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n-STRONGLY GORENSTEIN GRADED MODULES

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Abstract. Let R be a graded ring and $n \geqslant 1$ an integer. We introduce and study n-strongly Gorenstein gr-projective, gr-injective and gr-flat modules. Some examples are given to show that n-strongly Gorenstein gr-injective (gr-projective, gr-flat, respectively) modules need not be m-strongly Gorenstein gr-injective (gr-projective, gr-flat, respectively) modules whenever $n > m$. Many properties of the n-strongly Gorenstein gr-injective and gr-flat modules are discussed, some known results are generalized. Then we investigate the relations between the graded and the ungraded n-strongly Gorenstein injective (or flat) modules. In addition, the connections between the n-strongly Gorenstein gr-projective, gr-injective and gr-flat modules are considered.

Keywords: n-strongly Gorenstein gr-injective module; n-strongly Gorenstein gr-flat module; n-strongly Gorenstein gr-projective module

MSC 2010: 16W50, 18G25, 16E05

1. Introduction

Auslander and Bridger [7] introduced the notion of finitely generated modules having Gorenstein dimension zero over a two-sided Noetherian ring. Enochs, Jenda and Torrecillas in [13], [15] introduced the notions of Gorenstein projective, injective and flat modules for any modules over a general ring. These Gorenstein homological modules have been studied extensively by many authors (cf. [8], [10], [12], [13], [14], [19], [25]). In 2007, Bennis and Mahdou introduced and studied in [8] strongly Gorenstein projective, injective and flat modules, which situate between projective, injective, flat modules and Gorenstein projective, injective, flat modules, respectively.

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Furthermore, they discussed a generalization of strongly Gorenstein projective, injective and flat modules, named n-strongly Gorenstein projective, injective and flat modules, respectively. Zhao and Huang in [27] continued the study of homological behavior of the n-strongly Gorenstein projective, injective and flat modules.

As we know, graded rings and modules are a classical topic in algebra, and the homological theory of graded rings has very important applications in algebraic geometry (see [18], [21], [22], [23]). It seems to be natural to establish relative homological theory for graded rings. In [17], García Rozas et al. proved the existence of flat covers in the category of graded modules over a graded ring. Also, the homological properties of FP-gr-injective modules over a gr-coherent ring were investigated in [4], [26]. On the other hand, Asensio, López-Ramos and Torrecillas in [1], [2] introduced the notions of Gorenstein gr-projective, gr-injective and gr-flat modules. In the recent years, the Gorenstein homological theory for graded rings have become an important area of research (cf. [1], [2], [3], [5], [6], [16]). In particular, Mao in [20] introduced the notions of strongly Gorenstein gr-projective, gr-injective and gr-flat modules, and gave many nice characterizations of them. Along the same lines, it is natural to generalize the notion of "strongly Gorenstein graded modules" to "n-strongly Gorenstein graded modules" and study homological properties of the n-strongly Gorenstein graded modules.

In this paper, we introduce and study n-strongly Gorenstein gr-projective, gr-injective and gr-flat modules over a graded ring. In Section 2, we give some notation and collect some preliminary results. Then in Section 3, we give the definition of n-strongly Gorenstein gr-injective and gr-projective modules and generalize some principal results of [9], [27] to the n-strongly Gorenstein gr-injective modules. An example is given to show that n-strongly Gorenstein gr-injective modules need not be m-strongly Gorenstein gr-injective modules whenever $n > m$. The relations between the graded and the ungraded n-strongly Gorenstein injective modules are also discussed. Section 4 is devoted to investigating n-strongly Gorenstein gr-flat modules. Some characterizations of the n-strongly Gorenstein gr-flat modules are given. We also investigate the relations between n-strongly Gorenstein gr-flat modules and n-strongly Gorenstein gr-projective (or gr-injective) modules. In addition, we consider the relations between the graded and the ungraded n-strongly Gorenstein flat modules.

2. Preliminaries

Throughout this paper, all rings considered are associative with an identity element and the R-modules are unital. By R-Mod we will denote the Grothendieck category of all left R-modules. Let G be a multiplicative group with a neutral element e.

A *graded ring* R is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus R_e is a subring of R, $1 \in R_e$ and R_σ is an R_e-bimodule for every $\sigma \in G$. A *graded left R-module* is a left R-module M endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where each M_σ is a subgroup of the additive group M such that $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For any graded left R-modules M and N, set

$$\mathrm{Hom}_{R\text{-gr}}(M, N) := \{f\colon M \to N;\ f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma \text{ for any } \sigma \in G\},$$

which is the group of all morphisms from M to N in the category R-gr of all graded left R-modules (gr-R will denote the category of all graded right R-modules). It is well known that R-gr is a Grothendieck category. An R-linear map $f\colon M \to N$ is said to be a *graded morphism of degree* τ with $\tau \in G$ if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree σ build an additive subgroup $\mathrm{HOM}_R(M, N)_\sigma$ of $\mathrm{Hom}_R(M, N)$. Then $\mathrm{HOM}_R(M, N) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M, N)_\sigma$ is a graded abelian group of type G. We will denote by $\mathrm{Ext}^i_{R\text{-gr}}$ and EXT^i_R the right derived functors of $\mathrm{Hom}_{R\text{-gr}}$ and HOM_R, respectively. Given a graded left R-module M, the *graded character module* of M is defined as $M^+ := \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the rational numbers field and $\mathbb{Z}$ is the integers ring. It is easy to see that $M^+ = \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

Let M be a graded right R-module and N a graded left R-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma$ with $\sigma \in G$ to be the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$ and $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the *graded tensor product* of M and N.

If M is a graded left R-module and $\sigma \in G$, then $M(\sigma)$ is the graded left R-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for any $\tau \in G$. The graded module $M(\sigma)$ is called the *σ-suspension* of M. We may regard the σ-suspension as an isomorphism of categories $T_\sigma\colon R\text{-gr} \to R\text{-gr}$, given on objects as $T_\sigma(M) = M(\sigma)$ for any $M \in R$-gr.

