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BOUNDEDNESS AND SQUARE INTEGRABILITY OF SOLUTIONS OF CERTAIN THIRD-ORDER DIFFERENTIAL EQUATIONS

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Abstract. We establish some new sufficient conditions which guarantee the boundedness and square integrability of solutions of certain third order differential equation. Example is included to illustrate the results. By this work, we extend and improve some results in the literature.

Keywords: third-order differential equation; boundedness; square integrability

MSC 2010: 34C11, 34D05

1. Introduction

In recent years, the qualitative theory of differential equations and their applications have received intensive attention. Many works have been written by several authors on the properties of solutions of ordinary differential equations of the second, third, fourth, fifth and higher order in which the unknown functions and its derivatives are all evaluated at the same instant t, see for instance Reissig et. al. [11], a survey book and Afuwape [1], [2], Graef [4], [5], Qian [9], [10], Omeike [6]–[8], Remili [12]–[20], Sadek [21], Tunç [22]–[27], Zhu [28] and the references cited therein, to mention few.

The purpose of this paper is to obtain criteria for boundedness and square integrability of solutions of the equation

$$(x' + g(x))'' + a(t)x'' + b(t)x' + c(t)h(x) = 0. \tag{1.1}$$

In the sequel we will assume that the functions $a(t), b(t), c(t) \in C\ ([0,\infty))$ are positive. In addition, it is also supposed that $g(x)$ is non-negative and the derivatives $g'(x)$, $g''(x)$, $h'(x)$ exist and are continuous functions for all x.

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Equation (1.1) is equivalent to the following system

$$
\begin{cases} x' = y - g(x), \\ y' = z, \\ z' = -a(t)z + a(t)\theta(t) - b(t)U - c(t)h(x), \end{cases} \tag{1.2}
$$

where

$$
y(t) - g(x(t)) = U(t), \tag{1.3}
$$

and $\theta(t) = g'(x(t))x'(t)$. The continuity of the functions a, b, c, g, g' and h guarantees the existence of the solutions of (1.1) (see [3], page 15). It is assumed that the right-hand side of system (1.2) satisfies a Lipschitz condition in $x(t)$, $y(t)$, and $z(t)$. This assumption guarantees the uniqueness of solutions of (1.1) (see [3], page 15).

Equation (1.1) can be rewritten as

$$
x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x''), \tag{1.4}
$$

where

$$
\psi(x, x') = g'(x), \quad f(x, x') = g''(x)x'^2,
$$

and

$$
p(t, x, x', x'') = -a(t)x'' - b(t)x' - c(t)h(x).
$$

Equations of these types are encountered in the control of a flying apparatus in cosmic space. Particularly, the linear case of the above equation states the motion $x(t)$ of a steam supply control slide valve. In recent years, many results have been obtained on boundedness of equation (1.4); we refer the reader to the papers Qian [10], Tunç [26] and Omeike [7]. It should be noted that in [10] (Theorem 1, page 192) the author gave some sufficient conditions for boundedness of all solutions of (1.4) under conditions:

$$
\int_0^x f(u, 0)\,\mathrm{d}u > 0 \quad \text{for } x \neq 0, \tag{1}
$$

$$
\psi(x, x') \geqslant B \tag{2}
$$

with B positive number. Qian [10] obtained sufficient conditions for every solution of equation (1.4) to be bounded, he also established criteria for every solution of equation (1.4) to converge to zero. After that [26], Theorem 2.1 and [7], Theorem 1 proved the same result obtained in [10] under less restrictive conditions:

$$
\frac{f(x, 0)}{x} \geqslant \delta_0 > 0, \tag{i}
$$

$$
\int_0^x p(s, x(s), x'(s), x''(s))\,\mathrm{d}s < \infty. \tag{ii}
$$

However, in our theorem this conditions are not true since we have $f(x,0) = \psi(x,0) = 0$. This is a significant difference between our paper and the papers above.

The motivation for the present work has been inspired by the results established in the above mentioned papers.

The paper is organized as follows. In Section 2 we establish sufficient conditions under which all solutions of equation (1.1) are bounded. In Section 3 we establish conditions for the square integrability of solutions of equation (1.1). Finally, we give an example to illustrate the results.

