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Dong Su; Shilin Yang

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AUTOMORPHISM GROUP OF REPRESENTATION RING OF THE WEAK HOPF ALGEBRA $\widetilde{H_8}$

Dong Su, Shilin Yang, Beijing

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Abstract. Let H_8 be the unique noncommutative and noncocommutative eight dimensional semi-simple Hopf algebra. We first construct a weak Hopf algebra $\widetilde{H_8}$ based on H_8, then we investigate the structure of the representation ring of $\widetilde{H_8}$. Finally, we prove that the automorphism group of $r(\widetilde{H_8})$ is just isomorphic to D_6, where D_6 is the dihedral group with order 12.

Keywords: automorphism group; representation ring; weak Hopf algebra

MSC 2010: 16W20, 19A22

1. Introduction

As is well known, many researches have focused on studying automorphisms of algebras. For examples, van der Kulk in [17], Zhao in [21], Yu in [20], Vesselin and Yu in [8] have made some significant contributions to the automorphisms of polynomial algebras. Alperin in [2] gave the homology of the group of automorphisms of $k[x, y]$ over a field k. Furthermore, Dicks in [7] researched automorphisms of polynomial ring in two variables. Chen in [3] consider the coalgebra automorphism group of Hopf algebra $k_q[x; x^{-1}; y]$. Han and Su in [9] studied the automorphism group of Witt algebras. Jia et al. in [10] proved that the automorphism group of the Green ring of the Sweedler Hopf algebra over the field $\mathbb{F}$ is isomorphic to the Klein group, and the automorphism group of the Green algebra of the Sweedler Hopf algebra is just the semidirect product of $\mathbb{Z}_2$ and G, where the group $G = \mathbb{F} \setminus \{1/2\}$ with multiplication given by $a \cdot b = 1 - a - b + 2ab$.

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Recently, Chen, Van Oystaeyen and Zhang in [4] described the structure of the Green rings of the Taft algebra $H_n(q)$. Li and Zhang in [12] extended these results to the case of the generalized Taft Hopf algebras $H_{n,d}(q)$ and determined all nilpotent elements in the Green ring of $H_{n,d}(q)$. It is noted that for generalized Taft Hopf algebras Yang in [18] classified their indecomposable modules and gave the multiplication of their representation rings. In this paper, we first construct the weak Hopf algebra $\widetilde{H_8}$ corresponding to the unique 8-dimensional noncommutative and noncocommutative semi-simple Hopf algebra H_8. Then we describe the structure of the representation ring $r(\widetilde{H_8})$ of $\widetilde{H_8}$ by the generators and relations. Finally, we investigate the automorphism group of the representation ring $r(\widetilde{H_8})$.

The paper is organized as follows. We first introduce some notation and the concept of the 8-dimensional semi-simple Hopf algebra H_8. Then we introduce a class of weak Hopf algebras $\widetilde{H_8}$ based on H_8. The structure of its representation ring $r(\widetilde{H_8})$ is investigated. Finally we show that the automorphism group of $r(\widetilde{H_8})$ is isomorphic to D_6, where D_6 is the dihedral group with order 12. It is interesting to describe the corresponding results for restricted forms of general quantum groups. It is noted that our approach is very straightforward.

2. Preliminaries

Throughout, we work over the complex field $\mathbb{C}$ unless otherwise stated. All algebras, Hopf algebras and modules are defined over $\mathbb{C}$; all modules are left modules and finite dimensional; all maps are $\mathbb{C}$-linear; dim, $\otimes$ and hom stand for $\dim_{\mathbb{C}}$, $\otimes_{\mathbb{C}}$ and $\hom_{\mathbb{C}}$, respectively. For the theory of Hopf algebras, we refer to [14], [16].

All 8-dimensional Hopf algebras are described in [13], [15]. One of them contains a unique neither commutative nor cocommutative semisimple Hopf algebra H_8. In detail, as an algebra over $\mathbb{C}$, H_8 is generated by g, h and x subject to the relations

$$g^2 = 1, \quad h^2 = 1, \quad gh = hg, \quad xg = hx, \quad gx = xh, \quad x^2 = \tfrac{1}{2}(1 + g + h - gh).$$

The coalgebra structure Δ, ε and the antipode S are given by

$$\begin{gathered}\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \varepsilon(g) = 1, \quad \varepsilon(h) = 1,\\ \Delta(x) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes g + h \otimes 1 - h \otimes g)(x \otimes x), \quad \varepsilon(x) = 1,\\ S(g) = g^{-1}, \quad S(h) = h^{-1}, \quad S(x) = x.\end{gathered}$$

Note that the set

$$\{1, g, h, x, gh, gx, xg, xgh\}$$

forms a basis of H_8 and

$$H_8 \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}). \tag{2.1}$$

Definition 2.1. The $\mathbb{C}$-algebra $\widetilde{H_8}$ is the associative algebra generated by g, h and x subject to the relations

$$g^3 = g, \quad h^3 = h, \quad g^2 = h^2, \quad gh = hg, \quad x = hxg,$$
$$x = gxh, \quad x^2 = \tfrac{1}{2}(g^2 + g + h - gh).$$

We set $J = g^2 = h^2$, it is easy to see that J and $1 - J$ are a pair of orthogonal central idempotents in $\widetilde{H_8}$. Let $W_1 = \widetilde{H_8}J$, $W_2 = \widetilde{H_8}(1 - J)$; we have

Proposition 2.2. *$\widetilde{H_8} = W_1 \oplus W_2$, as two-sided ideals. Moreover, $W_1 \cong H_8$ and $W_2 \cong \mathbb{C}$ as algebras.*

Proof. The first statement is easy to see. Let us prove the second statement. Note that W_1 is generated by g, h and x and subject to the relations

$$g^2 = h^2 = J, \quad gh = hg, \quad gx = xh, \quad xg = hx, \quad x^2 = \tfrac{1}{2}(g^2 + g + h - gh).$$

Let $\varphi\colon W_1 \to H_8$ be the map defined by

$$\varphi(J) = 1, \quad \varphi(x) = x, \quad \varphi(g) = g, \quad \varphi(h) = h.$$

It is easy to see that φ is an algebraic isomorphism.

W_2 is generated by $(1 - J)g$, $(1 - J)h$ and $(1 - J)x$. Note that

$$Jg = gJ = g, \quad Jh = hJ = h,$$

moreover, $x = hxg$ or $x = gxh$. It follows that

$$Jx = Jhxg = hxg = x = xJ, \quad \text{or} \quad Jx = Jgxh = gxh = x = xJ.$$

Hence $g(1 - J) = 0$, $h(1 - J) = 0$, $x(1 - J) = 0$ and $W_2 \cong \mathbb{C}$. $\square$

By Proposition 2.2, it is easy to see that $\widetilde{H_8}$ is semi-simple, and the set

$$\{1, g, h, x, gh, gx, xg, J, xgh\}$$

forms a basis of $\widetilde{H_8}$.