The injective objects of R-gr will be called gr-*injective modules*. Projective (flat) objects of R-gr will be called *projective* (*flat*) graded modules because M is gr-projective (gr-flat) if and only if it is a projective (flat, respectively) graded module. By $\text{gr-id}_R M$, $\mathrm{pd}_R M$ and $\mathrm{fd}_R M$ we will denote the gr-injective, projective and flat dimension of a graded module M, respectively. We denote by $\text{l.gr-gl.}\dim(R)$ ($\text{gr-w.gl.}\dim(R)$) the left global (weak global, respectively) dimension of a graded ring R. A graded R-module M is called FP-gr-*injective* if $\mathrm{EXT}^1_R(N, M) = 0$ for any finitely presented graded R-module N. It can be proved that if R is gr-coherent (i.e., a graded ring R such that, given a family of gr-flat R-modules $\{F_i\}_{i \in I}$, the graded R-module $\prod^{R\text{-gr}}_{i \in I} F_i$ is flat), then M is FP-gr-injective if and only if M^+ is flat.

In the following, we collect some basic concepts on Gorenstein graded homological modules which will be useful in the article.

Definition 2.1 ([1], [2]).

(1) A graded left R-module M is called *Gorenstein* gr-*injective* if there exists an exact sequence of gr-injective left R-modules

$$\ldots \to E_1 \to E_0 \to E^0 \to E^1 \to \ldots$$

in R-gr with $M = \mathrm{Ker}(E^0 \to E^1)$ such that $\mathrm{Hom}_{R\text{-gr}}(E, -)$ leaves the sequence exact whenever E is a gr-injective left R-module.

The Gorenstein gr-projective modules are defined dually.

(2) A graded left R-module N is called *Gorenstein* gr-*flat* if there exists an exact sequence of gr-flat left R-modules

$$\ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$$

in R-gr with $N = \mathrm{Ker}(F^0 \to F^1)$ such that $E \otimes_R -$ leaves the sequence exact whenever E is a gr-injective right R-module.

Definition 2.2 ([20]).

(1) A graded left R-module M is called *strongly Gorenstein* gr-*injective* if there exists an exact sequence of gr-injective left R-modules

$$\ldots \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \ldots$$

in R-gr with $M \cong \mathrm{Ker}(f)$ such that $\mathrm{Hom}_{R\text{-gr}}(I, -)$ leaves the sequence exact whenever I is a gr-injective left R-module.

The strongly Gorenstein gr-projective modules are defined dually.

(2) A graded left R-module N is called *strongly Gorenstein* gr-*flat* if there exists an exact sequence of gr-flat left R-modules

$$\ldots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \ldots$$

in R-gr with $N \cong \mathrm{Ker}(f)$ such that $E \otimes_R -$ leaves the sequence exact whenever E is a gr-injective right R-module.

Proposition 2.3 ([20]). *Let R be a graded ring.*

(1) *M is a strongly Gorenstein* gr-*projective left R-module if and only if there is an exact sequence $0 \to M \to P \to M \to 0$ in R-*gr *with P* gr-*projective and $\mathrm{Ext}^1_{R\text{-gr}}(M, Q) = 0$ for any* gr-*projective left R-module Q.*

(2) *M is a strongly Gorenstein* gr-*injective left R-module if and only if there is an exact sequence $0 \to M \to E \to M \to 0$ in R-*gr *with E* gr-*injective and $\mathrm{Ext}^1_{R\text{-gr}}(I, M) = 0$ for any* gr-*injective left R-module I.*

Remark 2.4. It has been shown in [20] that strongly Gorenstein gr-injective (gr-projective, gr-flat) modules lie strictly between gr-injective (gr-projective, gr-flat) modules and Gorenstein gr-injective (gr-projective, gr-flat, respectively) modules.

3. n-strongly Gorenstein gr-injective and gr-projective modules

In this section, we study the properties of n-strongly Gorenstein gr-injective and gr-projective modules. Some principal results of [9], [27] are generalized to n-strongly Gorenstein gr-injective or gr-projective modules.

Definition 3.1. Let n be a positive integer. A graded left R-module M is called *n-strongly Gorenstein* gr*-injective* (n-SG-gr-*injective* for short), if there is an exact sequence of graded left R-modules

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} M \longrightarrow 0 \tag{3.1}$$

in R-gr with each E_i gr-injective for any $0 \leqslant i \leqslant n-1$, such that $\mathrm{Hom}_{R\text{-gr}}(E, -)$ leaves the sequence (3.1) exact whenever E is a gr-injective left R-module.

Dually, a graded left R-module M is called *n-strongly Gorenstein* gr*-projective* (n-SG-gr-*projective* for short), if there is an exact sequence of graded left R-modules

$$0 \longrightarrow M \xrightarrow{g_n} P_{n-1} \xrightarrow{g_{n-1}} P_{n-2} \xrightarrow{g_{n-2}} \dots \xrightarrow{g_1} P_0 \xrightarrow{g_0} M \longrightarrow 0 \tag{3.2}$$

in R-gr with each P_i gr-projective for any $0 \leqslant i \leqslant n-1$, such that $\mathrm{Hom}_{R\text{-gr}}(-, P)$ leaves the sequence (3.2) exact whenever P is a gr-projective left R-module.

Remark 3.2.

(1) It is easy to see that the 1-SG-gr-injective (1-SG-gr-projective) left R-modules are just the strongly Gorenstein gr-injective (strongly Gorenstein gr-projective, respectively) left R-modules in [20]. Moreover, for any $1 \leqslant i \leqslant n$, each $\mathrm{Im}(f_i)$ in the sequence (3.1) is also n-SG-gr-injective and each $\mathrm{Im}(g_i)$ in the sequence (3.2) is also n-SG-gr-projective.

(2) Let m and n be positive integers with $n \leqslant m$. If $n \mid m$, then all n-strongly Gorenstein gr-injective (n-strongly Gorenstein gr-projective) left R-modules are m-strongly Gorenstein gr-injective (m-strongly Gorenstein gr-projective, respectively) by definition.

For any $n \geqslant 1$, we denote by n-SG-gr-Inj(R) (n-SG-gr-Proj(R)) the subcategory of R-gr consisting of all n-SG-gr-injective modules (n-SG-gr-projective modules, respectively). In what follows, we only prove the Gorenstein gr-injective case, and the dual results hold for Gorenstein gr-projective case.

