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SOME BEREZIN NUMBER INEQUALITIES  
FOR OPERATOR MATRICES

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*Abstract.* The Berezin symbol  $\tilde{A}$  of an operator  $A$  acting on the reproducing kernel Hilbert space  $\mathcal{H} = \mathcal{H}(\Omega)$  over some (nonempty) set is defined by  $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ ,  $\lambda \in \Omega$ , where  $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|$  is the normalized reproducing kernel of  $\mathcal{H}$ . The Berezin number of the operator  $A$  is defined by  $\mathbf{ber}(A) = \sup_{\lambda \in \Omega} |\tilde{A}(\lambda)| = \sup_{\lambda \in \Omega} |\langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle|$ . Moreover,  $\mathbf{ber}(A) \leq w(A)$  (numerical radius). We present some Berezin number inequalities. Among other inequalities, it is shown that if  $\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ , then

$$\mathbf{ber}(\mathbf{T}) \leq \frac{1}{2}(\mathbf{ber}(A) + \mathbf{ber}(D)) + \frac{1}{2}\sqrt{(\mathbf{ber}(A) - \mathbf{ber}(D))^2 + (\|B\| + \|C\|)^2}.$$

*Keywords:* reproducing kernel; Berezin number; numerical radius; operator matrix

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## 1. INTRODUCTION

Let  $\mathcal{H}$  be a complex Hilbert space and let  $\mathbb{B}(\mathcal{H})$  denote the  $C^*$ -algebra of all bounded linear operators on  $\mathcal{H}$  with the identity  $I$ . In the case when  $\dim \mathcal{H} = n$ , we identify  $\mathbb{B}(\mathcal{H})$  with the matrix algebra  $\mathbb{M}_n$  of all  $n \times n$  matrices having entries in the complex field. An operator  $A \in \mathbb{B}(\mathcal{H})$  is called positive if  $\langle Ax, x \rangle \geq 0$  for all  $x \in \mathcal{H}$ , and then we write  $A \geq 0$ . Let  $r(\cdot)$  denote the spectral radius. The numerical range and numerical radius of  $A \in \mathbb{B}(\mathcal{H})$  are defined by

$$W(A) := \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\} \quad \text{and} \quad w(A) := \sup\{|\lambda| : \lambda \in W(A)\},$$

respectively. It is well known that  $w(\cdot)$  defines a norm on  $\mathbb{B}(\mathcal{H})$ , which is equivalent to the usual operator norm  $\|\cdot\|$ . In fact, for any  $A \in \mathbb{B}(\mathcal{H})$ , we have  $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$

(see [4], page 9). For further information about numerical radius we refer the reader to [1], [5], [15].

A functional Hilbert space is a Hilbert space  $\mathcal{H} = \mathcal{H}(\Omega)$  of complex-valued functions on a (nonempty) set  $\Omega$ , which has the property that point evaluations are continuous, i.e., for each  $\lambda \in \Omega$  the map  $f \rightarrow f(\lambda)$  is a continuous linear functional on  $\mathcal{H}$ . Then the Riesz representation theorem ensures that for each  $\lambda \in \Omega$  there is a unique element  $k_\lambda$  of  $\mathcal{H}$  such that  $f(\lambda) = \langle f, k_\lambda \rangle$  for all  $f \in \mathcal{H}$ . The collection  $\{k_\lambda : \lambda \in \Omega\}$  is called the reproducing kernel of  $\mathcal{H}$ . If  $\{e_n\}$  is an orthonormal basis for a functional Hilbert space  $\mathcal{H}$ , then the reproducing kernel of  $\mathcal{H}$  is given by [6], Problem 37:

$$k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z).$$

For  $\lambda \in \Omega$ , let  $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|$  be the normalized reproducing kernel of  $\mathcal{H}$ . For a bounded linear operator  $A$  on  $\mathcal{H}$ , the function  $\tilde{A}$  defined on  $\Omega$  by  $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$  is the Berezin symbol of  $A$ , which was first introduced by Berezin [2], [3]. Berezin set and Berezin number of the operator  $A$  are defined by (see [11])

$$\mathbf{Ber}(A) := \{\tilde{A}(\lambda) : \lambda \in \Omega\} \quad \text{and} \quad \mathbf{ber}(A) := \sup\{|\tilde{A}(\lambda)| : \lambda \in \Omega\},$$

respectively. It is clear that the Berezin symbol  $\tilde{A}$  is a bounded function on  $\Omega$  whose values lie in the numerical range of the operator  $A$  and hence