The definition of the weak Hopf algebra was introduced by Li (see [11]). Many examples of weak Hopf algebras can be found in [1], [19], [6], [5]. Recall that a k-bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ is called a weak Hopf algebra if there exists a map $T \in \hom(H, H)$ such that $T * \mathrm{id} * T = T$ and $\mathrm{id} * T * \mathrm{id} = \mathrm{id}$, where $*$ is the convolution map in $\hom(H, H)$. Now, we introduce the coalgebra structure maps on $\widetilde{H_8}$ as follows.

The comultiplication $\Delta\colon \widetilde{H_8} \to \widetilde{H_8} \otimes \widetilde{H_8}$ and the counit $\varepsilon\colon \widetilde{H_8} \to k$ are given by

$$\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \varepsilon(1) = \varepsilon(g) = \varepsilon(h) = 1,$$
$$\Delta(x) = \tfrac{1}{2}(g^2 \otimes g^2 + g^2 \otimes g + h \otimes g^2 - h \otimes g)(x \otimes x), \quad \varepsilon(x) = 1.$$

It is obvious that $\widetilde{H_8}$ is indeed a coalgebra by the definition of Δ and ε.

The $\mathbb{C}$-map $T\colon \widetilde{H_8} \to \widetilde{H_8}$ is given by

$$T(1) = 1, \quad T(g) = g, \quad T(h) = h, \quad T(x) = x.$$

Theorem 2.3. *$\widetilde{H_8}$ is a noncommutative and noncocommutative weak Hopf algebra with the weak antipode T.*

P r o o f. (1) It is straightforward to check that $\widetilde{H_8}$ is a bialgebra.

(2) The map T can define a weak antipode in $\widetilde{H_8}$ naturally. First, the map $T\colon \widetilde{H_8} \to \widetilde{H_8}^{\mathrm{op}}$ keeps the defining relations. Indeed,

$$(T(g))^3 = T(g), \quad (T(h))^3 = T(h), \quad (T(g))^2 = (T(h))^2, \quad T(g)T(h) = T(h)T(g).$$

When $x = hxg$, we have

$$T(g)T(x)T(h) = gxh = x = T(x),$$

when $x = gxh$, we have

$$T(h)T(x)T(g) = hxg = x = T(x).$$

Therefore the map T can define an anti-algebra homomorphism $T\colon \widetilde{H_8} \to \widetilde{H_8}$.

Secondly, it is easy to see that in $\widetilde{H_8}$ we have

$$\begin{aligned}
T * \mathrm{id} * T(g) &= m(T \otimes \mathrm{id} \otimes T)(g \otimes g \otimes g) = g^3 = g = T(g),\\
\mathrm{id} * T * \mathrm{id}(g) &= m(\mathrm{id} \otimes T \otimes \mathrm{id})(g \otimes g \otimes g) = g^3 = g = \mathrm{id}(g),\\
T * \mathrm{id} * T(h) &= m(T \otimes \mathrm{id} \otimes T)(h \otimes h \otimes h) = h^3 = h = T(h),\\
\mathrm{id} * T * \mathrm{id}(h) &= m(\mathrm{id} \otimes T \otimes \mathrm{id})(h \otimes h \otimes h) = h^3 = h = \mathrm{id}(h),
\end{aligned}$$

$$
\begin{aligned}
T * \mathrm{id} * T(x) &= m(T \otimes \mathrm{id} \otimes T)((g^2x + hx) \otimes (g^2x + hx) \\
&\qquad \otimes g^2x + (g^2x + hx) \otimes (g^2x - hx) \otimes gx + (g^2x - hx) \\
&\qquad \otimes (gx + ghx) \otimes gx + (g^2x - hx) \otimes (gx - ghx) \otimes g^2x) \\
&= \tfrac{1}{2}(g^2 + g + h - gh)x^3 = x^5 = x = T(x), \\
\mathrm{id} * T * \mathrm{id}(x) &= m(\mathrm{id} \otimes T \otimes \mathrm{id})((g^2x + hx) \otimes (g^2x + hx) \\
&\qquad \otimes g^2x + (g^2x + hx) \otimes (g^2x - hx) \otimes gx + (g^2x - hx) \\
&\qquad \otimes (gx + ghx) \otimes gx + (g^2x - hx) \otimes (gx - ghx) \otimes g^2x) \\
&= \tfrac{1}{2}(g^2 + g + h - gh)x^3 = x^5 = x = \mathrm{id}(x).
\end{aligned}
$$

On the other hand, we have

$$
\begin{gathered}
\mathrm{id} * T(g) = J = T * \mathrm{id}(g), \quad \mathrm{id} * T(h) = J = T * \mathrm{id}(h), \\
\mathrm{id} * T(x) = \tfrac{1}{2}x(g^2 + g + h - gh)x = x^4 = J = T * \mathrm{id}(x).
\end{gathered}
$$

These arguments show that for any $z \in \widetilde{H_8}$ that we have $\mathrm{id} * T(z)$ and $T * \mathrm{id}(z)$ are the elements of the center of $\widetilde{H_8}$. Now, if $a, b \in \widetilde{H_8}$ and

$$
\begin{gathered}
T * \mathrm{id} * T(a) = T(a), \quad T * \mathrm{id} * T(b) = T(b), \\
\mathrm{id} * T * \mathrm{id}(a) = a, \quad \mathrm{id} * T * \mathrm{id}(b) = b,
\end{gathered}
$$

it is easy to see that

$$
T * \mathrm{id} * T(ab) = T(ab), \quad \mathrm{id} * T * \mathrm{id}(ab) = ab.
$$

Hence T is indeed a weak antipode of $\widetilde{H_8}$ and $\widetilde{H_8}$ is a weak Hopf algebra, which is noncommutative and noncocommutative. □

3. The representation ring $r(\widetilde{H_8})$ of $\widetilde{H_8}$

Assume that A is an algebra, and let irr-A denote the set of finite dimensional irreducible A-modules.

One sees that $\widetilde{H_8}$ is semisimple. By Proposition 2.2 we have $\widetilde{H_8} = W_1 \oplus W_2$ as algebras, where $W_1 \cong H_8$ and $W_2 \cong \mathbb{C}$. Finite dimensional irreducible representations of $\widetilde{H_8}$ are described as follows.