Proposition 3.3. *For any $n \geqslant 1$, n-SG-gr-Inj(R) is closed under direct products.*

P r o o f. Let $\{M_j\}_{j\in J}$ be a family of n-strongly Gorenstein gr-injective left R-modules. Then, for any $j \in J$, there exists an exact sequence

$$0 \longrightarrow M_j \longrightarrow E_0^{(j)} \longrightarrow E_1^{(j)} \longrightarrow \ldots \longrightarrow E_{n-1}^{(j)} \longrightarrow M_j \longrightarrow 0$$

in R-gr with each $E_i^{(j)}$ gr-injective for any $0 \leqslant i \leqslant n-1$, such that $\mathrm{Hom}_{R\text{-gr}}(E,-)$ leaves the sequence exact whenever E is a gr-injective left R-module. Thus we have the exact sequence

$$0 \longrightarrow \prod_{j\in J} M_j \longrightarrow \prod_{j\in J} E_0^{(j)} \longrightarrow \prod_{j\in J} E_1^{(j)} \longrightarrow \ldots \longrightarrow \prod_{j\in J} E_{n-1}^{(j)} \longrightarrow \prod_{j\in J} M_j \longrightarrow 0$$

in R-gr. Since $\prod\limits_{j\in J} E_0^{(j)}, \ldots, \prod\limits_{j\in J} E_{n-1}^{(j)}$ are gr-injective and the sequence above remains exact after applying the functor $\mathrm{Hom}_{R\text{-gr}}(E,-)$ whenever E is a gr-injective left R-module, it follows that $\prod\limits_{j\in J} M_j$ is n-strongly Gorenstein gr-injective. $\square$

Proposition 3.4. *Let n be a positive integer. Then:*

(1) *Every strongly Gorenstein* gr-*injective left R-module is n*-SG-gr-*injective.*

(2) *Every n*-SG-gr-*injective left R-module is Gorenstein* gr-*injective.*

P r o o f. (1) Let M be a strongly Gorenstein gr-injective left R-module. By [20], Proposition 2.2, there exists a short exact sequence: $0 \to M \overset{f}{\longrightarrow} E \overset{g}{\longrightarrow} M \to 0$ with E gr-injective, and the sequence $0 \to \mathrm{Hom}_{R\text{-gr}}(I,M) \to \mathrm{Hom}_{R\text{-gr}}(I,E) \to \mathrm{Hom}_{R\text{-gr}}(I,M) \to 0$ is exact for any gr-injective left R-module I. So we get an exact sequence

$$0 \longrightarrow M \overset{f}{\longrightarrow} E \overset{f\circ g}{\longrightarrow} E \overset{f\circ g}{\longrightarrow} \ldots \overset{f\circ g}{\longrightarrow} E \overset{g}{\longrightarrow} M \longrightarrow 0$$

in R-gr such that $\mathrm{Hom}_{R\text{-gr}}(I,-)$ leaves the sequence exact whenever I is a gr-injective left R-module. Thus M is an n-SG-gr-injective left R-module.

(2) Let M be an n-SG-gr-injective left R-module. Then there exists an exact sequence

$$0 \longrightarrow M \overset{f_0}{\longrightarrow} E_0 \overset{f_1}{\longrightarrow} E_1 \overset{f_2}{\longrightarrow} \ldots \overset{f_{n-1}}{\longrightarrow} E_{n-1} \overset{f_n}{\longrightarrow} M \longrightarrow 0$$

in R-gr with each E_i gr-injective for any $0 \leqslant i \leqslant n-1$, such that $\mathrm{Hom}_{R\text{-gr}}(I,-)$ leaves the sequence exact whenever I is a gr-injective left R-module. Thus we obtain the following exact commutative diagram:

$$\mathbf{E} = \cdots \xrightarrow{f_{n-2}} E_{n-2} \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_0\circ f_n} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_0\circ f_n} E_0 \xrightarrow{f_1} \cdots$$

$$\begin{array}{ccccccccc} E_{n-1} & \searrow & & \nearrow & E_0 & E_{n-1} & \searrow & & \nearrow & E_0 \\ & & M & & & & & M & & \\ 0 & \nearrow & & \searrow & 0 & 0 & \nearrow & & \searrow & 0 \end{array}$$

in R-gr with each E_i gr-injective, and such that $\mathrm{Hom}_{R\text{-gr}}(I,-)$ leaves the sequence $\mathbf{E}$ exact whenever I is a gr-injective left R-module. Therefore M is Gorenstein gr-injective. □

In general, n-SG-gr-injective modules need not be m-SG-gr-injective modules whenever $n > m$ as shown by the following example.

Example 3.5. Consider a Noetherian local ring $R = k[[X,Y]]/(XY)$, where k is a field. Then the two ideals $(\overline{X})$ and $(\overline{Y})$ of R are 2-SG-flat R-modules, but not 1-SG-flat by [27], Example 4.8, where $(\overline{X})$ and $(\overline{Y})$ are the residue classes in R of X and Y respectively. By [27], Proposition 4.9, $(\overline{X})^+$ and $(\overline{Y})^+$ are 2-SG-injective, and neither of them are 1-SG-injective by [25], Theorem 2.4. Since R may be viewed as a trivially graded ring, $(\overline{X})^+$ and $(\overline{Y})^+$ are 2-SG-gr-injective, which are not 1-SG-gr-injective.

It is well known that a strongly Gorenstein injective module is injective if and only if it has finite injective dimension (dual version of [19], Proposition 2.27). Now we have:

Proposition 3.6. *For any $n \geqslant 1$, an n-strongly Gorenstein* gr-*injective left R-module is* gr-*injective if and only if it has finite* gr-*injective dimension.*

P r o o f. "only if" part is trivial.