2. Assumptions and main results

We shall state here some assumptions which will be used for the functions that appeared in equation (1.1). Suppose that there are positive constants a_0, b_0, c_0, d, a_1, b_1, c_1, δ_0, δ_1, Δ_1 and ε, such that the following conditions are satisfied:

(i) $\quad 0 < a_0 \leqslant a(t) \leqslant a_1; \quad 0 < b_0 \leqslant b(t) \leqslant b_1; \quad 0 < c_0 \leqslant c(t) \leqslant c_1;$

(ii) $\quad c(t) \leqslant b(t); \quad b'(t) \leqslant c'(t) \leqslant 0 \quad \forall\, t \geqslant 0;$

(iii) $\quad \dfrac{h(x)}{x} \geqslant \delta_0 > 0 \quad (x \neq 0), \quad$ and $\quad |h'(x)| \leqslant \delta_1 < a_0 \quad \forall\, x;$

(iv) $\quad \dfrac{1}{2}da'(t) - b_0(d - \delta_1) \leqslant -\varepsilon < 0;$

(v) $\quad \displaystyle\int_{-\infty}^{\infty} |g'(u)|\,\mathrm{d}u \leqslant \Delta_1 < \infty.$

Theorem 2.1. *Suppose that assumptions* (i) *through* (v) *hold. Then any solution* $x(t)$ *of* (1.1) *and its derivatives* $x'(t)$ *and* $x''(t)$ *satisfy*

$$|x(t)| \leqslant D_2, \quad |x'(t)| \leqslant D_2, \quad |x''(t)| \leqslant D_2 \quad \forall\, t \geqslant 0, \tag{2.1}$$

where $D_2 > 0$.

Proof. We define the function $V = V(t, x(t), y(t), z(t))$ as

$$V = dc(t)H(x) + c(t)h(x)U + \frac{b(t)}{2}U^2 + \frac{d}{2}a(t)U^2 + \frac{1}{2}z^2 + dzU, \tag{2.2}$$

so that $H(x) = \int_0^x h(u)\,\mathrm{d}u$. Let d be a positive constant that will be specified later in the proof. From (2.2) we have the following estimates

$$V \geqslant dc(t)\Big(\int_0^x h(u)\,\mathrm{d}u - \frac{c(t)}{2db(t)}h^2(x)\Big) + \frac{b(t)}{2}\Big(U + \frac{c(t)}{b(t)}h(x)\Big)^2 + \frac{d}{2}(a_0 - d)U^2 + \frac{1}{2}(z + dU)^2$$

$$\geqslant dc(t)\Big(\int_0^x h(u)\,\mathrm{d}u - \frac{c(t)}{2db(t)}h^2(x)\Big) + \frac{d}{2}(a_0 - d)U^2 + \frac{1}{2}(z + dU)^2.$$

In view of condition (iii) and after some rearrangements we have

$$\begin{aligned} dc(t)\Big(\int_0^x h(u)\,\mathrm{d}u - \frac{c(t)}{2db(t)}h^2(x)\Big) &= dc(t)\int_0^x \Big(1 - \frac{h'(u)}{d}h(u)\Big)\,\mathrm{d}u \\ &\geqslant dc_0\Big(1 - \frac{\delta_1}{d}\Big)\int_0^x h(u)\,\mathrm{d}u \geqslant \frac{dc_0\delta_0}{2}\Big(1 - \frac{\delta_1}{d}\Big)x^2. \end{aligned}$$

By the choice of

$$\delta_1 < d < a_0, \tag{2.3}$$

it follows that

$$V \geqslant \frac{dc_0\delta_0}{2}\Big(1 - \frac{\delta_1}{d}\Big)x^2 + \frac{d}{2}(a_0 - d)U^2 + \frac{1}{2}(z + dU)^2.$$

It is evident from the terms contained in the last inequality that there exists a sufficiently small positive constant k such that

$$V \geqslant k(x^2 + U^2 + z^2). \tag{2.4}$$

For the time derivative of the function V along the trajectories of system (1.2), a straightforward calculation yields

$$V'_{(1.2)} = H_1(t,x,y) + H_2(t,x,y) + H_3(t,x,y),$$

where

$$\begin{aligned} H_1(t,x,y) &= dc'(t)H(x) + c'(t)Uh(x) + \frac{b'(t)}{2}U^2, \\ H_2(t,x,y) &= \Big(\frac{d}{2}a'(t) - db(t) + c(t)h'(x)\Big)U^2 + (d - a(t))z^2, \\ H_3(t,x,y) &= \theta(t)((a(t) - d)z - c(t)h(x) - b(t)U). \end{aligned}$$

Now, we verify that $H_1(t,x,y) \leqslant 0$. To show this, two cases are considered.