Lemma 3.1. *There are six classes of non-isomorphic irreducible $\widetilde{H_8}$-modules S_n, $n \in \mathbb{Z}_5$, and S, the actions of $\widetilde{H_8}$ on them are defined as follows:*

$$\begin{aligned}
S_m\colon\ & g\cdot v^{(m)} = (-1)^m v^{(m)}, && h\cdot v^{(m)} = (-1)^m v^{(m)},\\
& x\cdot v^{(m)} = \mathrm{i}^m v^{(m)}, && v^{(m)} \in S_m,\ m\in\mathbb{Z}_4,\\
S_4\colon\ & g\cdot v^{(4)} = 0, && h\cdot v^{(4)} = 0,\\
& x\cdot v^{(4)} = 0, && v^{(4)} \in S_4,\\
S\colon\ & g\cdot v_j = (-1)^j v_j, && h\cdot v_j = (-1)^{j+1} v_j,\\
& x\cdot v_j = v_{3-j}, && v_j\in S,\ j=1,2,
\end{aligned}$$

where $v^{(n)}$ is the basis of S_n and v_1, v_2 is the basis of S.

Proof. It is obvious by (2.1) and Proposition 2.2. In fact, S_n, $n \in \mathbb{Z}_4$, S and S_4 are just irreducible $\widetilde{H_8}$-modules lifting by those of H_8-modules and $\mathbb{C}$-modules. $\square$

Let H be a finite dimensional semisimple bialgebra and M and N two finite dimensional H-modules. Then $M\otimes N$ is also an H-module defined by

$$h\cdot(m\otimes n) = \sum_{(h)} h_{(1)}\cdot m \otimes h_{(2)}\cdot n$$

for all $h\in H$ and $m\in M$, $n\in N$, where $\Delta(h) = \sum_{(h)} h_{(1)}\otimes h_{(2)}$. By the Krull-Schmidt theorem, $M\otimes N$ can be decomposed into the direct sum of irreducible H-modules. The decomposition formulas of the tensor product of two irreducible $\widetilde{H_8}$-modules are as follows.

Lemma 3.2. *Let $n\in\mathbb{Z}_5$, then as $\widetilde{H_8}$-modules we have*

(1) *provided that $m, m' \in \mathbb{Z}_4$,*
   (a) *if $m+m'$ is odd, then $S_m\otimes S_{m'} \cong S_{m+m' \pmod 4}$;*
   if $m+m'$ is even, then $S_m\otimes S_{m'} \cong S_{m-m' \pmod 4}$;
   (b) *$S\otimes S_m \cong S_m\otimes S \cong S$;*

(2) *$S\otimes S \cong \bigoplus_{i=0}^{3} S_i$;*

(3) *$S_n\otimes S_4 \cong S_4\otimes S_n \cong S_4$;*

(4) *$S\otimes S_4 \cong S_4\otimes S \cong S_4\oplus S_4$.*

Proof. (1) (a) Considering the tensor product $S_m\otimes S_{m'}$, where $m, m'\in\mathbb{Z}_4$, we have

$$\begin{aligned}
g\cdot(v^{(m)}\otimes v^{(m')}) &= (-1)^{m+m'}(v^{(m)}\otimes v^{(m')}) = (-1)^{m-m'}(v^{(m)}\otimes v^{(m')}),\\
h\cdot(v^{(m)}\otimes v^{(m')}) &= (-1)^{m+m'}(v^{(m)}\otimes v^{(m')}) = (-1)^{m-m'}(v^{(m)}\otimes v^{(m')});
\end{aligned}$$

if $m+m'$ is odd, then

$$x \cdot (v^{(m)} \otimes v^{(m')}) = \mathrm{i}^{m+m'}(v^{(m)} \otimes v^{(m')}),$$

if $m+m'$ is even, then

$$x \cdot (v^{(m)} \otimes v^{(m')}) = \mathrm{i}^{m-m'}(v^{(m)} \otimes v^{(m')}).$$

It follows that if $m+m'$ is odd, then $S_m \otimes S_{m'} \cong S_{m+m' \pmod 4}$; if $m+m'$ is even, then $S_m \otimes S_{m'} \cong S_{m-m' \pmod 4}$.

(1) (b) Considering the tensor products $S_m \otimes S$ and $S \otimes S_m$, where $m \in \mathbb{Z}_4$, we have for given $j = 1, 2$

$$g \cdot (v^{(m)} \otimes v_j) = (-1)^{m+j}(v^{(m)} \otimes v_j),$$
$$h \cdot (v^{(m)} \otimes v_j) = (-1)^{m+1+j}(v^{(m)} \otimes v_j);$$

if $m-j$ is odd, then

$$x \cdot (v^{(m)} \otimes v_j) = \mathrm{i}^{-m}(v^{(m)} \otimes v_{3-j}),$$

if $m-j$ is even, then

$$x \cdot (v^{(m)} \otimes v_j) = \mathrm{i}^{m}(v^{(m)} \otimes v_{3-j}).$$

Further,

$$g \cdot (v_j \otimes v^{(m)}) = (-1)^{m+j}(v_j \otimes v^{(m)}),$$
$$h \cdot (v_j \otimes v^{(m)}) = (-1)^{m+1+j}(v_j \otimes v^{(m)});$$

if $m-j$ is odd, then

$$x \cdot (v_j \otimes v^{(m)}) = \mathrm{i}^{m}(v_{3-j} \otimes v^{(m)}),$$

if $m-j$ is even, then

$$x \cdot (v_j \otimes v^{(m)}) = \mathrm{i}^{-m}(v_{3-j} \otimes v^{(m)}).$$

Obviously, if $m = 0$, then

$$S_0 \otimes S \cong S \otimes S_0 \cong S.$$

If $m = 1$, we set

$$w_1 = \mathrm{i}v^{(1)} \otimes v_2, \quad w_2 = v^{(1)} \otimes v_1.$$

It is easy to check that $\{w_1, w_2\}$ is also a basis of the $\widetilde{H_8}$-module $S_1 \otimes S$, and

$$g \cdot w_k = (-1)^k w_k, \quad h \cdot w_k = (-1)^{k+1} w_k, \quad x \cdot w_k = w_{3-k}, \quad k = 1, 2.$$

Hence $S_1 \otimes S \cong S$. We set

$$w_1' = v_2 \otimes v^{(1)}, \quad w_2' = \mathrm{i} v_1 \otimes v^{(1)}.$$

It is easy to check that $\{w_1', w_2'\}$ is also a basis of the $\widetilde{H_8}$-module $S \otimes S_1$, and

$$g \cdot w_k' = (-1)^k w_k', \quad h \cdot w_k' = (-1)^{k+1} w_k', \quad x \cdot w_k' = w_{3-k}', \quad k = 1, 2.$$

Then $S \otimes S_1 \cong S$. The same arguments are applied to the case $m = 2$ and $m = 3$, we show that $S \otimes S_m \cong S_m \otimes S \cong S$.