"if" part. Suppose M is an n-strongly Gorenstein gr-injective left R-module with finite gr-injective dimension. Then there exists an exact sequence

$$0 \to M \to E_0 \to E_1 \to \ldots \to E_{n-1} \to M \to 0$$

in R-gr with each E_i gr-injective. Let F be a graded left R-module. Since $\mathrm{Ext}^i_{R\text{-gr}}(F, E_j) = 0$ for all $i \geqslant 1$ and $0 \leqslant j \leqslant n-1$, we deduce that $\mathrm{Ext}^i_{R\text{-gr}}(F, M) \cong \mathrm{Ext}^{i+n}_{R\text{-gr}}(F, M)$. Note that $\text{gr-id}_R(M) < \infty$, hence it follows that $\mathrm{Ext}^i_{R\text{-gr}}(F, M) = 0$ for all $i \geqslant 1$, and so M is gr-injective, as desired. □

Remark 3.7. It is easy to see that if there exists a non-gr-injective n-SG-gr-injective left R-module in R-gr for some $n \geqslant 1$, then $\text{l.gr-gl.}\dim(R) = \infty$.

By Proposition 3.4 and [20], Proposition 2.7, we immediately get the following result.

Proposition 3.8. *The following statements are equivalent:*

(1) *Every Gorenstein* gr-*injective left R-module is* gr-*injective.*

(2) *Every n-strongly Gorenstein* gr-*injective left R-module is* gr-*injective.*

(3) *Every strongly Gorenstein* gr-*injective left R-module is* gr-*injective.*

The following proposition is a generalization of [20], Proposition 2.2, which gives a characterization of the n-strongly Gorenstein gr-injective modules.

Proposition 3.9. *The following are equivalent for a graded left R-module M.*

(1) *M is n-strongly Gorenstein* gr-*injective.*

(2) *There exists an exact sequence $0 \to M \to E_0 \to E_1 \to \ldots \to E_{n-1} \to M \to 0$ in R-*gr *with each E_i* gr-*injective and $\mathrm{Ext}^j_{R\text{-gr}}(E, M) = 0$ for any* gr-*injective left R-module E and any $1 \leqslant j \leqslant n$.*

P r o o f. (1) $\Rightarrow$ (2): This follows from the definition of n-strongly Gorenstein gr-injective modules.

(2) $\Rightarrow$ (1): There is an exact sequence $0 \to M \to E_0 \to E_1 \to \ldots \to E_{n-1} \to M \to 0$ in R-gr with each E_i gr-injective for any $0 \leqslant i \leqslant n-1$. Next it suffices to show that $\mathrm{Hom}_{R\text{-gr}}(I, -)$ leaves the sequence exact whenever I is a gr-injective left R-module. Let $L_i = \mathrm{Im}(E_{i-1} \to E_i)$, $i = 1, \ldots, n-1$. Then we get the short exact sequences

$$\begin{aligned} &0 \to M \to E_0 \to L_1 \to 0, \\ &0 \to L_1 \to E_1 \to L_2 \to 0, \\ &\quad\vdots \\ &0 \to L_{n-1} \to E_{n-1} \to M \to 0. \end{aligned}$$

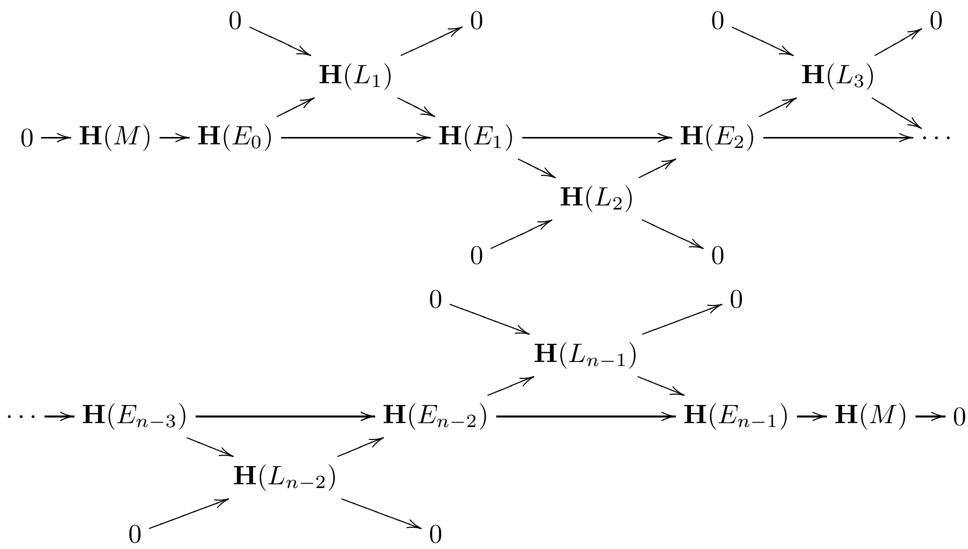
For every gr-injective left R-module I, we have

$$\mathrm{Ext}^i_{R\text{-gr}}(I, M) \cong \mathrm{Ext}^{i+1}_{R\text{-gr}}(I, L_{n-1}) \cong \mathrm{Ext}^{i+2}_{R\text{-gr}}(I, L_{n-2}) \cong \ldots \cong \mathrm{Ext}^{i+n}_{R\text{-gr}}(I, M)$$

for any $i \geqslant 1$. It follows that $\mathrm{Ext}^i_{R\text{-gr}}(I, M) = 0$ for all $i \geqslant 1$ by assumption. Thus we obtain the exactness of the sequences

$$\begin{gathered} 0 \to \mathrm{Hom}_{R\text{-gr}}(I, M) \to \mathrm{Hom}_{R\text{-gr}}(I, E_0) \to \mathrm{Hom}_{R\text{-gr}}(I, L_1) \to 0, \\ 0 \to \mathrm{Hom}_{R\text{-gr}}(I, L_1) \to \mathrm{Hom}_{R\text{-gr}}(I, E_1) \to \mathrm{Hom}_{R\text{-gr}}(I, L_2) \to 0, \\ \vdots \\ 0 \to \mathrm{Hom}_{R\text{-gr}}(I, L_{n-1}) \to \mathrm{Hom}_{R\text{-gr}}(I, E_{n-1}) \to \mathrm{Hom}_{R\text{-gr}}(I, M) \to 0. \end{gathered}$$

So we have the following exact commutative diagram (with substitution $\mathbf{H}(X) = \mathrm{Hom}_{R\text{-gr}}(I, X)$ to save space):



which gives rise to the exactness of

$$0 \to \mathrm{Hom}_{R\text{-gr}}(I, M) \to \mathrm{Hom}_{R\text{-gr}}(I, E_0) \to \dots \\ \to \mathrm{Hom}_{R\text{-gr}}(I, E_{n-1}) \to \mathrm{Hom}_{R\text{-gr}}(I, M) \to 0.$$

Therefore, M is n-strongly Gorenstein gr-injective, as desired. □

Recall that a graded ring R is called gr-n-*Gorenstein* (or simply a gr-*Gorenstein ring*) if R is left and right gr-Noetherian with self gr-injective dimension on either side at most n for an integer $n \geqslant 0$ (see [1]).

