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A NOTE ON A CONJECTURE ON NICHE HYPERGRAPHS

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Abstract. For a digraph D, the niche hypergraph $N\mathcal{H}(D)$ of D is the hypergraph having the same set of vertices as D and the set of hyperedges $E(N\mathcal{H}(D)) = \{e \subseteq V(D) \colon |e| \geqslant 2$ and there exists a vertex v such that $e = N_D^-(v)$ or $e = N_D^+(v)\}$. A digraph is said to be acyclic if it has no directed cycle as a subdigraph. For a given hypergraph $\mathcal{H}$, the niche number $\hat{n}(\mathcal{H})$ is the smallest integer such that $\mathcal{H}$ together with $\hat{n}(\mathcal{H})$ isolated vertices is the niche hypergraph of an acyclic digraph. C. Garske, M. Sonntag and H. M. Teichert (2016) conjectured that for a linear hypercycle $\mathcal{C}_m$, $m \geqslant 2$, if $\min\{|e| \colon e \in E(\mathcal{C}_m)\} \geqslant 3$, then $\hat{n}(\mathcal{C}_m) = 0$. In this paper, we prove that this conjecture is true.

Keywords: niche hypergraph; digraph; linear hypercycle

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1. Introduction

All hypergraphs in this note are finite and might have isolated vertices but no loops or multiple edges. For a hypergaph $\mathcal{H}$, let $V(\mathcal{H})$ denote the set of all vertices and $E(\mathcal{H})$ denote the set of all hyperedges. Moreover, we let $\underline{d}(\mathcal{H}) = \min\{|e| \colon e \in E(\mathcal{H})\}$. For an integer $m \geqslant 3$, a *linear hypercycle* $\mathcal{C}_m$ of length m is the hypergraph induced by the hyperedges $e_1, e_2, \ldots, e_{m-1}$ and e_m such that

$$|e_i \cap e_j| = \begin{cases} 1 & \text{if } j = i+1 \text{ for } 1 \leqslant i \leqslant m-1 \text{ or } i = m \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

However, a linear hypercycle $\mathcal{C}_2$ of length two is induced by the two hyperedges e_1 and e_2 such that $|e_1 \cap e_2| = 2$. For a digraph D in this note we assume that D might have isolated vertices or loops but no multiple edges. Moreover, we let $V(D)$ denote the set of all vertices and $A(D)$ denote the set of all arcs. The *in-neighborhood* and the *out-neighborhood* are denoted by $N^-_D(v)$ and $N^+_D(v)$, respectively. For a digraph D, the *niche hypergraph* $N\mathcal{H}(D)$ of D is the hypergraph having the same set of vertices as D and the set of hyperedges

$$E(N\mathcal{H}(D)) = \{e \subseteq V(D)\colon\ |e| \geqslant 2 \text{ and there exists a vertex } v \text{ such that } e = N^-_D(v) \text{ or } e = N^+_D(v)\}.$$

A digraph is said to be *acyclic* if it has no directed cycle as a subdigraph. For a given hypergraph $\mathcal{H}$, the *niche number* $\hat{n}(\mathcal{H})$ of $\mathcal{H}$ is the smallest integer such that $\mathcal{H}$ together with $\hat{n}(\mathcal{H})$ isolated vertices is the niche hypergraph of an acyclic digraph. For a vertex $x \in V(D)$ and a set of vertices $X \subseteq V(D)$ we use $x \to X$ to denote the set of all arcs from x to every vertex in X and we use $X \to x$ to denote the set of all arcs from every vertex in X to x.

Garske et al. in [1] conjectured that if $\underline{d}(\mathcal{C}_m) \geqslant 3$, then $\hat{n}(\mathcal{C}_m) = 0$ for each integer $m \geqslant 2$. In this paper, we prove that this conjecture is true.