If $c'(t) = 0$, then from condition (ii) we get

$$H_1(t,x,y) = \frac{b'(t)}{2}U^2 \leqslant 0.$$

If $c'(t) < 0$, we can write $H_1(t,x,y)$ as

$$\begin{aligned} H_1(t,x,y) &= dc'(t)\Big(H(x) + \frac{1}{d}Uh(x) + \frac{b'(t)}{2dc'(t)}U^2\Big) \\ &= dc'(t)\Big(H(x) + \frac{b'(t)}{2dc'(t)}\Big(y + \frac{c'(t)}{b'(t)}h(x)\Big)^2 - \frac{c'(t)}{2db'(t)}h^2(x)\Big). \end{aligned}$$

By the help of assumption (ii) it follows that

$$0 < \frac{c'(t)}{b'(t)} \leqslant 1 \quad \forall\, t \geqslant 0,$$

thus,

$$\begin{aligned} H_1(t,x,y) &\leqslant dc'(t)\Big(H(x) - \frac{1}{2d}h^2(x)\Big) \\ &\leqslant dc'(t)\int_0^x \Big(1 - \frac{\delta_1}{d}\Big)h(u)\,\mathrm{d}u \leqslant c'(t)(d-\delta_1)H(x). \end{aligned}$$

Now, from (2.3) we get

$$H_1(t,x,y) \leqslant 0.$$

Assumption (iv) with (2.3) yield

$$H_2(t,x,y) \leqslant \Big(\frac{d}{2}a'(t) - b_0(d-\delta_1)\Big)U^2 + (d-a_0)z^2 \leqslant -\varepsilon U^2 + (d-a_0)z^2 \leqslant 0.$$

Since $u \leqslant |u| \leqslant u^2 + 1$, we have

$$\begin{aligned} H_3(t,x,y) &\leqslant |\theta(t)|((d+a(t))|z| + c(t)|h(x)| + b(t)|U|) \\ &\leqslant |\theta(t)|((d+a(t))(z^2+1) + c(t)(h^2(x)+1) + b(t)(U^2+1)) \\ &\leqslant |\theta(t)|MV + |\theta(t)|(d+a(t)+b(t)+c(t)) \leqslant |\theta(t)|(MV+\alpha), \end{aligned}$$

where

$$M = \frac{1}{k}(d + a_1 + c_1\delta_1^2 + b_1) \quad \text{and} \quad \alpha = d + a_1 + b_1 + c_1.$$

Thus

$$\begin{aligned} V'_{(1.2)} &\leqslant -\varepsilon U^2 - (a_0 - d)z^2 + |\theta(t)|(MV+\alpha) \\ &\leqslant -L(U^2 + z^2) + M|\theta(t)|V + \alpha|\theta(t)|, \end{aligned} \tag{2.5}$$

where $L = \min\{\varepsilon, (a_0 - d)\}$. Let

$$W = V\Delta(t), \tag{2.6}$$

where

$$\Delta(t) = \exp\Big(-\frac{1}{\mu_1}\int_0^t |\theta(s)|\,\mathrm{d}s\Big),$$

and μ_1 is a positive constant which will be determined later. It is straightforward to verify that

$$\frac{1}{\mu_1}\int_0^t |\theta(s)|\,\mathrm{d}s = \frac{1}{\mu_1}\int_0^t |g'(x(s)x'(s)|\,\mathrm{d}s \leqslant \frac{1}{\mu_1}\int_{\alpha_1(t)}^{\alpha_2(t)} |g'(u)|\,\mathrm{d}u \leqslant \frac{1}{\mu_1}\int_{-\infty}^{\infty} |g'(u)|\,\mathrm{d}u,$$

where

$$\alpha_1(t) = \min\{x(0), x(t)\}, \quad \text{and} \quad \alpha_2(t) = \max\{x(0), x(t)\}.$$

