Aleksandra Kostić; Zoran Z. Petrović; Zoran S. Pucanović; Maja Roslavcev
Note on strongly nil clean elements in rings

Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 1, 87–92

Persistent URL: `http://dml.cz/dmlcz/147619`

Czechoslovak Mathematical Journal, 69 (144) (2019), 87–92

NOTE ON STRONGLY NIL CLEAN ELEMENTS IN RINGS

Aleksandra Kostić, Zoran Z. Petrović, Zoran S. Pucanović,
Maja Roslavcev, Belgrade

Received April 10, 2017. Published online June 22, 2018.

Abstract. Let R be an associative unital ring and let $a \in R$ be a strongly nil clean element. We introduce a new idea for examining the properties of these elements. This approach allows us to generalize some results on nil clean and strongly nil clean rings. Also, using this technique many previous proofs can be significantly shortened. Some shorter proofs concerning nil clean elements in rings in general, and in matrix rings in particular, are presented, together with some generalizations of these results.

Keywords: nilpotent element; nil clean element

MSC 2010: 13B25, 15B33, 16U99

1. Introduction

All rings in this paper are assumed to be associative with identity. An element $a \in R$ is said to be nil clean (strongly nil clean) if there exist an idempotent element e and nilpotent element n such that $a = e + n$ (and $en = ne$). A ring itself is (strongly) nil clean if every element in this ring is (strongly) nil clean. This class of rings was introduced in [5] as a particularly interesting subclass of (strongly) clean rings.

It is well known that a is strongly nil clean if and only if $a(a-1)$ is nilpotent. Although this condition was known even before the introduction of this terminology (probably the first proof can be found in [6]), it has not been widely used in investigation of nil clean elements. Some recent uses may be found in [4]. For other proofs of this condition, the reader may wish to consult the recent papers [7] and [8].

It is the purpose of this short note to advertise the use of this condition by showing how one can use it to find much shorter proofs of some of the previous results

The first, second and fourth author are partially supported by Ministry of Science and Environmental Protection of Republic of Serbia Project #174032.

DOI: 10.21136/CMJ.2018.0167-17

concerning nil clean elements, even in a more general case. We also give a new proof of this condition which we think is interesting per se.

2. Strongly nil clean elements

We start this section with a simple lemma.

Lemma 2.1. *Let $S = \mathbb{Z}[X]/\langle (X(X-1))^n \rangle$. If we denote by x the class of X in this quotient ring, then x is (strongly) nil clean.*

Proof. Let us consider two ideals $I = \langle X \rangle$ and $J = \langle X-1 \rangle$ in the ring $\mathbb{Z}[X]$. Since $1 = X + (1 - X)$, these ideals are co-prime. Therefore I^n and J^n are co-prime as well. Chinese remainder theorem gives us

$$\mathbb{Z}[X]/(I^n \cdot J^n) \cong \mathbb{Z}[X]/I^n \times \mathbb{Z}[X]/J^n,$$

where the isomorphism is given by: $x = X + I^n \cdot J^n \longmapsto (X + I^n, X + J^n)$. But

$$(X + I^n, X + J^n) = (0 + I^n, 1 + J^n) + (X + I^n, X - 1 + J^n)$$

shows that the element $(X + I^n, X + J^n) \in \mathbb{Z}[X]/I^n \times \mathbb{Z}[X]/J^n$ is nil clean and so is the element x. Since S is commutative, it is trivially strongly nil clean. $\square$

From this lemma it is easy to prove that an element $r \in R$ is strongly nil clean if and only if $r(r-1)$ is nilpotent.

Theorem 2.1. *An element $r \in R$ is strongly nil clean if and only if $r(r-1)$ is nilpotent.*

Proof. Of course, if r is strongly nil clean, it is easy to show by direct computation that $r(r-1)$ is nilpotent. The other direction is the interesting one.