2. Main results

In this section, for each integer $m \geqslant 2$ we give constructions of acyclic digraphs having $\mathcal{C}_m$ as the niche hypergraph without adding any isolated vertex. First of all, we set up the notation of a linear hypercycle $\mathcal{C}_m$. In the following, we let $\mathcal{C}_m$ be a linear hypercycle such that

$$V(\mathcal{C}_m) = \bigcup_{i=1}^{m} \{a^i_1, a^i_2, \ldots, a^i_{n_i}\} \quad \text{and} \quad E(\mathcal{C}_m) = \{e_1, e_2, \ldots, e_m\},$$

where

$$|e_i| = n_i \geqslant 2 \text{ and } e_i = \{a^i_1, a^i_2, \ldots, a^i_{n_i} = a^{i+1}_1\} \text{ for } 1 \leqslant i \leqslant m-1$$
$$\text{and } e_m = \{a^m_1, a^m_2, \ldots, a^m_{n_m} = a^1_1\}.$$

From the assumption $\underline{d}(\mathcal{C}_m) \geqslant 3$ we obtain $a^i_{n_i-1} \neq a^i_1$ for all $1 \leqslant i \leqslant m$. The following lemma provides constructions of acyclic digraphs having $\mathcal{C}_m$ as the niche hypergraphs when m is small.

Lemma 2.1. *For an integer $2 \leqslant m \leqslant 4$, let $\mathcal{C}_m$ be a linear hypercycle such that $\underline{d}(\mathcal{C}_m) \geqslant 3$. Then there exists an acyclic digraph D with $V(D) = V(\mathcal{C}_m)$ having $\mathcal{C}_m$ as the niche hypergraph.*

P r o o f. For $m \in \{2,3,4\}$ we construct an acyclic digraph $D=(V,A)$ having the niche hypergraph $N\mathcal{H}(D)=\mathcal{C}_m=(V,\{e_1,e_2,\ldots,e_m\})$. Obviously, it suffices to give $A(D)=A$ in each case. We let

$$A(D)=\begin{cases}(e_1\to a^2_{n_2-1})\cup(a^1_{n_1-1}\to e_2) & \text{if } m=2,\\ (e_1\to a^2_{n_2})\cup(a^1_1\to e_2)\cup(e_3\to a^2_{n_2-1}) & \text{if } m=3,\\ (e_1\to a^2_{n_2-1})\cup(a^1_{n_1-1}\to e_2)\cup(e_3\to a^4_{n_4-1})\cup(a^3_{n_3-1}\to e_4) & \text{if } m=4.\end{cases}$$

Clearly, D is acyclic and has $\mathcal{C}_m$ as the niche hypergraph. □

Lemma 2.2. *For an odd integer $m\geqslant 5$, let $\mathcal{C}_m$ be a linear hypercycle such that $\underline{d}(\mathcal{C}_m)\geqslant 3$. Then there exists an acyclic digraph D with $V(D)=V(\mathcal{C}_m)$ having $\mathcal{C}_m$ as the niche hypergraph.*

P r o o f. For $m=5$ and $m\geqslant 7$, respectively, we construct an acyclic digraph $D=(V,A)$ having the niche hypergraph $N\mathcal{H}(D)=\mathcal{C}_m=(V,\{e_1,e_2,\ldots,e_m\})$.

Case 1: $m=5$. Obviously, it suffices to give $A(D)=A$. Let

$$A(D)=(e_1\to a^3_{n_3-1})\cup(e_2\to a^4_1)\cup(a^2_1\to e_3)\cup(a^3_1\to e_4)\cup(e_3\to a^5_1)\cup(a^4_1\to e_5).$$

Case 2: $m\geqslant 7$. Let D be a digraph with $V(D)=V(\mathcal{C}_m)$ and $A(D)$ be the union of the following sets:

$$\begin{aligned}&(e_{2i-1}\to a^{2i}_{n_{2i}-1})\cup(a^{2i-1}_{n_{2i-1}-1}\to e_{2i})\quad \text{for } 1\leqslant i\leqslant \tfrac12(m-5),\\ &(e_{m-4}\to a^{m-2}_{n_{m-2}-1})\cup(e_{m-3}\to a^{m-1}_1)\cup(a^{m-3}_1\to e_{m-2})\\ &\qquad\qquad\cup(a^{m-2}_1\to e_{m-1})\cup(e_{m-2}\to a^m_1)\cup(a^{m-1}_1\to e_m).\end{aligned}$$

It is not difficult to see that a^{m-1}_1 gives two hyperedges e_{m-3} and e_m and there are two vertices a^{m-3}_1 and a^m_1 giving a hyperedge e_{m-2}. Figure 1 illustrates an example