$$\mathbf{Ber}(A) \subseteq W(A) \quad \text{and} \quad \mathbf{ber}(A) \leq w(A)$$

for all  $A \in \mathbb{B}(\mathcal{H})$ . Karaev in [12] showed that if  $\mathcal{H}^2$  is the Hardy space, then we take  $A = \langle \cdot, z \rangle z$  in  $\mathcal{H}^2$ ; an elementary calculation shows that  $\tilde{A}(\lambda) = |\lambda|^2(1 - |\lambda|^2)$ , and thus

$$\mathbf{Ber}(A) = \left[0, \frac{1}{4}\right] \subsetneq [0, 1] = W(A) \quad \text{and} \quad \mathbf{ber}(A) = \frac{1}{4} \leq 1 = w(A).$$

Moreover, Berezin number of an operator  $A$  satisfies the following properties:

- (a)  $\mathbf{ber}(\alpha A) = |\alpha| \mathbf{ber}(A)$  for all  $\alpha \in \mathbb{C}$ .
- (b)  $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$ .

Let  $T_i \in \mathbb{B}(\mathcal{H}(\Omega))$ ,  $1 \leq i \leq n$ . Then we define the generalized Euclidean Berezin number of  $T_1, \dots, T_n$  as

$$\mathbf{ber}_p(T_1, \dots, T_n) := \sup_{\lambda \in \Omega} \left( \sum_{i=1}^n |\langle T_i \hat{k}_\lambda, \hat{k}_\lambda \rangle|^p \right)^{1/p}.$$

The generalized Euclidean Berezin number  $\mathbf{ber}_p$ ,  $p \geq 1$ , has the following properties:

- (a)  $\mathbf{ber}_p(\alpha T_1, \dots, \alpha T_n) = |\alpha| \mathbf{ber}_p(T_1, \dots, T_n)$  for all  $\alpha \in \mathcal{C}$ ;
  - (b)  $\mathbf{ber}_p(T_1 + S_1, \dots, T_n + S_n) \leq \mathbf{ber}_p(T_1, \dots, T_n) + \mathbf{ber}_p(S_1, \dots, S_n)$ ,
- where  $T_i, S_i \in \mathbb{B}(\mathcal{H}(\Omega))$ ,  $1 \leq i \leq n$ .

The Berezin symbol has been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it has wide applications in various questions of analysis and uniquely determines the operator (i.e.,  $\tilde{A}(\lambda) = \tilde{B}(\lambda)$  for all  $\lambda \in \Omega$  implies  $A = B$ ). For further information about Berezin symbol we refer the reader to [9], [10], [12], [14], [16] and references therein.

Let  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$  be Hilbert spaces, and consider the direct sum  $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j$ . With respect to this decomposition, every operator  $T \in \mathbb{B}(\mathcal{H})$  has an  $n \times n$  operator matrix representation  $T = [T_{ij}]$  with entries  $T_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ , the space of all bounded linear operators from  $\mathcal{H}_j$  to  $\mathcal{H}_i$ . Let  $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_1)$ ,  $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ ,  $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$  and  $D \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_2)$ . The operator matrix  $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$  is called the diagonal part of  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  and  $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$  is the off-diagonal part. Operator matrices provide a usual tool for studying Hilbert space operators, which have been extensively studied in the literature. Hou et al. in [8] and Omer et al. in [1] established useful estimates for the spectral radius, the numerical radius, and the operator norm of an  $n \times n$  operator matrix  $\mathbf{T} = [T_{ij}]$ . In particular, they proved that

$$r(\mathbf{T}) \leq r(\| \|T_{ij}\| \|), \quad w(\mathbf{T}) \leq w(\| \|T_{ij}\| \|), \quad \|\mathbf{T}\| \leq \| \| \|T_{ij}\| \| \|$$

and  $w(\mathbf{T}) \leq w([t_{ij}])$ , where

$$t_{ij} = \begin{cases} w(T_{ij}) & \text{if } i = j, \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$$

The Berezin number is named in honor of Felix Berezin, who introduced this concept in [2]. In this paper, we establish some inequalities involving the Berezin number of operators. By using the ideas of [1], [15] we give several upper bounds for the Berezin number and the generalized Euclidean Berezin number of Hilbert space operators.

## 2. MAIN RESULTS

Now we are in a position to present our first result.

