{G}(\mathcal{A})$  by Theorem 4.1. Hence  $C \in \mathcal{G}(\tilde{\mathcal{A}})$  by Theorem 3.5.  $\square$

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