Observe that by condition (v) we have

$$\frac{1}{\mu_1}\int_0^t |\theta(s)|\,\mathrm{d}s \leqslant \frac{\Delta_1}{\mu_1}. \tag{2.7}$$

Thus, we can deduce that

$$\mathrm{e}^{-\Delta_1/\mu_1} \leqslant \Delta(t) \leqslant 1. \tag{2.8}$$

Now, the time derivative of the functional W along the system (1.2) leads to

$$\begin{aligned} W'_{(1.2)} &= \Big(V'_{(1.2)} - \frac{1}{\mu_1}|\theta(t)|V\Big)\Delta(t) \\ &\leqslant \Big(-L(U^2+z^2) + M|\theta(t)|V + \alpha|\theta(t)| - \frac{1}{\mu_1}|\theta(t)|V\Big)\Delta(t). \end{aligned}$$

Let $\mu_1 = M^{-1}$, hence

$$W'_{(1.2)} \leqslant \alpha|\theta(t)|\Delta(t) - L(U^2+z^2)\Delta(t).$$

Combining (2.8) and the last inequality, we get

$$W'_{(1.2)} \leqslant \alpha|\theta(t)| - \gamma(U^2+z^2), \tag{2.9}$$

where $\gamma = L\mathrm{e}^{-\Delta_1/\mu_1}$. Integrating (2.9) from 0 to t, we obtain

$$W(t) \leqslant W(0) + \alpha\int_0^t |\theta(s)|\,\mathrm{d}s.$$

Hence

$$W(t) \leqslant W(0) + \Delta_2, \tag{2.10}$$

where $\Delta_2 = \alpha\Delta_1$. Therefore W is bounded. From (2.6) we get

$$V = W\Delta^{-1}(t).$$

From (2.8) and (2.10) we obtain

$$V \leqslant (W(0) + \Delta_2)\mathrm{e}^{\Delta_1/\mu_1}. \tag{2.11}$$

So V is bounded. Hence, from (2.4) we conclude that there exists a positive constant D_1 such that

$$|x(t)| \leqslant D_1, \quad |U(t)| \leqslant D_1, \quad |z(t)| \leqslant D_1. \tag{2.12}$$

In view of (1.2) and (2.12) it follows that

$$|x'| = |y - g(x)| \leqslant D_1.$$

Observe that the boundedness of $x(t)$ and the continuity of g' imply that there exists a positive constant g_1 such that

$$g'(x) \leqslant g_1. \tag{2.13}$$

By an analogous reasoning it can be shown that x'' is also bounded. Indeed,

$$|x''| = |z - g'(x)x'| \leqslant |z| + |g'(x)||x'| \leqslant D_2,$$

where $D_2 = D_1\big(1 + g_1\big)$.

So,

$$|x(t)| \leqslant D_2, \quad |x'(t)| \leqslant D_2, \quad |x''(t)| \leqslant D_2 \quad \forall t \geqslant 0. \tag{2.14}$$

□

3. Square integrability of solutins

Our next result concerns the square integrability of solutions of equation (1.1).

Theorem 3.1. *In addition to the assumptions of Theorem* 2.1, *if we assume that*

$$c_0\delta_0 - \frac{a_1}{2} > 0, \tag{vi}$$

then for any solution $x(t)$ of equation (1.1),

$$\int_0^t (x''^2(s) + x'^2(s) + x^2(s))\,\mathrm{d}s < \infty. \tag{3.1}$$

P r o o f. Let $x(t)$ be a solution of (1.1). Define

$$P(t) = W(t) + \lambda \int_0^t (U^2 + z^2)\,\mathrm{d}s, \tag{3.2}$$

where $\lambda > 0$ is a constant which will be specified later and $W(t)$ is given in (2.6). We have

$$P'(t) = W'(t) + \lambda(U^2 + z^2).$$

Inequality (2.9) implies

$$P'(t) \leqslant \alpha|\theta(t)| + (\lambda - \gamma)(U^2 + z^2).$$

If we choose $\lambda \leqslant \gamma$, we obtain

$$P'(t) \leqslant \alpha|\theta(t)|.$$

Integrating the last inequality from 0 to t, we obtain

$$P(t) = P(0) + \int_0^t P'(s)\,\mathrm{d}s \leqslant P(0) + \alpha \int_0^t |\theta(s)|\,\mathrm{d}s \leqslant P(0) + \Delta_2.$$