**Theorem 2.1.** *Let  $\mathbf{T} = [T_{ij}]$  be an  $n \times n$  operator matrix with  $T_{ij} \in \mathbb{B}(\mathcal{H}(\Omega_j), \mathcal{H}(\Omega_i))$ ,  $1 \leq i, j \leq n$ . Then*

$$\mathbf{ber}(\mathbf{T}) \leq w([t_{ij}]),$$

$$\text{where } t_{ij} = \begin{cases} \mathbf{ber}(T_{ij}) & \text{if } i = j, \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$$

*Proof.* Let  $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}(\Omega_i)$ . For every  $(\lambda_1, \dots, \lambda_n) \in \Omega_1 \times \dots \times \Omega_n$ , let  $\hat{\mathbf{k}}_{(\lambda_1, \dots, \lambda_n)} = \begin{bmatrix} k_{\lambda_1} \\ \vdots \\ k_{\lambda_n} \end{bmatrix}$  be the normalized reproducing kernel of  $\mathcal{H}$ . Then

$$\begin{aligned} |\tilde{\mathbf{T}}(\lambda_1, \dots, \lambda_n)| &= |\langle \mathbf{T} \hat{\mathbf{k}}_{(\lambda_1, \dots, \lambda_n)}, \hat{\mathbf{k}}_{(\lambda_1, \dots, \lambda_n)} \rangle| = \left| \sum_{i,j=1}^n \langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \rangle \right| \\ &\leq \sum_{i,j=1}^n |\langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \rangle| = \sum_{i=1}^n |\langle T_{ii} k_{\lambda_i}, k_{\lambda_i} \rangle| + \sum_{\substack{i,j=1 \\ i \neq j}}^n |\langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \rangle| \\ &\leq \sum_{i=1}^n \mathbf{ber}(T_{ii}) \|k_{\lambda_i}\|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^n \|T_{ij}\| \|k_{\lambda_j}\| \|k_{\lambda_i}\| \\ &= \sum_{i,j=1}^n t_{ij} \|k_{\lambda_j}\| \|k_{\lambda_i}\| = \langle [t_{ij}] y, y \rangle, \end{aligned}$$

where  $y = \begin{bmatrix} \|k_{\lambda_1}\| \\ \vdots \\ \|k_{\lambda_n}\| \end{bmatrix}$ . It follows from  $\|y\| = 1$  that  $|\tilde{\mathbf{T}}(\lambda_1, \dots, \lambda_n)| \leq w([t_{ij}])$ . Hence

$$\mathbf{ber}(\mathbf{T}) = \sup_{(\lambda_1, \dots, \lambda_n) \in \Omega_1 \times \dots \times \Omega_n} |\tilde{\mathbf{T}}(\lambda_1, \dots, \lambda_n)| \leq w([t_{ij}])$$

as required. □

**Corollary 2.2.** *If  $\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ , then*

$$\mathbf{ber}(\mathbf{T}) \leq \frac{1}{2}(\mathbf{ber}(A) + \mathbf{ber}(D)) + \frac{1}{2} \sqrt{(\mathbf{ber}(A) - \mathbf{ber}(D))^2 + (\|B\| + \|C\|)^2}.$$

In particular, for  $\mathbf{T} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$  we have

$$(2.1) \quad \mathbf{ber}(\mathbf{T}) \leq \max \{ \mathbf{ber}(A), \mathbf{ber}(D) \}.$$

*Proof.* Using Theorem 2.1 we get the inequality

$$\begin{aligned} \mathbf{ber} \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leq w \left( \begin{bmatrix} \mathbf{ber} A & \|B\| \\ \|C\| & \mathbf{ber} D \end{bmatrix} \right) \\ &\leq r \left( \begin{bmatrix} \mathbf{ber} A & \frac{1}{2}(\|B\| + \|C\|) \\ \frac{1}{2}(\|B\| + \|C\|) & \mathbf{ber} D \end{bmatrix} \right) \quad (\text{by [7], page 44}) \\ &= \frac{1}{2}(\mathbf{ber}(A) + \mathbf{ber}(D)) + \frac{1}{2} \sqrt{(\mathbf{ber}(A) - \mathbf{ber}(D))^2 + (\|B\| + \|C\|)^2}. \end{aligned}$$

In particular, if  $\mathbf{T} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ , then

$$\begin{aligned} \mathbf{ber}(\mathbf{T}) &\leq \frac{1}{2}(\mathbf{ber}(A) + \mathbf{ber}(D)) + \frac{1}{2} \sqrt{(\mathbf{ber}(A) - \mathbf{ber}(D))^2} \\ &= \frac{1}{2}(\mathbf{ber}(A) + \mathbf{ber}(D)) + \frac{1}{2} |\mathbf{ber}(A) - \mathbf{ber}(D)| \\ &= \max \{ \mathbf{ber}(A), \mathbf{ber}(D) \}. \end{aligned}$$

□

We need the following lemmas for our results. The next lemma follows from the spectral theorem for positive operators and the Jensen inequality; see [13].