Corollary 3.10. *Let R be a* gr-n-*Gorenstein ring. Then the following conditions are equivalent for a graded left R-module M.*

(1) *M is n-strongly Gorenstein* gr-*injective.*

(2) *There exists an exact sequence* $0 \to M \to E_0 \to E_1 \to \dots \to E_{n-1} \to M \to 0$ *in R-*gr *with each E_i* gr-*injective.*

P r o o f. (1) ⇒ (2): holds by definition.

(2) ⇒ (1): By assumption, there exists an exact sequence $0 \to M \to E_0 \to E_1 \to \dots \to E_{n-1} \to M \to 0$ in R-gr with each E_j gr-injective for any $0 \leqslant j \leqslant n-1$. One easily deduces that $\mathrm{Ext}^i_{R\text{-gr}}(N, M) \cong \mathrm{Ext}^{i+n}_{R\text{-gr}}(N, M)$ for all graded left R-modules N. For any gr-injective left R-module I, we have that $\text{gr-pd}_R(I) \leqslant n$ by [1], Theorem 2.8, since R is a gr-n-Gorenstein ring. It follows that $\mathrm{Ext}^i_{R\text{-gr}}(I, M) = 0$ for any gr-injective left R-module I and any $1 \leqslant i \leqslant n$. So M is n-strongly Gorenstein gr-injective by Proposition 3.9. □

The next result generalizes [27], Theorem 3.9, which gives a method how to construct a 1-SG-gr-injective module from n-SG-gr-injective modules.

Theorem 3.11. *For any graded left R-module M and $n \geqslant 1$, the following statements are equivalent.*

(1) *M is n-SG-gr-injective.*

(2) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} M \longrightarrow 0$$

in R-gr with E_i gr-injective for any $0 \leqslant i \leqslant n-1$, such that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is 1-SG-gr-injective.

(3) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} M \longrightarrow 0$$

in R-gr with E_i gr-injective for any $0 \leqslant i \leqslant n-1$, such that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is Gorenstein gr-injective.

(4) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} M \longrightarrow 0$$

in R-gr, where E_i has finite gr-injective dimension for any $0 \leqslant i \leqslant n-1$, such that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is 1-SG-gr-injective.

(5) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} M \longrightarrow 0$$

in R-gr, where E_i has finite gr-injective dimension for any $0 \leqslant i \leqslant n-1$, such that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is Gorenstein gr-injective.

P r o o f. (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2)$\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. Next we will show that (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Let M be an n-SG-gr-injective left R-module. Then we have an exact sequence

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} M \longrightarrow 0$$

in R-gr with E_i gr-injective for any $0 \leqslant i \leqslant n-1$, such that $\mathrm{Hom}_{R\text{-gr}}(I,-)$ leaves the sequence exact whenever I is a gr-injective left R-module. Thus, for any $1 \leqslant i \leqslant n$, we have an exact sequence

$$0 \longrightarrow \mathrm{Im}(f_i) \xrightarrow{\alpha_i} E_i \xrightarrow{f_{i+1}} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_0 f_n} E_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} E_{i-1} \xrightarrow{f_i} \mathrm{Im}(f_i) \longrightarrow 0$$

in R-gr, which gives rise to the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Im}(f_i) \xrightarrow{\alpha} E_{n-1} \oplus E_0 \oplus \dots \oplus E_{n-2} \xrightarrow{f} E_0 \oplus \dots \oplus E_{n-2} \oplus E_{n-1} \longrightarrow \dots$$

where $\alpha = \mathrm{diag}\{\alpha_{n-1}, \alpha_0, \dots, \alpha_{n-2}\}$ and $f = \mathrm{diag}\{f_0 f_n, f_1, \dots, f_{n-1}\}$. One checks readily that $\mathrm{Im}(f) \cong \bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ and $\mathrm{Ext}^1_{R\text{-gr}}\Big(I, \bigoplus_{i=1}^{n} \mathrm{Im}(f_i)\Big) = 0$ for any gr-injective left R-module I. It follows that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is 1-SG-gr-injective by [20], Proposition 2.2.

(5) $\Rightarrow$ (1): Let $0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} M \longrightarrow 0$ be an exact sequence in R-gr, where E_i has finite gr-injective dimension for any $0 \leqslant i \leqslant n-1$, such that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is Gorenstein gr-injective. Then, for any $0 \leqslant i \leqslant n-1$, we have an exact sequence

$$0 \to \mathrm{Im}(f_i) \to E_i \to \mathrm{Im}(f_{i+1}) \to 0$$

in R-gr. Note that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is Gorenstein gr-injective; one easily gets that each $\mathrm{Im}(f_i)$ is Gorenstein gr-injective by analogy with the ungraded case, and so is each E_i. Thus E_i is gr-injective since E_i has finite gr-injective dimension for any $0 \leqslant i \leqslant n-1$. In particular, M is Gorenstein gr-injective, and hence $\mathrm{Ext}^i_{R-gr}(I, M) = 0$ for any gr-injective left R-module I and $i \geqslant 1$. Therefore M is n-SG-gr-injective by Proposition 3.9. □

Let U be the forgetful functor from R-gr to the category R-Mod of the left R-modules. This functor has a right adjoint F which associates M in R-Mod with the graded R-module

$$F(M) = \bigoplus_{\sigma \in G} ({}^{\sigma}M),$$

where each ${}^{\sigma}M$ is a copy of M written as $\{{}^{\sigma}x\colon x \in M\}$ with R-module structure defined by $r * {}^{\tau}x = {}^{\sigma\tau}(rx)$ for any $r \in R_\sigma$. If $f\colon M \to N$ is R-linear, then $F(f)\colon F(M) \to F(N)$ is a graded morphism given by $F(f)({}^{\sigma}x) = {}^{\sigma}f(x)$. In particular, if G is a finite group, then (F, U) is an adjoint pair by [23], Theorem 2.5.1.