(2) Considering the tensor product $S \otimes S$, we have for given $j, j' = 1, 2$

$$g \cdot (v_j \otimes v_{j'}) = (-1)^{j+j'} (v_j \otimes v_{j'}),$$
$$h \cdot (v_j \otimes v_{j'}) = (-1)^{j+j'} (v_j \otimes v_{j'});$$

if $j + j'$ is odd, then

$$x \cdot (v_j \otimes v_{j'}) = \mathrm{i}^{2j} (v_{3-j} \otimes v_{3-j'}),$$

if $j + j'$ is even, then

$$x \cdot (v_j \otimes v_{j'}) = v_{3-j} \otimes v_{3-j'}.$$

Set

$$u_0 = v_1 \otimes v_1 + v_2 \otimes v_2, \quad u_1 = -\mathrm{i} v_1 \otimes v_2 + v_2 \otimes v_1,$$
$$u_2 = v_1 \otimes v_1 - v_2 \otimes v_2, \quad u_3 = \mathrm{i} v_1 \otimes v_2 + v_2 \otimes v_1.$$

It is easy to check that $\{u_k\}$, $k \in \mathbb{Z}_4$, is also a basis of the $\widetilde{H_8}$-module $S \otimes S$, and

$$g \cdot u_k = (-1)^k u_k, \quad h \cdot u_k = (-1)^k u_k, \quad x \cdot u_k = \mathrm{i}^k u_k.$$

Hence $S \otimes S \cong \bigoplus_{i=0}^{3} S_i$.

(3) Considering the tensor products $S_n \otimes S_4$ and $S_4 \otimes S_n$, where $n \in \mathbb{Z}_5$, we have

$$g \cdot (v^{(n)} \otimes v^{(4)}) = 0, \quad h \cdot (v^{(n)} \otimes v^{(4)}) = 0, \quad x \cdot (v^{(n)} \otimes v^{(4)}) = 0,$$

and

$$g \cdot (v^{(4)} \otimes v^{(n)}) = 0, \quad h \cdot (v^{(4)} \otimes v^{(n)}) = 0, \quad x \cdot (v^{(4)} \otimes v^{(n)}) = 0.$$

Hence $S_4 \otimes S_n \cong S_n \otimes S_4 \cong S_4$.

(4) Considering the tensor products $S \otimes S_4$ and $S_4 \otimes S$, we have for given $j = 1, 2$

$$g \cdot (v_j \otimes v^{(4)}) = 0, \quad h \cdot (v_j \otimes v^{(4)}) = 0, \quad x \cdot (v_j \otimes v^{(4)}) = 0,$$

and

$$g \cdot (v^{(4)} \otimes v_j) = 0, \quad h \cdot (v^{(4)} \otimes v_j) = 0, \quad x \cdot (v^{(4)} \otimes v_j) = 0.$$

Hence $S \otimes S_4 \cong S_4 \otimes S \cong S_4 \oplus S_4$. □

Corollary 3.3. *For any $\widetilde{H_8}$-modules M and N, we have the isomorphism*

$$M \otimes N \cong N \otimes M$$

as $\widetilde{H_8}$-modules.

Let H be a semisimple bialgebra; the representation ring $r(H)$ of H is defined as follows. As a group $r(H)$ is the free abelian group generated by the isomorphism classes $[V]$ of finite dimensional H-modules V modulo the relations

$$[M \oplus V] = [M] + [V].$$

The multiplication of $r(H)$ is given by the tensor product of H-modules, that is,

$$[M][V] = [M \otimes V].$$

Note that the representation ring $r(H)$ is an associative ring with a $\mathbb{Z}$-basis $\{[V]\colon V \in \text{irr-}H\}$.

Theorem 3.4. *The representation ring $r(\widetilde{H_8})$ of $\widetilde{H_8}$ is isomorphic to the quotient ring of the polynomial ring $\mathbb{Z}[x_1, x_2, x_3, x_4]$ modules the ideal I generated by the elements*

$$x_1^2-1, \quad x_2^2-1, \quad x_1x_3-x_3, \quad x_2x_3-x_3, \quad 1+x_1+x_2+x_1x_2-x_3^2, \quad x_4^2-x_4, \quad x_3x_4-2x_4.$$

P r o o f. Let $\pi\colon \mathbb{Z}[x_1, x_2, x_3, x_4] \to \mathbb{Z}[x_1, x_2, x_3, x_4]/I$ be the natural epimorphism and $\overline{v} = \pi(v)$ for any $v \in \mathbb{Z}[x_1, x_2, x_3, x_4]$. In $\mathbb{Z}[x_1, x_2, x_3, x_4]/I$, we have

$$\overline{x_1}^2 = \overline{x_2}^2 = 1, \quad \overline{x_1x_3} = \overline{x_2x_3} = \overline{x_3},$$
$$\overline{x_3}^2 = 1 + \overline{x_1} + \overline{x_2} + \overline{x_1x_2}, \quad \overline{x_4}^2 = \overline{x_4}, \quad \overline{x_3x_4} = 2\overline{x_4}.$$

It is straightforward to check that the ring $\mathbb{Z}[x_1, x_2, x_3, x_4]/I$ is $\mathbb{Z}$-spanned by

$$\{1, \overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_1 x_2}\}.$$

This also means that the rank of $\mathbb{Z}[x_1, x_2, x_3, x_4]/I$ is at most 6.

Let $a_1 = [S_1]$, $a_2 = [S_2]$, $a_3 = [S]$, $a_4 = [S_4]$. Since $[S_0]$ is the identity element in $r(\widetilde{H_8})$, the ring $r(\widetilde{H_8})$ is generated by a_1, a_2, a_3, a_4 by Lemma 3.2. Therefore there is a unique ring epimorphism

$$\varphi\colon\ \mathbb{Z}[x_1, x_2, x_3, x_4] \to r(\widetilde{H_8})$$

such that

$$\varphi(x_i) = a_i, \quad i = 1, 2, 3, 4.$$

On the other hand, from Lemma 3.2 we have

$$\begin{gathered} a_1^2 = a_2^2 = 1, \quad a_1 a_3 = a_2 a_3 = a_3, \\ 1 + a_1 + a_2 + a_1 a_2 = a_3^2, \quad a_4^2 = a_4, \quad a_3 a_4 = 2a_4. \end{gathered}$$