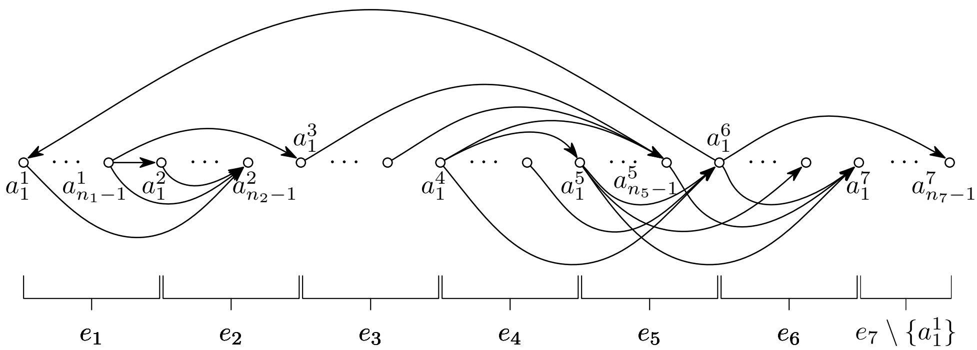

Figure 1. A digraph D having $\mathcal{C}_7$ as the niche hypergraph.

of the digraph D when $m = 7$. Remark that the hyperedge e_7 is obtained by $N_D^+(a_1^6)$. We remark also that for an odd integer $m \geqslant 9$, the subdigraph induced by $e_{2i-1} \cup e_{2i}$ for $1 \leqslant i \leqslant \frac{1}{2}(m-5)$ is isomorphic to the subdigraph induced by $e_1 \cup e_2$ in Figure 1. It is not difficult to see that D is acyclic and has $\mathcal{C}_m$ as the niche hypergraph. □

Lemma 2.3. *For an even integer $m \geqslant 6$, let $\mathcal{C}_m$ be a linear hypercycle such that $\underline{d}(\mathcal{C}_m) \geqslant 3$. Then there exists an acyclic digraph D with $V(D) = V(\mathcal{C}_m)$ having $\mathcal{C}_m$ as the niche hypergraph.*

Proof. Again, we construct an acyclic digraph $D = (V, A)$ having the niche hypergraph $N\mathcal{H}(D) = \mathcal{C}_m = (V, \{e_1, e_2, \ldots, e_m\})$, where now A is the union of the following sets:

$$(e_{2i-1} \to a_{n_{2i}-1}^{2i}) \cup (a_{n_{2i-1}-1}^{2i-1} \to e_{2i}) \quad \text{for } 1 \leqslant i \leqslant \tfrac{1}{2}(m-4)$$

and

$$(e_{m-3} \to a_1^{m-1}) \cup (e_{m-2} \to a_1^m) \cup (a_1^{m-2} \to e_{m-1}) \cup (a_1^{m-1} \to e_m).$$

Figure 2 illustrates an example of D when $m = 6$. □

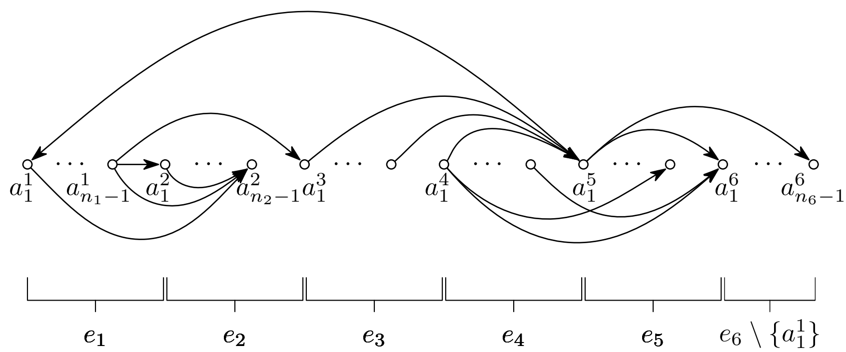

Figure 2. A digraph D having $\mathcal{C}_6$ as the niche hypergraph.

Summarizing the results of Lemmas 2.1–2.3, we obtain the following theorem.

Theorem 2.1. *For an integer $m \geqslant 2$, let $\mathcal{C}_m$ be a linear hypercycle with $\underline{d}(\mathcal{C}_m) \geqslant 3$. Then $\hat{n}(\mathcal{C}_m) = 0$.*

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References

[1] *C. Garske, M. Sonntag, H. M. Teichert*: Niche Hypergraphs. Discuss. Math., Graph Theory *36* (2016), 819–832. zbl MR doi

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