Let us consider a homomorphism $f\colon \mathbb{Z}[X] \to R$ given by $f(X) = r$. By the assumption, there exists $n \geqslant 1$ such that $(r(r-1))^n = 0$. This means that f induces a homomorphism $\overline{f}\colon \mathbb{Z}[X]/\langle (X(X-1))^n \rangle \to R$ such that $\overline{f}(x) = r$. Since x is strongly nil clean, so is r. $\square$

We now prove a key lemma. If R is a ring, by $N(R)$ we denote the set of all nilpotent elements in R and by $Z(R)$ we denote the center of that ring.

Lemma 2.2. *If an element $a \in R$ can be written in the form $a = d + zc$, where $c \in R$, $d \in N(R)$ and $z \in N(R) \cap Z(R)$, then $a \in N(R)$.*

P r o o f. Since d and z are nilpotent elements, there exist positive integers m, n such that $d^m = z^n = 0$. Let us show that $a^N = 0$, where $N = mn$. When we develop $a^N = (d + zc)^N$ in the sum, any term of this sum is a product of d's and (zc)'s of various order. Since z is central, any term is of the form z^k times the product of d's and c's. If $k \geqslant n$, this term is 0. So, let us assume that $k < n$. In the 'worst' case, our term is of the form $z^{n-1}d^{k_1}cd^{k_2}c \ldots d^{k_{n-1}}cd^{k_n}$, where $k_1 + \ldots + k_n = N - (n - 1) = mn - n + 1 = (m - 1)n + 1$. By Dirichlet's (or pigeon hole) principle, at least one of k_i is greater than or equal to m and then this term is equal to zero. So, $a^N = 0$. □

The following corollary points to the way this lemma will be used in the discussion of strongly nil clean elements.

Corollary 2.1. *If $a = b + zc$, where $c \in R$ and $z \in N(R) \cap Z(R)$, then a is strongly nil clean if and only if b is strongly nil clean.*

P r o o f. We compute:

$$a(a - 1) = (b + zc)(b + zc - 1) = b(b - 1) + z(bc + cb + zc^2 - c).$$

The claim now follows from Lemma 2.2. □

As an interesting corollary we also give the following one (cf. Lemma 2.4 and its proof in [2]).

Corollary 2.2. *Suppose that R is a commutative ring and $n \geqslant 1$. Then*

$$M_n(N(R)) \subseteq N(M_n(R)).$$

P r o o f. We express matrix $A = (a_{ij}) \in M_n(N(R))$ as the sum: $A = \sum\limits_{i,j} a_{ij}E_{ij} = \sum\limits_{i,j}(a_{ij}I_n)E_{ij}$, where E_{ij} are elementary matrices (having 1 at the position (i, j) and 0 elsewhere). Since $a_{ij}I_n$ is a central nilpotent, the result follows by the repeated use of Lemma 2.2, starting with the zero matrix. □

3. Strongly nil clean matrices

The criterion we used is also rather convenient for discussing the problem whether a given matrix is strongly nil clean. Let us give a few examples.

In [2] this question was discussed for matrix rings over the commutative ring $R[X]/\langle X^2-1\rangle$, where R is a commutative unital local ring such that $\mathrm{char}(R)=2$. See Theorem 2.5 and Theorem 3.2 from [2]. In the next theorem, we give generalization of these results without the assumption on the ring being local and for all n (in [2] the result was proved for $n \in \{2,3,4\}$).

Theorem 3.1. *Let R be a commutative unital ring with $\mathrm{char}(R)=2$ and $A(x) \in M_n(R[X]/\langle X^2-1\rangle)$, where x is the class of X in the quotient ring. Then $A(x)$ is strongly nil clean if and only if $A(1) \in M_n(R)$ is strongly nil clean.*

Proof. Every element in $R[X]/\langle X^2-1\rangle$ is of the form $c+dx$ for some $c,d \in R$. Of course, $A(1)$ stands for the matrix $(a_{ij}(1))$. Since R has characteristic 2, $0 = x^2-1 = (x-1)^2$, so the element $x-1$ is nilpotent. Therefore

$$A(x) = (c_{ij} + d_{ij}x) = (c_{ij}+d_{ij}) + (x-1)(d_{ij}).$$

So, $A(x) = A(1) + ((x-1)I_n)D$, where D is the matrix (d_{ij}). Since $(x-1)I_n$ is a central nilpotent, the result follows from Corollary 2.1. □

Remark 3.1. Since the characteristic of R is 2, the ring $R[X]/\langle X^2-1\rangle$ is actually isomorphic to the ring $R[X]/\langle X^2\rangle$.