In the rest of this section, we consider the relations between the graded and the ungraded n-strongly Gorenstein injective modules, which generalizes [20], Propositions 2.9 and 2.10.

Proposition 3.12. *Let R be a graded ring by a finite group G.*

(1) *If M is an n-strongly Gorenstein injective left R-module, then $F(M)$ is n-strongly Gorenstein* gr*-injective.*

(2) *If $M \in R$-*gr *is an n-strongly Gorenstein* gr*-injective left R-module, then $U(M)$ is n-strongly Gorenstein injective.*

Proof. (1) Let M be an n-strongly Gorenstein injective left R-module. Then there exists an exact sequence

$$0 \to M \to E_0 \to E_1 \to \ldots \to E_{n-1} \to M \to 0 \tag{3.3}$$

in R-Mod with each E_i injective and such that $\mathrm{Hom}_R(E, -)$ leaves the sequence (3.3) exact whenever E is an injective left R-module. Since the functor F is exact, we get the exact sequence

$$0 \to F(M) \to F(E_0) \to F(E_1) \to \ldots \to F(E_{n-1}) \to F(M) \to 0 \tag{3.4}$$

in R-gr. Because F preserves injective objects by [24], Proposition 9.5 C.IV, each $F(E_i)$ is gr-injective. Let I be a gr-injective left R-module. Note that U and F are a pair of adjoint functors since G is a finite group, hence one has the isomorphism

$$\mathrm{Hom}_{R\text{-gr}}(F(-), I) \cong \mathrm{Hom}_R(-, U(I)).$$

So $U(I)$ is an injective left R-module. On the other hand, one has the isomorphism

$$\mathrm{Hom}_{R\text{-gr}}(I, F(-)) \cong \mathrm{Hom}_R(U(I), -).$$

Then the the sequence (3.4) remains exact when the functor $\mathrm{Hom}_{R\text{-gr}}(I, -)$ is applied. Thus $F(M)$ is an n-strongly Gorenstein gr-injective left R-module.

(2) Since $M \in R$-gr is n-strongly Gorenstein gr-injective, there is an exact sequence

$$0 \to M \to I_0 \to I_1 \to \ldots \to I_{n-1} \to M \to 0 \tag{3.5}$$

in R-gr with each I_i gr-injective and such that $\mathrm{Hom}_{R\text{-gr}}(I, -)$ leaves the sequence (3.5) exact whenever I is a gr-injective left R-module. Now since the functor U is exact, one gets the exact sequence

$$0 \to U(M) \to U(I_0) \to U(I_1) \to \ldots \to U(I_{n-1}) \to U(M) \to 0 \tag{3.6}$$

in R-Mod with $U(I_i)$ injective by [23], Corollary 2.5.2, since G is a finite group. Let E be an injective left R-module. Then $F(E)$ is gr-injective by [24], Proposition 9.5 C.IV. Since U and F are a pair of adjoint functors, one has the isomorphism

$$\operatorname{Hom}_{R\text{-gr}}(F(E), -) \cong \operatorname{Hom}_R(E, U(-)).$$

Thus the functor $\operatorname{Hom}_R(E, -)$ leaves the sequence (3.6) exact. So the assertion follows. □

4. n-strongly Gorenstein gr-flat modules

In this section, we introduce and study n-strongly Gorenstein gr-flat modules. The relations between these modules and n-strongly Gorenstein gr-projective (or gr-injective) modules are also considered.

Definition 4.1. Let n be a positive integer. A graded left R-module M is called *n-strongly Gorenstein* gr-*flat* (n-SG-gr-*flat* for short), if there is an exact sequence of graded left R-modules

$$0 \longrightarrow M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \dots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \longrightarrow 0 \tag{4.1}$$

in R-gr with each F_i gr-flat for any $0 \leqslant i \leqslant n-1$, such that $E \otimes_R -$ leaves the sequence (4.1) exact whenever E is a gr-injective right R-module.

Remark 4.2.
(1) It is obvious that the 1-SG-gr-flat left R-modules are just the strongly Gorenstein gr-flat left R-modules. For any $1 \leqslant i \leqslant n$, each $\operatorname{Im}(h_i)$ in the sequence (4.1) is also n-SG-gr-flat.
(2) Let $n \geqslant 1$ be an integer. It is trivial that every 1-SG-gr-flat module is n-SG-gr-flat. Moreover, If m and n are positive integers and $n \mid m$, then all n-SG-gr-flat left R-modules are m-SG-gr-flat by definition.
(3) If there exists a non-gr-flat n-SG-gr-flat left R-module in R-gr for some $n \geqslant 1$, then $\text{gr-w.gl.}\dim(R) = \infty$.

Example 4.3. From Example 3.5 we can see easily that the ideals $(\overline{X})$ and $(\overline{Y})$ of R are 2-SG-flat R-modules, but not 1-SG-flat. Since R may be viewed as a trivially graded ring, we have $(\overline{X})$ and $(\overline{Y})$ are 2-SG-gr-flat which are not 1-SG-gr-flat.

For any $n \geqslant 1$, we use n-SG-gr-Flat(R) to denote the subcategory of R-gr consisting of all n-SG-gr-flat left R-modules.