**Lemma 2.3** (The McCarty inequality). *Let  $T \in \mathbb{B}(\mathcal{H})$ ,  $T \geq 0$  and  $x \in \mathcal{H}$  such that  $\|x\| \leq 1$ . Then*

$$\langle Tx, x \rangle^r \leq \langle T^r x, x \rangle$$

for  $r \geq 1$ .

*Proof.* Let  $r \geq 1$  and  $x \in \mathcal{H}$  such that  $\|x\| \leq 1$ . Fix  $u = x/\|x\|$ . Using the McCarty inequality we have  $\langle Tu, u \rangle^r \leq \langle T^r u, u \rangle$ , whence

$$\begin{aligned} \langle Tx, x \rangle^r &\leq \|x\|^{2r-2} \langle T^r x, x \rangle \\ &\leq \langle T^r x, x \rangle \quad \text{since } \|x\| \leq 1 \text{ and } 2r - 2 \geq 0. \end{aligned}$$

Hence, we get the desired result. □

**Lemma 2.4** ([13], Theorem 1). *Let  $T \in \mathbb{B}(\mathcal{H})$  and let  $x, y \in \mathcal{H}$  be any vectors. If  $f, g$  are nonnegative continuous functions on  $[0, \infty)$  which satisfy the relation  $f(t)g(t) = t, t \in [0, \infty)$ , then*

$$|\langle Tx, y \rangle|^2 \leq \langle f^2(|T|)x, x \rangle \langle g^2(|T^*|)y, y \rangle,$$

in which  $|T| = (T^*T)^{1/2}$ .

**Theorem 2.5.** *Let  $\mathbf{T} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ ,  $r \geq 1$ , and let  $f, g$  be nonnegative continuous functions on  $[0, \infty)$  satisfying the relation  $f(t)g(t) = t, t \in [0, \infty)$ . Then*

$$\mathbf{ber}^r(\mathbf{T}) \leq 2^{r-2} \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|)) \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|)).$$

*Proof.* For every  $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$ , let  $\hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$  be the normalized reproducing kernel of  $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$  (i.e.,  $\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2 = 1$ ). Then

$$\begin{aligned} |\tilde{\mathbf{T}}(\lambda_1, \lambda_2)|^r &= |\langle \mathbf{T} \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} \rangle|^r \\ &= |\langle X k_{\lambda_2}, k_{\lambda_1} \rangle + \langle Y k_{\lambda_1}, k_{\lambda_2} \rangle|^r \\ &\leq (|\langle X k_{\lambda_2}, k_{\lambda_1} \rangle| + |\langle Y k_{\lambda_1}, k_{\lambda_2} \rangle|)^r \quad (\text{by the triangular inequality}) \\ &\leq \frac{2^r}{2} (|\langle X k_{\lambda_2}, k_{\lambda_1} \rangle|^r + |\langle Y k_{\lambda_1}, k_{\lambda_2} \rangle|^r) \quad (\text{by the convexity } f(t) = t^r) \\ &\leq \frac{2^r}{2} ((\langle f^2(|X|)k_{\lambda_2}, k_{\lambda_2} \rangle^{1/2} \langle g^2(|X^*|)k_{\lambda_1}, k_{\lambda_1} \rangle^{1/2})^r \\ &\quad + (\langle f^2(|Y|)k_{\lambda_1}, k_{\lambda_1} \rangle^{1/2} \langle g^2(|Y^*|)k_{\lambda_2}, k_{\lambda_2} \rangle^{1/2})^r) \quad (\text{by Lemma 2.4}) \\ &\leq \frac{2^r}{2} (\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2} \rangle^{1/2} \langle g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1} \rangle^{1/2} \\ &\quad + \langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1} \rangle^{1/2} \langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2} \rangle^{1/2}) \quad (\text{by Lemma 2.3}) \\ &\leq \frac{2^r}{2} (\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2} \rangle + \langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2} \rangle)^{1/2} (\langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1} \rangle \\ &\quad + \langle g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1} \rangle)^{1/2} \quad (\text{by the Cauchy-Schwarz inequality}) \\ &= \frac{2^r}{2} (\langle f^{2r}(|X|) + g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2} \rangle^{1/2} \langle f^{2r}(|Y|) + g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1} \rangle^{1/2} \\ &\leq \frac{2^r}{2} \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|)) \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|)) \|k_{\lambda_1}\| \|k_{\lambda_2}\| \\ &\leq \frac{2^r}{2} \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|)) \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|)) \\ &\quad \times \frac{\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2}{2} \quad (\text{by the arithmetic-geometric mean inequality}) \\ &= \frac{2^r}{4} \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|)) \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|)). \end{aligned}$$