Remark 3.2. Note how the use of Corollary 2.1 allows us to give a short proof for all n avoiding complicated calculations from [2].

The following theorem shows the importance of Lemma 2.2 and Corollary 2.1, in the sense that one easily proves the results from [3] (cf. Theorem 4.3), with a slight generalization. Namely, the condition that the ring is projective-free can be dropped.

Theorem 3.2. *Let R be a commutative ring with identity, $S = M_n(R[[X]]/\langle X^k\rangle)$, where $k \geqslant 1$, and $A(x) \in M_n(R[[X]]/\langle X^k\rangle)$, where x is the class of X in the quotient ring. Then $A(x)$ is strongly nil clean if and only if $A(0) \in M_n(R)$ is strongly nil clean.*

Proof. We have: $A(x) = A(0) + xB(x) = A(0) + (xI_n)B(x)$ for some $B(x) \in S$. Since $xI_n \in N(S) \cap Z(S)$, the claim follows from Corollary 2.1. □

Remark 3.3. Actually, $R[[X]]/\langle X^k\rangle \cong R[X]/\langle X^k\rangle$.

Let us now investigate triangular matrices. By $UT_n(R)$ we denote the subring of $M_n(R)$ consisting of all upper triangular matrices, i.e. $A = (a_{ij}) \in UT_n(R)$ if and only if $a_{ij} = 0$ for $i > j$. Of course, analogous results hold for lower triangular matrices as well.

Theorem 3.3. *Let R be an associative unital ring. A matrix $A \in UT_n(R)$ is strongly nil clean if and only if all of its diagonal components are strongly nil clean.*

P r o o f. ($\Rightarrow$) If the matrix $A = (a_{ij})$ is strongly nil clean, then $(A(A - I_n))^m = O$ for some $m \geqslant 1$. But the diagonal components of this matrix are $(a_{ii}(a_{ii} - 1))^m$ and so the elements a_{ii} are also (strongly) nil clean.

($\Leftarrow$) Let us assume that all of a_{ii}'s are strongly nil clean. Therefore there exists m_i such that $(a_{ii}(a_{ii} - 1))^{m_i} = 0$ for all $1 \leqslant i \leqslant n$. If we take $m = \max\{m_1, \ldots, m_n\}$, we get that the matrix $(A(A - I_n))^m$ is an upper triangular matrix with zeros on the diagonal. This matrix is nilpotent: we have that $((A(A - I_n))^m)^n = O$. $\square$

Corollary 3.1 (Proposition 3.1 from [1]). *Let R be an associative unital local ring. A matrix $A = (a_{ij}) \in UT_n(R)$ is strongly nil clean if and only if for all i: $a_{ii} \in N(R)$ or $a_{ii} - 1 \in N(R)$.*

P r o o f. From the previous theorem, we know that A is strongly nil clean if and only if for all i: $a_{ii}(a_{ii} - 1) \in N(R)$, so for all i there exists n_i such that

$$a_{ii}^{n_i}(a_{ii} - 1)^{n_i} = 0. \tag{3.1}$$

Since R is local, it follows that for all i: $a_{ii} \in M$ or $a_{ii} - 1 \in M$, where M is the unique maximal (left) ideal. So, it follows that for all i: $a_{ii} - 1 \in U(R)$ or $a_{ii} \in U(R)$. Therefore from (3.1) it follows that $a_{ii} \in N(R)$ or $a_{ii} - 1 \in N(R)$. $\square$

The fact that we investigate whether or not the matrix $A(A - I)$ is nilpotent points out to another useful criterion for matrices.