Hence, using (3.2) we get

$$\int_0^t (U^2 + z^2)\,\mathrm{d}s \leqslant \frac{1}{\lambda} P(t) \leqslant \frac{1}{\lambda}(P(0) + \Delta_2). \tag{3.3}$$

We conclude the existence of positive constants η_1 and η_2 such that

$$\int_0^t x'^2(s)\,\mathrm{d}s = \int_0^t U^2(s)\,\mathrm{d}s \leqslant \eta_1 \quad \text{and} \quad \int_0^t z^2(s)\,\mathrm{d}s \leqslant \eta_2. \tag{3.4}$$

Now, we will prove that $\int_0^t x''^2(s)\,\mathrm{d}s < \infty$.

From (1.2) we have

$$z^2(t) = y'^2(t) = (x''(t) + \theta(t))^2 = x''^2(t) + g'^2(t)x'^2(t) + 2x''(t)\theta(t).$$

Thus

$$\int_0^t x''^2(s)\,\mathrm{d}s \leqslant \int_0^t z^2(s)\,\mathrm{d}s + 2\int_0^t |x''(s)||\theta(s)|\,\mathrm{d}s.$$

Taking into account (2.7), (2.14) and (3.4), we get

$$\int_0^t x''^2(s)\,\mathrm{d}s \leqslant l_0,$$

where $l_0 = \eta_2 + 2D_2\Delta_1$. Next, multiplying (1.1) by $x(t)$, we get

$$(x' + g(x))''x + a(t)x''x + b(t)x'x + c(t)h(x)x = 0. \tag{3.5}$$

Integrating by parts from 0 to t all the terms on the left-hand side of (3.5), we obtain

$$\int_0^t c(s)x(s)h(x(s))\,\mathrm{d}s = I_1(t) + I_2(t) + I_3(t), \tag{3.6}$$

where

$$\begin{aligned} I_1(t) &= -\int_0^t b(s)x(s)x'(s)\,\mathrm{d}s, \\ I_2(t) &= -\int_0^t a(s)x(s)x''(s)\,\mathrm{d}s, \\ I_3(t) &= -\int_0^t x(s)(x'(s) + g(x(s)))''\,\mathrm{d}s. \end{aligned}$$

Note that (2.14) is a crucial step for our estimates of $I_1(t)$, $I_2(t)$, and $I_3(t)$. Integrating I_1 by parts and using conditions (i)–(ii), we get

$$I_1(t) = -\frac{1}{2}b(t)x^2(t) + \frac{1}{2}\int_0^t b'(s)x^2(s)\,\mathrm{d}s + l_1 \leqslant l_1,$$

where

$$l_1 = \frac{1}{2}b(0)x^2(0).$$

Next, it is clear from (i) and Cauchy Schwartz inequality that

$$I_2(t) \leqslant \int_0^t a(s)|x(s)x''(s)|\,\mathrm{d}s \leqslant \frac{a_1}{2}\int_0^t (x^2(s) + x''^2(s))\,\mathrm{d}s.$$

Integrating I_3 by parts and using condition assumption (v), we obtain

$$\begin{aligned} I_3(t) = &\; -x(t)(x'(t) + g(x(t)))' + \int_0^t (x'(s) + g(x(s)))'x'(s)\,\mathrm{d}s + l_2 \\ = &\; -x(t)(x'(t) + g(x(t)))' + x'(t)(x'(t) + g(x(t))) \\ &+ \int_0^t (x'(s) + g(x(s)))x''(s)\,\mathrm{d}s + l_2 + l_3 \\ = &\; -x(t)(x''(t) + \theta(t)) + x'(t)(x'(t) + g(x(t))) \\ &- \frac{1}{2}x'^2(t) - x'(t)g(x(t)) + \int_0^t \theta(s)(s)\,\mathrm{d}s + l_2 + l_3 + l_4 \\ \leqslant &\; |x(t)|(|x''(t)| + |\theta(t)|) + |x'(t)|(|x'(t)| + |g(x(t))|) \\ &+ \frac{1}{2}|x'(t)|^2 + |x'(t)||g(x(t))| + \int_0^t |\theta(s)|\,\mathrm{d}s + l_2 + l_3 + l_4 \end{aligned}$$