Proposition 4.4. *For any $n \geqslant 1$, n-SG-gr-Flat(R) is closed under direct sums.*

P r o o f. Let $\{M_j\}_{j\in J}$ be a family of n-strongly Gorenstein gr-flat left R-modules. Then, for any $j \in J$, there exists an exact sequence

$$0 \longrightarrow M_j \longrightarrow F^{(j)}_{n-1} \longrightarrow F^{(j)}_{n-2} \longrightarrow \ldots \longrightarrow F^{(j)}_0 \longrightarrow M_j \longrightarrow 0$$

in R-gr with each $F^{(j)}_i$ gr-flat for any $0 \leqslant i \leqslant n-1$, such that $E \otimes_R -$ leaves the sequence exact whenever E is a gr-injective right R-module. Thus we have the exact sequence

$$0 \longrightarrow \bigoplus_{j\in J} M_j \longrightarrow \bigoplus_{j\in J} F^{(j)}_{n-1} \longrightarrow \bigoplus_{j\in J} F^{(j)}_{n-2} \longrightarrow \ldots \longrightarrow \bigoplus_{j\in J} F^{(j)}_0 \longrightarrow \bigoplus_{j\in J} M_j \longrightarrow 0$$

in R-gr. Since $\bigoplus_{j\in J} F^{(j)}_0, \ldots, \bigoplus_{j\in J} F^{(j)}_{n-1}$ are gr-flat and the sequence above remains exact after applying the functor $E \otimes_R -$ whenever E is a gr-injective right R-module, it follows that $\bigoplus_{j\in J} M_j$ is n-strongly Gorenstein gr-flat. □

Proposition 4.5. *Let n be a positive integer. Then:*
(1) *Every strongly Gorenstein* gr-*flat left R-module is n*-SG-gr-*flat.*
(2) *Every n*-SG-gr-*flat left R-module is Gorenstein* gr-*flat.*

P r o o f. The proof is similar to that of Proposition 3.4, so we omit it. □

Bennis and Mahdou showed in [8], Proposition 3.7 that a strongly Gorenstein flat module is flat if and only if it has finite flat dimension. Hence we have

Proposition 4.6. *For any $n \geqslant 1$, an n-strongly Gorenstein* gr-*flat left R-module is* gr-*flat if and only if it has finite* gr-*flat dimension.*

P r o o f. "only if" part is trivial.
"if" part. Let M be an n-strongly Gorenstein gr-flat left R-module with finite gr-flat dimension. Then there is an exact sequence $0 \to M \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ in R-gr with each F_i gr-flat. Let N be a graded right R-module. Note that $\mathrm{Tor}^R_i(N, F_j) = 0$ for all $i \geqslant 1$ and $0 \leqslant j \leqslant n-1$, so we have $\mathrm{Tor}^R_i(N, M) \cong \mathrm{Tor}^R_{i+n}(N, M)$. Since $\mathrm{fd}_R(M) < \infty$, it follows that $\mathrm{Tor}^R_i(N, M) = 0$ for all $i \geqslant 1$, and hence M is gr-flat. □

The following proposition gives a simple characterization of the n-strongly Gorenstein gr-flat modules.

Proposition 4.7. *The following assertions are equivalent for a graded left R-module M.*

(1) *M is n-strongly Gorenstein* gr-*flat.*
(2) *There exists an exact sequence* $0 \to M \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ *in* R-gr *with each* F_i gr-*flat and* $\mathrm{Tor}_j^R(E, M) = 0$ *for any* gr-*injective right* R-*module* E *and any* $1 \leqslant j \leqslant n$.

Proof. Similar to the proof of Proposition 3.9. □

Recall that a graded ring R is called a gr-n-FC *ring* (see [2]) if R is left and right gr-coherent with self FP-gr-injective dimension on either side at most n for an integer $n \geqslant 0$.

Corollary 4.8. *Let R be a* gr-n-FC *ring. Then the following assertions are equivalent for a graded left R-module M.*
(1) *M is n-strongly Gorenstein* gr-*flat.*
(2) *There exists an exact sequence* $0 \to M \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ *in* R-gr *with each* F_i gr-*flat.*

Proof. (1) $\Rightarrow$ (2): is trivial.
(2) $\Rightarrow$ (1): Since R is a gr-n-FC ring, we have that $\mathrm{fd}_R(N) \leqslant n$ for every gr-injective module N by [2], Proposition 2.8. Similarly to the proof of (2) $\Rightarrow$ (1) in Corollary 3.10, we get the assertion. □

Corollary 4.9. *If R is a* gr-n-FC *ring, then every n-strongly Gorenstein* gr-*projective module is n-strongly Gorenstein* gr-*flat.*

Proof. Clear. □

The following theorem gives a method how to construct a 1-SG-gr-flat module from n-SG-gr-flat modules, which generalizes [27], Theorem 4.4.

Theorem 4.10. *For a module $M \in R$-gr and $n \geqslant 1$, we consider the following conditions.*
(1) *M is n-*SG-gr-*flat.*
(2) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \ldots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \longrightarrow 0$$

in R-gr *with each* F_i gr-*flat for any* $0 \leqslant i \leqslant n-1$, *such that* $\bigoplus_{i=1}^{n} \mathrm{Im}(h_i)$ *is* 1-SG-gr-*flat.*
(3) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \ldots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \longrightarrow 0$$

in R-gr *with each* F_i gr-*flat for any* $0 \leqslant i \leqslant n-1$, *such that* $\bigoplus_{i=1}^{n} \mathrm{Im}(h_i)$ *is Gorenstein* gr-*flat.*

(4) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \dots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \longrightarrow 0$$

in R-gr, *where* F_i *has finite* gr-*flat dimension for any* $0 \leqslant i \leqslant n-1$, *such that* $\bigoplus_{i=1}^{n} \mathrm{Im}(h_i)$ *is* 1-SG-gr-*flat.*

(5) *There exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \dots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \longrightarrow 0$$

in R-gr, *where* F_i *has finite* gr-*flat dimension for any* $0 \leqslant i \leqslant n-1$, *such that* $\bigoplus_{i=1}^{n} \mathrm{Im}(h_i)$ *is Gorenstein* gr-*flat.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). *If* R *is a* gr-n-FC *ring, then also* (5) $\Rightarrow$ (1), *and hence all of these conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2): The proof is similar to that of (1) $\Rightarrow$ (2) in Theorem 3.11 and omitted.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. It is easy to check that (3) $\Rightarrow$ (1).