It follows that

$$\begin{gathered} \varphi(x_1^2 - 1) = 0, \quad \varphi(x_2^2 - 1) = 0, \quad \varphi(x_3 x_1 - x_3) = 0, \quad \varphi(x_3 x_2 - x_3) = 0, \\ \varphi(1 + x_1 + x_2 + x_1 x_2 - x_3^2) = 0, \quad \varphi(x_4^2 - x_4) = 0, \quad \varphi(x_3 x_4 - 2x_4) = 0. \end{gathered}$$

Hence, $\varphi(I) = 0$ and φ induces a ring epimorphism

$$\overline{\varphi}\colon\ \mathbb{Z}[x_1, x_2, x_3, x_4]/I \to r(\widetilde{H_8}),$$

such that $\overline{\varphi}(\overline{v}) = \varphi(v)$ for all $v \in \mathbb{Z}[x_1, x_2, x_3, x_4]$. Noting that the $\mathbb{Z}$-rank of $r(\widetilde{H_8})$ is 6, we get that $\overline{\varphi}$ is in fact a ring isomorphism. $\square$

Remark 3.5. Argument similar to the proof of Theorem 3.4 shows that

$$r(H_8) \cong \mathbb{Z}[x_1, x_2, x_3]/I,$$

where I is the ideal generated by the elements

$$x_1^2 - 1, \quad x_2^2 - 1, \quad x_1 x_3 - x_3, \quad x_2 x_3 - x_3, \quad 1 + x_1 + x_2 + x_1 x_2 - x_3^2.$$

4. Automorphism group of representation ring $r(\widetilde{H_8})$

In this section, let $\mathbf{A}_g$ denote the corresponding coefficient matrix of a $\mathbb{Z}$-linear map $g\colon r(\widetilde{H_8}) \to r(\widetilde{H_8})$, and let $|\mathbf{A}_g|$ denote the determinant of $\mathbf{A}_g$.

Let g_i, $i \in \mathbb{Z}_{12}$ be $\mathbb{Z}$-linear maps of $r(\widetilde{H_8})$ determined by the following relations:

$$
\begin{array}{rllllll}
g_0\colon & 1 \to 1 & x_1 \to x_1, & x_2 \to x_2, & x_3 \to x_3, & x_1x_2 \to x_1x_2, & x_4 \to x_4,\\
g_1\colon & 1 \to 1 & x_1 \to x_1x_2, & x_2 \to x_1, & x_3 \to x_3, & x_1x_2 \to x_2, & x_4 \to x_4,\\
g_2\colon & 1 \to 1 & x_1 \to x_1x_2, & x_2 \to x_1, & x_3 \to -x_3+4x_4, & x_1x_2 \to x_2, & x_4 \to x_4,\\
g_3\colon & 1 \to 1 & x_1 \to x_1x_2, & x_2 \to x_2, & x_3 \to x_3, & x_1x_2 \to x_1, & x_4 \to x_4,\\
g_4\colon & 1 \to 1 & x_1 \to x_1x_2, & x_2 \to x_2, & x_3 \to -x_3+4x_4, & x_1x_2 \to x_1, & x_4 \to x_4,\\
g_5\colon & 1 \to 1 & x_1 \to x_1, & x_2 \to x_1x_2, & x_3 \to x_3, & x_1x_2 \to x_2, & x_4 \to x_4,\\
g_6\colon & 1 \to 1 & x_1 \to x_1, & x_2 \to x_1x_2, & x_3 \to -x_3+4x_4, & x_1x_2 \to x_2, & x_4 \to x_4,\\
g_7\colon & 1 \to 1 & x_1 \to x_1, & x_2 \to x_2, & x_3 \to -x_3+4x_4, & x_1x_2 \to x_1x_2, & x_4 \to x_4,\\
g_8\colon & 1 \to 1 & x_1 \to x_2, & x_2 \to x_1x_2, & x_3 \to x_3, & x_1x_2 \to x_1, & x_4 \to x_4,\\
g_9\colon & 1 \to 1 & x_1 \to x_2, & x_2 \to x_1x_2, & x_3 \to -x_3+4x_4, & x_1x_2 \to x_1, & x_4 \to x_4,\\
g_{10}\colon & 1 \to 1 & x_1 \to x_2, & x_2 \to x_1, & x_3 \to x_3, & x_1x_2 \to x_1x_2, & x_4 \to x_4,\\
g_{11}\colon & 1 \to 1 & x_1 \to x_2, & x_2 \to x_1, & x_3 \to -x_3+4x_4, & x_1x_2 \to x_1x_2, & x_4 \to x_4.
\end{array}
$$

It is easy to check that g_i, $i \in \mathbb{Z}_{12}$, are automorphisms of $r(\widetilde{H_8})$ and g_0 is the identity map. The set $\{g_i\colon\ i \in \mathbb{Z}_{12}\}$ is a group under the composition of functions. The multiplication is described as follows:

$\circ$	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
g_0	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
g_1	g_1	g_8	g_9	g_{10}	g_{11}	g_3	g_4	g_2	g_0	g_7	g_5	g_6
g_2	g_2	g_9	g_8	g_{11}	g_{10}	g_4	g_3	g_1	g_7	g_0	g_6	g_5
g_3	g_3	g_5	g_6	g_0	g_7	g_1	g_2	g_4	g_{10}	g_{11}	g_8	g_9
g_4	g_4	g_6	g_5	g_7	g_0	g_2	g_1	g_3	g_{11}	g_{10}	g_9	g_8
g_5	g_5	g_{10}	g_{11}	g_8	g_9	g_0	g_7	g_6	g_3	g_4	g_1	g_2
g_6	g_6	g_{11}	g_{10}	g_9	g_8	g_7	g_0	g_5	g_4	g_3	g_2	g_1
g_7	g_7	g_2	g_1	g_4	g_3	g_6	g_5	g_0	g_9	g_8	g_{11}	g_{10}
g_8	g_8	g_0	g_7	g_5	g_6	g_{10}	g_{11}	g_9	g_1	g_2	g_3	g_4
g_9	g_9	g_7	g_0	g_6	g_5	g_{11}	g_{10}	g_8	g_2	g_1	g_4	g_3
g_{10}	g_{10}	g_3	g_4	g_1	g_2	g_8	g_9	g_{11}	g_5	g_6	g_0	g_7
g_{11}	g_{11}	g_4	g_3	g_2	g_1	g_9	g_8	g_{10}	g_6	g_5	g_7	g_0

It follows that $\{g_i\colon\ i \in \mathbb{Z}_{12}\}$ is a subgroup of $\mathrm{Aut}(r(\widetilde{H_8}))$. Also we have

$$
\begin{gathered}
g_2^2 = g_8, \quad g_2^3 = g_7, \quad g_2^4 = g_1, \quad g_2^5 = g_9, \quad g_2^6 = g_0, \quad g_3^2 = g_0,\\
g_2g_3 = g_{11}, \quad g_3g_2 = g_6, \quad g_2^2g_3 = g_5, \quad g_2^3g_3 = g_4, \quad g_2^4g_3 = g_{10}.
\end{gathered}
$$

Hence, $\{g_i\colon\ i \in \mathbb{Z}_{12}\} \cong D_6$ as groups, where

$$D_6 = \langle u, v\colon\ u^6 = 1,\ v^2 = 1,\ v^{-1}uv = u^{-1}\rangle$$

is the dihedral group with order 12.