such that

$$
\begin{aligned}
l_2 &= x(0)(x''(0) + g'(x(0))x'(0)),\\
l_3 &= -x'(0)(x'(0) + g(x(0))),\\
l_4 &= \frac{1}{2}x^2(0) + x'(0)g(x(0)).
\end{aligned}
$$

Hence, from (2.13) and (2.14) we get

$$I_3(t) \leqslant l_5,$$

where

$$l_5 = D_2^2\Big(\frac{5}{2} + g_0\Big) + D_2 g_0 + \Delta_1 + l_2 + l_3 + l_4.$$

Gathering aforementioned estimates into (3.6), we obtain

$$\int_0^t c(s)x(s)h(x(s))\,\mathrm{d}s - \frac{a_1}{2}\int_0^t x^2(s)\,\mathrm{d}s < l_6, \tag{3.7}$$

where

$$l_6 = \frac{a_1}{2}l_0 + l_1 + l_5.$$

From condition (iii) it follows that

$$c_0\delta_0\int_0^t x^2(s)\,\mathrm{d}s \leqslant \int_0^t c(s)x(s)h(x(s))\,\mathrm{d}s.$$

Combining estimate (3.7) and condition (vi) we obtain

$$\int_0^t x^2(s)\,\mathrm{d}s < \frac{l_6}{c_0\delta_0 - \frac{1}{2}a_1}.$$

This completes the proof of the theorem. □

E x a m p l e 3.1. We consider the following third order differential equation

$$
\begin{aligned}
\Big(x' + \Big(\frac{\sin x}{1+x^2} + 2\Big)\Big)'' + \Big(\frac{21}{2} - \frac{1}{2}\mathrm{e}^{-t/2}\Big)x'' + \Big(\frac{1}{1+t} + 3\Big)x'&\\
+ \frac{7}{2}\Big(\frac{1}{1+t} + 2\Big)\Big(x + \frac{x}{1+x^2}\Big) &= 0.
\end{aligned} \tag{3.8}
$$

Now it is easy to see that for all $t \geqslant 0$,

$$
\begin{aligned}
&10 = a_0 \leqslant a(t) = \frac{21}{2} - \frac{1}{2}\mathrm{e}^{-t/2} \leqslant \frac{21}{2} = a_1, \quad a'(t) = \frac{1}{4}\mathrm{e}^{-t/2} \leqslant \frac{1}{4},\\
&3 = b_0 \leqslant b(t) = \frac{1}{1+t} + 3 \leqslant 4 = b_1, \quad 2 \leqslant c(t) = \frac{1}{1+t} + 2 \leqslant 3 = c_1,\\
&\frac{7}{2} \leqslant \frac{h(x)}{x} = \frac{7}{2}\Big(1 + \frac{1}{1+x^2}\Big) \quad \text{with } x \neq 0, \text{ and } |h'(x)| \leqslant 7 = \delta_1,\\
&7 = \delta_1 < d = 9 < a_0 = 10, \quad \frac{1}{2}da'(t) - b_0(d-\delta_1) = -\frac{9}{2} < 0,\\
&c_0\delta_0 - \frac{1}{2}a_1 = \frac{7}{4} > 0,
\end{aligned}
$$

and

$$
g(x) = \frac{\sin x}{1+x^2} + 2.
$$

A simple calculation shows

$$
\int_{-\infty}^{\infty} |g'(u)|\,\mathrm{d}u \leqslant \int_{-\infty}^{\infty} \Big(\Big|\frac{\cos u}{1+u^2}\Big| + \Big|\frac{2u\sin u}{(1+u^2)^2}\Big|\Big)\,\mathrm{d}u \leqslant \pi + 2.
$$

All the assumptions (i) through (vi) are satisfied, so every solution x of equation (3.8) and their derivatives satisfies (3.1).

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