Now, taking the supremum over all  $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$  we get the desired result.  $\square$

Theorem 2.5 includes a special case as follows.

**Corollary 2.6.** *Let  $\mathbf{T} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ ,  $0 \leq p \leq 1$  and  $r \geq 1$ . Then*

$$\mathbf{ber}^r(\mathbf{T}) \leq 2^{r-2} \mathbf{ber}^{1/2}(|X|^{2rp} + |Y^*|^{2r(1-p)}) \mathbf{ber}^{1/2}(|Y|^{2rp} + |X^*|^{2r(1-p)}).$$

*Proof.* The result follows immediately from Theorem 2.5 for  $f(t) = t^p$  and  $g(t) = t^{1-p}$ ,  $0 \leq p \leq 1$ .  $\square$

**Theorem 2.7.** *Let  $A, B, X \in \mathbb{B}(\mathcal{H}(\Omega))$ ,  $r \geq 1$  and let  $f, g$  be nonnegative continuous functions on  $[0, \infty)$  satisfying the relation  $f(t)g(t) = t$ ,  $t \in [0, \infty)$ . Then*

$$\mathbf{ber}^r(A^*XB) \leq \mathbf{ber} \left( \frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2} \right),$$

where  $1/p + 1/q = 1$  and  $pr \geq qr \geq 2$ .

*Proof.* For every  $\lambda \in \Omega$ , let  $\hat{\mathbf{k}}_\lambda$  be the normalized reproducing kernel of  $\mathcal{H}(\Omega)$ . Then

$$\begin{aligned} |\langle A^*XB\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle|^r &= |\langle X B\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda \rangle|^r \\ &\leq (\langle f^2(|X|)B\hat{\mathbf{k}}_\lambda, B\hat{\mathbf{k}}_\lambda \rangle \langle g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda \rangle)^{r/2} \\ &\quad \text{(by Lemma 2.4)} \\ &\leq \frac{1}{p} \langle f^2(|X|)B\hat{\mathbf{k}}_\lambda, B\hat{\mathbf{k}}_\lambda \rangle^{rp/2} + \frac{1}{q} \langle g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda \rangle^{rq/2} \\ &\quad \text{(by Young's inequality)} \\ &= \frac{1}{p} \langle B^*f^2(|X|)B\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle^{rp/2} + \frac{1}{q} \langle A^*g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle^{rq/2} \\ &\leq \frac{1}{p} \langle [B^*f^2(|X|)B]^{rp/2} \hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle + \frac{1}{q} \langle [A^*g^2(|X^*|)A]^{rq/2} \hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle \\ &\quad \text{(by Lemma 2.3)} \\ &= \left\langle \left( \frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2} \right) \hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \right\rangle \\ &\leq \mathbf{ber} \left( \frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2} \right), \end{aligned}$$

whence

$$\begin{aligned} \mathbf{ber}^r(A^*XB) &= \sup_{\lambda \in \Omega} |\langle A^*XB\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle|^r \\ &\leq \mathbf{ber} \left( \frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2} \right). \end{aligned}$$

$\square$



**Remark 2.8.** In Theorem 2.7, if we add the hypothesis of contraction for operators  $A$  and  $B$  (i.e.,  $A^*A \leq I$  and  $B^*B \leq I$ ) then by using Lemma 2.3 and in a fashion similar to the proof of Theorem 2.7 we get the inequality

$$(2.2) \quad \mathbf{ber}^r(A^*XB) \leq \mathbf{ber} \left( \frac{1}{p} B^* f^{rp}(|X|)B + \frac{1}{q} A^* g^{rq}(|X^*|)A \right),$$

where  $r \geq 1$ ,  $1/p + 1/q = 1$  are such that  $pr \geq qr \geq 2$  and  $f, g$  are nonnegative continuous functions on  $[0, \infty)$  satisfying the relation  $f(t)g(t) = t$ ,  $t \in [0, \infty)$ .