In the sequel, we will show the automorphism group $\mathrm{Aut}(r(\widetilde{H_8}))$ is just the group $\{g_i\colon\ i \in \mathbb{Z}_{12}\}$.

Lemma 4.1. *Let g be an automorphism of $r(\widetilde{H_8})$. Then*

(1) $g(x_1) = \pm x_1$ *or* $g(x_1) = \pm x_2$ *or* $g(x_1) = \pm x_1x_2$ *or* $g(x_1) = \pm 1 \mp 2x_4$ *or* $g(x_1) = \pm x_1 \mp 2x_4$ *or* $g(x_1) = \pm x_2 \mp 2x_4$ *or* $g(x_1) = \pm x_1x_2 \mp 2x_4$;

(2) $g(x_2) = \pm x_1$ *or* $g(x_2) = \pm x_2$ *or* $g(x_2) = \pm x_1x_2$ *or* $g(x_2) = \pm 1 \mp 2x_4$ *or* $g(x_2) = \pm x_1 \mp 2x_4$ *or* $g(x_2) = \pm x_2 \mp 2x_4$ *or* $g(x_2) = \pm x_1x_2 \mp 2x_4$;

(3) $g(x_4) = x_4$ *or* $g(x_4) = 1 - x_4$.

Proof. (1) Indeed, we have $(g(x_1))^2 = 1$ since g is an automorphism of $r(\widetilde{H_8})$ and $x_1^2 = 1$. Assume that

$$g(x_1) = \alpha_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_1x_2 + \alpha_5x_4, \quad \alpha_i \in \mathbb{Z},\ i = 0, 1, 2, 3, 4, 5.$$

Then we get

$$(\alpha_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_1x_2 + \alpha_5x_4)^2 = 1,$$

and we have

$$\begin{aligned}\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 &+ (2\alpha_0\alpha_1 + 2\alpha_2\alpha_4 + \alpha_3^2)x_1 + (2\alpha_0\alpha_2 + 2\alpha_1\alpha_4 + \alpha_3^2)x_2\\ &+ 2(\alpha_0\alpha_3 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_3\alpha_4)x_3 + (2\alpha_0\alpha_4 + 2\alpha_1\alpha_2 + \alpha_3^2)x_1x_2\\ &+ (2\alpha_0\alpha_5 + 2\alpha_1\alpha_5 + 2\alpha_2\alpha_5 + 4\alpha_3\alpha_5 + 2\alpha_4\alpha_5 + \alpha_5^2)x_4 = 1.\end{aligned}$$

Hence we get

$$\begin{cases}\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 1,\\ 2\alpha_0\alpha_1 + 2\alpha_2\alpha_4 + \alpha_3^2 = 0,\\ 2\alpha_0\alpha_2 + 2\alpha_1\alpha_4 + \alpha_3^2 = 0,\\ 2(\alpha_0\alpha_3 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_3\alpha_4) = 0,\\ 2\alpha_0\alpha_4 + 2\alpha_1\alpha_2 + \alpha_3^2 = 0,\\ 2\alpha_0\alpha_5 + 2\alpha_1\alpha_5 + 2\alpha_2\alpha_5 + 4\alpha_3\alpha_5 + 2\alpha_4\alpha_5 + \alpha_5^2 = 0.\end{cases}$$

Thanks to $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{Z}$, we obtain that $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is one of the solutions

$$(0, \pm 1, 0, 0, 0, 0), \quad (0, 0, \pm 1, 0, 0, 0), \quad (0, 0, 0, 0, \pm 1, 0),$$
$$(\pm 1, 0, 0, 0, 0, \mp 2), \quad (0, \pm 1, 0, 0, 0, \mp 2), \quad (0, 0, \pm 1, 0, 0, \mp 2), \quad (0, 0, 0, 0, \pm 1, \mp 2).$$

Therefore, $g(x_1) = \pm x_1$ or $g(x_1) = \pm x_2$ or $g(x_1) = \pm x_1 x_2$ or $g(x_1) = \pm 1 \mp 2x_4$ or $g(x_1) = \pm x_1 \mp 2x_4$ or $g(x_1) = \pm x_2 \mp 2x_4$ or $g(x_1) = \pm x_1 x_2 \mp 2x_4$. By similar arguments for $g(x_2)$ we can deduce the relation (2).

(3) Notice that $x_4^2 = x_4$, hence we have $(g(x_4))^2 = g(x_4)$.

Assume

$$g(x_4) = \gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \gamma_4 x_1 x_2 + \gamma_5 x_4, \qquad \gamma_i \in \mathbb{Z},\ i = 0, 1, 2, 3, 4, 5.$$

Then we have

$$\begin{aligned}(g(x_4))^2 = {} & \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2 + (2\gamma_0\gamma_1 + 2\gamma_2\gamma_4 + \gamma_3^2)x_1 + (2\gamma_0\gamma_2 + 2\gamma_1\gamma_4 + \gamma_3^2)x_2 \\ & + 2(\gamma_0\gamma_3 + \gamma_1\gamma_3 + \gamma_2\gamma_3 + \gamma_3\gamma_4)x_3 + (2\gamma_0\gamma_4 + 2\gamma_1\gamma_2 + \gamma_3^2)x_1 x_2 \\ & + (2\gamma_0\gamma_5 + 2\gamma_1\gamma_5 + 2\gamma_2\gamma_5 + 4\gamma_3\gamma_5 + 2\gamma_4\gamma_5 + \gamma_5^2)x_4.\end{aligned}$$

We get

$$\begin{cases} \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2 = \gamma_0, \\ 2\gamma_0\gamma_1 + 2\gamma_2\gamma_4 + \gamma_3^2 = \gamma_1, \\ 2\gamma_0\gamma_2 + 2\gamma_1\gamma_4 + \gamma_3^2 = \gamma_2, \\ 2(\gamma_0\gamma_3 + \gamma_1\gamma_3 + \gamma_2\gamma_3 + \gamma_3\gamma_4) = \gamma_3, \\ 2\gamma_0\gamma_4 + 2\gamma_1\gamma_2 + \gamma_3^2 = \gamma_4, \\ 2\gamma_0\gamma_5 + 2\gamma_1\gamma_5 + 2\gamma_2\gamma_5 + 4\gamma_3\gamma_5 + 2\gamma_4\gamma_5 + \gamma_5^2 = \gamma_5. \end{cases}$$