(5) $\Rightarrow$ (1): Suppose that R is an n-gr-FC ring. There exists an exact sequence

$$0 \longrightarrow M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \dots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \longrightarrow 0$$

in R-gr with F_i gr-flat for any $0 \leqslant i \leqslant n-1$, such that $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is Gorenstein gr-flat. Then, for any $0 \leqslant i \leqslant n-1$, we have an exact sequence

$$0 \to \mathrm{Im}(f_{i+1}) \to E_i \to \mathrm{Im}(f_i) \to 0$$

in R-gr. Since $\bigoplus_{i=1}^{n} \mathrm{Im}(f_i)$ is Gorenstein gr-flat, we have each $\mathrm{Im}(f_i)$ is Gorenstein gr-flat by [19], Proposition 1.4, and [2], Corollaries 2.11 and 2.12, since R is a gr-n-FC ring. In particular, M is Gorenstein gr-flat, and so $\mathrm{Tor}_i^R(I, M) = 0$ for any gr-injective right R-module I and $i \geqslant 1$. Therefore M is n-SG-gr-flat by Proposition 4.7. $\square$

Proposition 4.11. *The following statements hold.*

(1) *If* $M \in R$*-gr is* n-SG-gr-*flat, then* $M^+ \in$ gr-R *is* n-SG-gr-*injective.*

(2) *If* R *is a* gr-n-FC *ring and* $M \in R$*-gr is* n-SG-gr-*injective, then* M^+ *is* n-SG-gr-*flat.*

P r o o f. (1) Let M be an n-strongly Gorenstein gr-flat left R-module. Then there exists an exact sequence

$$0 \to M \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

in R-gr with each F_i gr-flat, which gives rise to the exact sequence

$$0 \to M^+ \to F_0^+ \to F_1^+ \to \ldots \to F_{n-1}^+ \to M^+ \to 0$$

in gr-R with each F_i^+ gr-injective by [26], Lemma 4.1. For any $X \in$ gr-R and $N \in R$-gr, we have $\mathrm{EXT}^1_R(N, X^+) \cong \mathrm{Tor}_1^R(X, N)^+$ by [17], Lemma 2.1. On the other hand, there exists an exact sequence $0 \to K \to P \to N \to 0$ in R-gr with P gr-projective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{EXT}^1_R(K, X^+) & \longrightarrow & \mathrm{EXT}^2_R(N, X^+) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Tor}_1^R(X, K)^+ & \longrightarrow & \mathrm{Tor}_2^R(X, N)^+ & \longrightarrow & 0. \end{array}$$

It follows that $\mathrm{EXT}^2_R(N, X^+) \cong \mathrm{Tor}_2^R(X, N)^+$ for any graded left R-module N. By using induction on i, one easily gets that $\mathrm{EXT}^i_R(N, X^+) \cong \mathrm{Tor}_i^R(X, N)^+$ for any $i \geqslant 1$. Now let E be a gr-injective right R-module, then we have that

$$\mathrm{EXT}^i_R(E, M^+) \cong \mathrm{Tor}_i^R(M, E)^+ = 0$$

for any $i \geqslant 1$ by assumption. It follows that $\mathrm{Ext}^i_{R\text{-gr}}(E, M^+) = 0$ for any $i \geqslant 1$. Therefore, M^+ is n-strongly Gorenstein gr-injective by Proposition 3.9.

(2) Assume that R is a gr-n-FC ring. Since M is an n-strongly Gorenstein gr-injective left R-module, we have an exact sequence

$$0 \to M \to E_0 \to E_1 \to \ldots \to E_{n-1} \to M \to 0$$

in R-gr with each E_i gr-injective, which induces an exact sequence

$$0 \to M^+ \to E_{n-1}^+ \to \ldots \to E_1^+ \to E_0^+ \to M^+ \to 0$$

in gr-R with each E_i^+ gr-flat. The assertion follows from Corollary 4.8. □

We finish this section by considering the relations between the graded and the ungraded n-strongly Gorenstein flat modules.

Proposition 4.12. *Let R be a* gr-n-FC *ring by a finite group G.*

(1) *If M is an n-SG-flat left R-module, then $F(M)$ is n-SG-gr-flat.*

(2) *If $M \in R$-gr is n-SG-gr-flat, then $U(M)$ is an n-SG-flat left R-module.*

Proof. (1) Let M be an n-SG-flat left R-module. Then there exists an exact sequence

$$0 \longrightarrow M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \dots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \longrightarrow 0$$

in R-Mod with each F_i flat for any $0 \leqslant i \leqslant n-1$. Since the functor F is exact, we get the exact sequence

$$0 \longrightarrow F(M) \xrightarrow{F(h_n)} F(F_{n-1}) \xrightarrow{F(h_{n-1})} F(F_{n-2}) \xrightarrow{F(h_{n-2})} \dots \xrightarrow{F(h_1)} F(F_0) \xrightarrow{F(h_0)} F(M) \longrightarrow 0$$

in R-gr. Note that G is a finite group, hence we have that each $F(F_i)$ is gr-flat. Thus $F(M)$ is an n-SG-gr-flat left R-module by Corollary 4.8.

(2) Let M be an n-SG-gr-flat left R-module. Then there exists an exact sequence

$$0 \longrightarrow M \xrightarrow{h_n} Q_{n-1} \xrightarrow{h_{n-1}} Q_{n-2} \xrightarrow{h_{n-2}} \dots \xrightarrow{h_1} Q_0 \xrightarrow{h_0} M \longrightarrow 0$$

in R-gr with each Q_i gr-flat for any $0 \leqslant i \leqslant n-1$. Since the functor U is exact, we obtain the exact sequence

$$(4.2) \qquad 0 \longrightarrow U(M) \xrightarrow{U(h_n)} U(Q_{n-1}) \xrightarrow{U(h_{n-1})} U(Q_{n-2}) \xrightarrow{U(h_{n-2})} \dots \xrightarrow{U(h_1)} U(Q_0) \xrightarrow{U(h_0)} U(M) \longrightarrow 0$$

in R-Mod such that each $U(Q_i)$ is flat. For every injective right R-module E, one easily gets that $\mathrm{Tor}_i^R(E, U(M)) \cong \mathrm{Tor}_{i+n}^R(E, U(M))$ for any $i \geqslant 1$. On the other hand, $\mathrm{fd}_R(E) \leqslant n$ by [11], Theorem 3.8. It follows that $\mathrm{Tor}_i^R(E, U(M)) = 0$ for any $i \geqslant 1$. Consequently, the functor $E \otimes_R -$ leaves the sequence (4.2) exact whenever E is an injective right R-module. So the assertion follows. □

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