The next result follows from Theorem 2.7 and inequality (2.2) for  $f(t) = t^\alpha$  and  $g(t) = t^{1-\alpha}$ ,  $0 \leq \alpha \leq 1$ .

**Corollary 2.9.** Let  $A, B, X \in \mathbb{B}(\mathcal{H}(\Omega))$ , let  $r \geq 1$ ,  $1/p + 1/q = 1$  be such that  $pr \geq qr \geq 2$  and  $0 \leq \alpha \leq 1$ . Then

$$\mathbf{ber}^r(A^*XB) \leq \mathbf{ber} \left( \frac{1}{p} [B^*|X|^{2\alpha}B]^{rp/2} + \frac{1}{q} [A^*|X^*|^{2(1-\alpha)}A]^{rq/2} \right).$$

In particular, if  $A$  and  $B$  are contractions, then

$$\mathbf{ber}^r(A^*XB) \leq \mathbf{ber} \left( \frac{1}{p} B^*|X|^{rp\alpha}B + \frac{1}{q} A^*|X^*|^{rp(1-\alpha)}A \right).$$

Now, we need the following lemma for the next result.

**Lemma 2.10.** Let  $X, Y \in \mathbb{B}(\mathcal{H}(\Omega))$ . If  $\mathbf{ber} \left( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \right) \leq \mathbf{ber} \left( \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \right)$ , then  $\mathbf{ber}(X) \leq \mathbf{ber}(Y)$ .

*Proof.* For every  $\lambda \in \Omega$ , let  $\hat{\mathbf{k}}_\lambda$  be the normalized reproducing kernel of  $\mathcal{H}(\Omega)$ . Then

$$\begin{aligned} |\langle X\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle| &= \left| \left\langle \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{k}}_\lambda \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{\mathbf{k}}_\lambda \\ 0 \end{bmatrix} \right\rangle \right| \\ &\leq \mathbf{ber} \left( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \right) \quad (\text{by the definition of } \mathbf{ber}) \\ &\leq \mathbf{ber} \left( \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &\leq \mathbf{ber}(Y) \quad (\text{by inequality (2.1)}). \end{aligned}$$

Hence

$$\mathbf{ber}(X) = \sup_{\lambda \in \Omega} |\langle X\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda \rangle| \leq \mathbf{ber}(Y).$$

□

**Corollary 2.11.** Let  $A_i, B_i, X_i \in \mathbb{B}(\mathcal{H}(\Omega))$ ,  $1 \leq i \leq n$ ,  $r \geq 1$  and let  $f, g$  be nonnegative continuous functions on  $[0, \infty)$  satisfying the relation  $f(t)g(t) = t$ ,  $t \in [0, \infty)$ . Then

$$\mathbf{ber}^r \left( \sum_{i=1}^n A_i^* X_i B_i \right) \leq \mathbf{ber} \left( \frac{1}{p} \left[ \sum_{i=1}^n B_i^* f^2(|X_i|) B_i \right]^{rp/2} + \frac{1}{q} \left[ \sum_{i=1}^n A_i^* g^2(|X_i^*|) A_i \right]^{rq/2} \right),$$

where  $1/p + 1/q = 1$  and  $pr \geq qr \geq 2$ .

In particular, if  $\sum_{i=1}^n A_i^* A_i \leq I$  and  $\sum_{i=1}^n B_i^* B_i \leq I$ , then

$$\mathbf{ber}^r \left( \sum_{i=1}^n A_i^* X_i B_i \right) \leq \mathbf{ber} \left( \frac{1}{p} \sum_{i=1}^n B_i^* f^{rp}(|X_i|) B_i + \frac{1}{q} \sum_{i=1}^n A_i^* g^{rq}(|X_i^*|) A_i \right).$$