Thanks to $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \mathbb{Z}$, we obtain that

$$(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0, 0, 0, 0, 0, 1) \quad \text{or} \quad (1, 0, 0, 0, 0, -1).$$

Therefore $g(x_4) = x_4$ or $g(x_4) = 1 - x_4$. $\square$

Lemma 4.2. *Let g be an automorphism of $r(\widetilde{H_8})$ and $g(x_4) = x_4$. Then*

(1) *$g(x_1) = x_1$ or $g(x_1) = x_2$ or $g(x_1) = x_1 x_2$ or $g(x_1) = -1 + 2x_4$ or $g(x_1) = -x_1 + 2x_4$ or $g(x_1) = -x_2 + 2x_4$ or $g(x_1) = -x_1 x_2 + 2x_4$;*

(2) *$g(x_2) = x_1$ or $g(x_2) = x_2$ or $g(x_2) = x_1 x_2$ or $g(x_2) = -1 + 2x_4$ or $g(x_2) = -x_1 + 2x_4$ or $g(x_2) = -x_2 + 2x_4$ or $g(x_2) = -x_1 x_2 + 2x_4$.*

P r o o f. (1) Noticing that $x_1x_3 = x_3$ and $x_3x_4 = 2x_4$, we have $x_1x_4 = x_4$ and

$$g(x_1)g(x_4) = g(x_4).$$

Under the condition that $g(x_4) = x_4$ and by Lemma 4.1, we obtain that $g(x_1)$ can only belong to one of the following 7 cases:

$$g(x_1) = x_1, \quad g(x_1) = x_2, \quad g(x_1) = x_1x_2, \quad g(x_1) = -1 + 2x_4,$$
$$g(x_1) = -x_1 + 2x_4, \quad g(x_1) = -x_2 + 2x_4, \quad g(x_1) = -x_1x_2 + 2x_4.$$

(2) Similar to the proof of (1). □

Remark 4.3. If we assume that g is an automorphism of $r(\widetilde{H_8})$ and $g(x_4) = x_4$ (see Lemma 4.2), we can exclude the following cases:
(1) $g(x_1) = x_1,\ g(x_2) = x_1$;
(2) $g(x_1) = x_2,\ g(x_2) = x_2$;
(3) $g(x_1) = x_1x_2,\ g(x_2) = x_1x_2$;
(4) $g(x_1) = -1 + 2x_4,\ g(x_2) = -1 + 2x_4$;
(5) $g(x_1) = -x_1 + 2x_4,\ g(x_2) = -x_1 + 2x_4$;
(6) $g(x_1) = -x_2 + 2x_4,\ g(x_2) = -x_2 + 2x_4$;
(7) $g(x_1) = -x_1x_2 + 2x_4,\ g(x_2) = -x_1x_2 + 2x_4$, since $x_1 \neq x_2$.

Lemma 4.4. *Let g be an automorphism of $r(\widetilde{H_8})$ and $g(x_4) = x_4$. Then $g \in \{g_i\colon\ i \in \mathbb{Z}_{12}\}$.*

P r o o f. Since g is an automorphism of $r(\widetilde{H_8})$ and

$$\begin{cases} x_1x_3 = x_3, \\ x_2x_3 = x_3, \\ x_3x_4 = 2x_4, \\ x_3^2 = 1 + x_1 + x_2 + x_1x_2 \end{cases}$$

we have

$$\begin{cases} g(x_1x_3) = g(x_1)g(x_3) = g(x_3), \\ g(x_2x_3) = g(x_2)g(x_3) = g(x_3), \\ g(x_3x_4) = g(x_3)g(x_4) = 2g(x_4), \\ g(x_3^2) = (g(x_3))^2 = 1 + g(x_1) + g(x_2) + g(x_1)g(x_2). \end{cases} \tag{4.1}$$

Assume

$$g(x_3) = \beta_0 + \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_1x_2 + \beta_5x_4, \quad \beta_j \in \mathbb{Z},\ j = 0,1,2,3,4,5.$$

Then we have

$$(g(x_3))^2 = \beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2+(2\beta_0\beta_1+2\beta_2\beta_4+\beta_3^2)x_1+(2\beta_0\beta_2+2\beta_1\beta_4+\beta_3^2)x_2 \\ +2(\beta_0\beta_3+\beta_1\beta_3+\beta_2\beta_3+\beta_3\beta_4)x_3+(2\beta_0\beta_4+2\beta_1\beta_2+\beta_3^2)x_1x_2+(2\beta_0\beta_5 \\ +2\beta_1\beta_5+2\beta_2\beta_5+4\beta_3\beta_5+2\beta_4\beta_5+\beta_5^2)x_4.$$

If $g(x_1)=x_1$, $g(x_2)=x_2$, then

$$\begin{cases} \beta_0x_1+\beta_1+\beta_2x_1x_2+\beta_3x_3+\beta_4x_2+\beta_5x_4 = \beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3 \\ \qquad\qquad +\beta_4x_1x_2+\beta_5x_4, \\ \beta_0x_2+\beta_1x_1x_2+\beta_2+\beta_3x_3+\beta_4x_1+\beta_5x_4 = \beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3 \\ \qquad\qquad +\beta_4x_1x_2+\beta_5x_4, \\ (\beta_0+\beta_1+\beta_2+2\beta_3+\beta_4+\beta_5)x_4 = 2x_4, \\ \beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2+(2\beta_0\beta_1+2\beta_2\beta_4+\beta_3^2)x_1+(2\beta_0\beta_2+2\beta_1\beta_4+\beta_3^2)x_2 \\ \quad +2(\beta_0\beta_3+\beta_1\beta_3+\beta_2\beta_3+\beta_3\beta_4)x_3+(2\beta_0\beta_4+2\beta_1\beta_2+\beta_3^2)x_1x_2 \\ \quad +(2\beta_0\beta_5+2\beta_1\beta_5+2\beta_2\beta_5+4\beta_3\beta_5+2\beta_4\beta_5+\beta_5^2)x_4 = 1+x_1+x_2+x_1x_2. \end{cases}$$