Theorem 3.4. *Let R be a commutative unital ring and $A \in M_n(R)$. Let us suppose that a polynomial $f(t) \in R[t]$ is such that*

$$f(t) \equiv t^k(t - 1)^l \bmod N(R)[t],$$

where $k + l > 0$ and that $f(A)$ is a nilpotent matrix. Then the matrix A is strongly nil clean.

P r o o f. Since $f(A) = A^k(A - I_n)^l + g(A)$, where all the coefficients of g are nilpotent elements, it follows that $A^k(A - I_n)^l$ is nilpotent. This implies that $A(A - I_n)$ is nilpotent as well and the matrix A is strongly nil clean. $\square$

This theorem is actually a slight generalization of Theorem 2.7 from [3].

Corollary 3.2. *Let R be a commutative unital ring and let $f(t) \in R[t]$ be a polynomial such that $f(t) = \prod\limits_{i=1}^{m}(t - \alpha_i)$, where $\{\alpha_1, \ldots, \alpha_m\} \subseteq N(R) \cup (1 + N(R))$. If $f(A)$ is nilpotent, then the matrix A is strongly nil clean.*

P r o o f. For simplicity of notation, let us assume that $\{\alpha_1, \ldots, \alpha_k\} \subset N(R)$ and $\{\alpha_{k+1}, \ldots, \alpha_m\} \subset 1 + N(R)$. Then for $1 \leqslant i \leqslant k$: $\alpha_i \equiv 0 \bmod N(R)$ and for $k+1 \leqslant i \leqslant m$: $\alpha_i \equiv 1 \bmod N(R)$. So, $f \equiv t^k(t-1)^{m-k} \bmod N(R)[t]$ and the result follows from Theorem 3.4. □

Corollary 2.8 and Corollary 2.9 from [3] follow as special cases (for $n = 2, 3$).

This method is especially useful when we have some information on the characteristic polynomial χ_A of the matrix A. Namely, the results in Theorem 3.4 and Corollary 3.5 from [2] and Corollary 2.10 from [3], which were proved there for matrices of order 3 or 4, are valid for matrices of any order. This follows directly from Theorem 3.4 and Corollary 3.2.

References

[1] *H. Chen*: Some strongly nil clean matrices over local rings. Bull. Korean Math. Soc. *48* (2011), 759–767. zbl MR doi

[2] *H. Chen*: Strongly nil clean matrices over $R[x]/(x^2 - 1)$. Bull. Korean Math. Soc. *49* (2012), 589–599. zbl MR doi

[3] *H. Chen*: On strongly nil clean matrices. Commun. Algebra *41* (2013), 1074–1086. zbl MR doi

[4] *H. Chen, M. Sheibani*: On strongly nil clean rings. Commun. Algebra *45* (2017), 1719–1726. zbl MR doi

[5] *A. J. Diesl*: Nil clean rings. J. Algebra *383* (2013), 197–211. zbl MR doi

[6] *Y. Hirano, H. Tominaga, A. Yaqub*: On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element. Math. J. Okayama Univ. *30* (1988), 33–40. zbl MR

[7] *T. Koşan, Z. Wang, Y. Zhou*: Nil-clean and strongly nil-clean rings. J. Pure Appl. Algebra *220* (2016), 633–646. zbl MR doi

[8] *J. Šter*: Rings in which nilpotents form a subring. Carpathian J. Math. *32* (2016), 251–258. MR

Authors' addresses: A l e k s a n d r a K o s t i ć, Z o r a n Z. P e t r o v i ć, M a j a R o s l a v c e v, Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia, e-mail: `alex@matf.bg.ac.rs`, `zoranp@matf.bg.ac.rs`, `roslavcev@matf.bg.ac.rs`; Z o r a n S. P u c a n o v i ć, Faculty of Civil Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia, e-mail: `pucanovic@grf.bg.ac.rs`.