*Proof.* If we replace  $A, B$  and  $X$  in Theorem 2.7 by operator matrices

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ A_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \dots & 0 \end{bmatrix}, \quad \begin{bmatrix} B_1 & 0 & \dots & 0 \\ B_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_n \end{bmatrix},$$

respectively, then we get

$$\begin{aligned} & \mathbf{ber}^r \left( \begin{bmatrix} \sum_{i=1}^n A_i^* X_i B_i & 0 \\ 0 & 0 \end{bmatrix} \right) \\ & \leq \mathbf{ber} \left( \begin{bmatrix} \frac{1}{p} \left[ \sum_{i=1}^n B_i^* f^2(|X_i|) B_i \right]^{rp/2} + \frac{1}{q} \left[ \sum_{i=1}^n A_i^* g^2(|X_i^*|) A_i \right]^{rq/2} & 0 \\ 0 & 0 \end{bmatrix} \right). \end{aligned}$$

Now, using Lemma 2.10 we have

$$\begin{aligned} & \mathbf{ber}^r \left( \sum_{i=1}^n A_i^* X_i B_i \right) \\ & \leq \mathbf{ber} \left( \frac{1}{p} \left[ \sum_{i=1}^n B_i^* f^2(|X_i|) B_i \right]^{rp/2} + \frac{1}{q} \left[ \sum_{i=1}^n A_i^* g^2(|X_i^*|) A_i \right]^{rq/2} \right), \end{aligned}$$

the first inequality. The second inequality follows from inequality (2.2) and this completes the proof.  $\square$

In the next theorem we present an inequality involving the generalized Euclidean Berezin number for off-diagonal operator matrices.

**Theorem 2.12.** Let  $T_i = \begin{bmatrix} 0 & X_i \\ Y_i & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ ,  $1 \leq i \leq n$ . Then

$$\begin{aligned} & \mathbf{ber}_p^p(T_1, T_2, \dots, T_n) \\ & \leq 2^{p-2} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)) \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)) \end{aligned}$$

for  $p \geq 1$ .

*Proof.* For every  $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$ , let  $\hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$  be the normalized reproducing kernel of  $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$  (i.e.,  $\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2 = 1$ ). Then

$$\begin{aligned} \sum_{i=1}^n |\langle T_i \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} \rangle|^p &= \sum_{i=1}^n |\langle X_i k_{\lambda_2}, k_{\lambda_1} \rangle + \langle Y_i k_{\lambda_1}, k_{\lambda_2} \rangle|^p \\ &\leq \sum_{i=1}^n (|\langle X_i k_{\lambda_2}, k_{\lambda_1} \rangle| + |\langle Y_i k_{\lambda_1}, k_{\lambda_2} \rangle|)^p \quad (\text{by the triangular inequality}) \\ &\leq \frac{2^p}{2} \sum_{i=1}^n (|\langle X_i k_{\lambda_2}, k_{\lambda_1} \rangle|^p + |\langle Y_i k_{\lambda_1}, k_{\lambda_2} \rangle|^p) \quad (\text{by the convexity } f(t) = t^p) \\ &\leq \frac{2^p}{2} \sum_{i=1}^n \langle f^2(|X_i|) k_{\lambda_2}, k_{\lambda_2} \rangle^{p/2} \langle g^2(|X_i^*|) k_{\lambda_1}, k_{\lambda_1} \rangle^{p/2} + \langle f^2(|Y_i|) k_{\lambda_1}, k_{\lambda_1} \rangle^{p/2} \\ &\quad \times \langle g^2(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2} \rangle^{p/2} \quad (\text{by Lemma 2.4}) \\ &\leq \frac{2^p}{2} \sum_{i=1}^n \langle f^{2p}(|X_i|) k_{\lambda_2}, k_{\lambda_2} \rangle^{1/2} \langle g^{2p}(|X_i^*|) k_{\lambda_1}, k_{\lambda_1} \rangle^{1/2} + \langle f^{2p}(|Y_i|) k_{\lambda_1}, k_{\lambda_1} \rangle^{1/2} \\ &\quad \times \langle g^{2p}(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2} \rangle^{1/2} \quad (\text{by Lemma 2.3}) \\ &\leq \frac{2^p}{2} \sum_{i=1}^n (\langle f^{2p}(|X_i|) k_{\lambda_2}, k_{\lambda_2} \rangle + \langle g^{2p}(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2} \rangle)^{1/2} (\langle f^{2p}(|Y_i|) k_{\lambda_1}, k_{\lambda_1} \rangle \\ &\quad + \langle g^{2p}(|X_i^*|) k_{\lambda_1}, k_{\lambda_1} \rangle)^{1/2} \quad (\text{by the Cauchy-Schwarz inequality}) \\ &\leq \frac{2^p}{2} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)) \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)) \|k_{\lambda_1}\| \|k_{\lambda_2}\| \\ &= \frac{2^p}{2} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)) \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)) \\ &\quad \times \left( \frac{\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2}{2} \right) \\ &= \frac{2^p}{4} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)) \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)). \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{ber}_p^p(T_1, T_2, \dots, T_n) &= \sup_{(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2} \sum_{i=1}^n |\langle T_i \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} \rangle|^p \\ &\leq 2^{p-2} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)) \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)) \end{aligned}$$