We get

$$\begin{cases} \beta_0=\beta_1=\beta_2=\beta_4, \\ \beta_0+\beta_1+\beta_2+2\beta_3+\beta_4+\beta_5=2, \\ \beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2=1, \\ 2\beta_0\beta_1+2\beta_2\beta_4+\beta_3^2=1, \\ 2\beta_0\beta_2+2\beta_1\beta_4+\beta_3^2=1, \\ 2(\beta_0\beta_3+\beta_1\beta_3+\beta_2\beta_3+\beta_3\beta_4)=0, \\ 2\beta_0\beta_4+2\beta_1\beta_2+\beta_3^2=1, \\ 2\beta_0\beta_5+2\beta_1\beta_5+2\beta_2\beta_5+4\beta_3\beta_5+2\beta_4\beta_5+\beta_5^2=0. \end{cases}$$

Thanks to $\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\in\mathbb{Z}$, we obtain that

$$(\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5)=(0,0,0,1,0,0) \quad \text{or} \quad (0,0,0,-1,0,4).$$

If $(\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5)=(0,0,0,1,0,0)$, then $g(1)=1$, $g(x_1)=x_1$, $g(x_2)=x_2$, $g(x_1x_2)=x_1x_2$, $g(x_3)=x_3$, $g(x_4)=x_4$, and

$$\mathbf{A}_g = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} = \mathbf{A}_g^{-1};$$

it follows that $g=g_0$.

If $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = (0, 0, 0, -1, 0, 4)$, then $g(1) = 1$, $g(x_1) = x_1$, $g(x_2) = x_2$, $g(x_1x_2) = x_1x_2$, $g(x_3) = -x_3 + 4x_4$, $g(x_4) = x_4$, and

$$\mathbf{A}_g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 \end{pmatrix} = \mathbf{A}_g^{-1};$$

it follows that $g = g_7$.

Similar arguments are applied to the remaining possibilities one by one. We get that there are only 10 possibilities such that g are automorphisms:

(1) if $g(x_1) = x_1x_2,\ g(x_2) = x_1$, then $g = g_1$ or g_2;
(2) if $g(x_1) = x_1x_2,\ g(x_2) = x_2$, then $g = g_3$ or $g = g_4$;
(3) if $g(x_1) = x_1,\ g(x_2) = x_1x_2$, then $g = g_5$ or $g = g_6$;
(4) if $g(x_1) = x_2,\ g(x_2) = x_1x_2$, then $g = g_8$ or $g = g_9$;
(5) if $g(x_1) = x_2,\ g(x_2) = x_1$, then $g = g_{10}$ or $g = g_{11}$.

Moreover, if $g(x_1) = -1 + 2x_4$ or $g(x_2) = -1 + 2x_4$ or $g(x_1x_2) = -1 + 2x_4$, then we obtain that $g(x_3) = 2x_4$ with $|\mathbf{A}_g| = 0$. It follows that g is not an automorphism of $r(\widetilde{H_8})$.

Finally, the 18 possible cases left are

(1) $g(x_1) = x_1,\ g(x_2) = -x_2 + 2x_4$;
(2) $g(x_1) = x_1,\ g(x_2) = -x_1x_2 + 2x_4$;
(3) $g(x_1) = x_2,\ g(x_2) = -x_1 + 2x_4$;
(4) $g(x_1) = x_2,\ g(x_2) = -x_1x_2 + 2x_4$;
(5) $g(x_1) = x_1x_2,\ g(x_2) = -x_1 + 2x_4$;
(6) $g(x_1) = x_1x_2,\ g(x_2) = -x_2 + 2x_4$;
(7) $g(x_1) = -x_1 + 2x_4,\ g(x_2) = x_2$;
(8) $g(x_1) = -x_1 + 2x_4,\ g(x_2) = x_1x_2$;
(9) $g(x_1) = -x_1 + 2x_4,\ g(x_2) = -x_2 + 2x_4$;
(10) $g(x_1) = -x_1 + 2x_4,\ g(x_2) = -x_1x_2 + 2x_4$;
(11) $g(x_1) = -x_2 + 2x_4,\ g(x_2) = x_1$;
(12) $g(x_1) = -x_2 + 2x_4,\ g(x_2) = x_1x_2$;
(13) $g(x_1) = -x_2 + 2x_4,\ g(x_2) = -x_1 + 2x_4$;
(14) $g(x_1) = -x_2 + 2x_4,\ g(x_2) = -x_1x_2 + 2x_4$;
(15) $g(x_1) = -x_1x_2 + 2x_4,\ g(x_2) = x_1$;
(16) $g(x_1) = -x_1x_2 + 2x_4,\ g(x_2) = x_2$;
(17) $g(x_1) = -x_1x_2 + 2x_4,\ g(x_2) = -x_1 + 2x_4$;

(18) $g(x_1) = -x_1x_2 + 2x_4,\ g(x_2) = -x_2 + 2x_4$.
It is easy to deduce that $g(x_3)$ has no reasonable solutions. Hence in these cases, g are not automorphisms of $r(\widetilde{H_8})$.

Consequently, $g \in \{g_i\colon\ i \in \mathbb{Z}_{12}\}$. □

Theorem 4.5. *Let* $\mathrm{Aut}(r(\widetilde{H_8}))$ *denote the automorphism group of* $r(\widetilde{H_8})$. *Then*

$$\mathrm{Aut}(r(\widetilde{H_8})) = \{g_i\colon\ i \in \mathbb{Z}_{12}\} \cong D_6,$$

where D_6 *is the dihedral group with order* 12.

Proof. If g is an automorphism of $r(\widetilde{H_8})$ then $g(x_4) = x_4$ or $g(x_4) = 1 - x_4$ by Lemma 4.1. Let g be an automorphism of $r(\widetilde{H_8})$ and $g(x_4) = 1 - x_4$. By $x_1x_4 = x_2x_4 = x_4$, we have

$$g(x_1)g(x_4) = g(x_4) \quad \text{and} \quad g(x_2)g(x_4) = g(x_4).$$

It follows that $g(x_1) = 1 - 2x_4$ and $g(x_2) = 1 - 2x_4$ by Lemma 4.1. Thus, $g(x_1) = g(x_2)$, which is impossible. Therefore, we have $g(x_4) = x_4$ and $g \in \{g_i\colon\ i \in \mathbb{Z}_{12}\}$ by Lemma 4.4. It follows that

$$\mathrm{Aut}(r(\widetilde{H_8})) = \{g_i\colon\ i \in \mathbb{Z}_{12}\} \cong D_6.$$

The proof is completed. □

Remark 4.6. Arguments similar to the proof of Theorem 4.5 show that the automorphism group of $r(H_8)$ is also isomorphic to D_6.

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Authors' address: Dong Su, Shilin Yang (corresponding author), College of Applied Sciences, Beijing University of Technology, 100 Ping Le Yuan, Beijing 100124, Chaoyang, P. R. China, e-mail: `ynsudong@emails.bujt.edu.cn`, `slyang@bjut.edu.cn`.