as required.  $\square$

**Theorem 2.13.** Let  $T_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ ,  $1 \leq i \leq n$  and  $p \geq 1$ .

Then

$$\begin{aligned} \mathbf{ber}_p^p(T_1, \dots, T_n) &\leq 2^{-p} \sum_{i=1}^n (\mathbf{ber}(A_i) + \mathbf{ber}(D_i) + \sqrt{(\mathbf{ber}(A_i) - \mathbf{ber}(D_i))^2 + (\|B_i\| + \|C_i\|)^2})^p. \end{aligned}$$

*Proof.* For every  $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$ , let  $\hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$  be the normalized reproducing kernel of  $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$ . It follows from

$$\begin{aligned} |\langle T_i \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} \rangle| &= \left| \left\langle \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}, \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \right\rangle \right| \\ &= \left| \left\langle \begin{bmatrix} A_i k_{\lambda_1} + B_i k_{\lambda_2} \\ C_i k_{\lambda_1} + D_i k_{\lambda_2} \end{bmatrix}, \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \right\rangle \right| \\ &= |\langle A_i k_{\lambda_1}, k_{\lambda_1} \rangle + \langle B_i k_{\lambda_2}, k_{\lambda_1} \rangle + \langle C_i k_{\lambda_1}, k_{\lambda_2} \rangle + \langle D_i k_{\lambda_2}, k_{\lambda_2} \rangle| \\ &\leq |\langle A_i k_{\lambda_1}, k_{\lambda_1} \rangle| + |\langle B_i k_{\lambda_2}, k_{\lambda_1} \rangle| + |\langle C_i k_{\lambda_1}, k_{\lambda_2} \rangle| + |\langle D_i k_{\lambda_2}, k_{\lambda_2} \rangle| \end{aligned}$$

that

$$\begin{aligned} \mathbf{ber}_p^p(T_1, \dots, T_n) &= \sup_{(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2} \sum_{i=1}^n |\langle T_i \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1, \lambda_2)} \rangle|^p \\ &\leq \sup_{(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2} \sum_{i=1}^n (|\langle A_i k_{\lambda_1}, k_{\lambda_1} \rangle| + |\langle B_i k_{\lambda_2}, k_{\lambda_1} \rangle| \\ &\quad + |\langle C_i k_{\lambda_1}, k_{\lambda_2} \rangle| + |\langle D_i k_{\lambda_2}, k_{\lambda_2} \rangle|)^p \\ &\leq \sum_{i=1}^n \left( \sup_{(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2} (|\langle A_i k_{\lambda_1}, k_{\lambda_1} \rangle| + |\langle B_i k_{\lambda_2}, k_{\lambda_1} \rangle| \right. \\ &\quad \left. + |\langle C_i k_{\lambda_1}, k_{\lambda_2} \rangle| + |\langle D_i k_{\lambda_2}, k_{\lambda_2} \rangle|) \right)^p \\ &\leq \sum_{i=1}^n \left( \sup_{(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2} (\mathbf{ber}(A_i) \|k_{\lambda_1}\|^2 + \mathbf{ber}(D_i) \|k_{\lambda_2}\|^2 \right. \\ &\quad \left. + (\|B_i\| + \|C_i\|) \|k_{\lambda_1}\| \|k_{\lambda_2}\|) \right)^p \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^n \left( \sup_{\theta \in [0, 2\pi]} (\mathbf{ber}(A_i) \cos^2 \theta + \mathbf{ber}(D_i) \sin^2 \theta \right. \\
&\quad \left. + (\|B_i\| + \|C_i\|) \cos \theta \sin \theta) \right)^p \\
&= 2^{-p} \sum_{i=1}^n \left( \mathbf{ber}(A_i) + \mathbf{ber}(D_i) \right. \\
&\quad \left. + \sqrt{(\mathbf{ber}(A_i) - \mathbf{ber}(D_i))^2 + (\|B_i\| + \|C_i\|)^2} \right)^p.
\end{aligned}$$

This completes the proof.  $\square$

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